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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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DECISION METHODS FOR ELEMENTARY ALGEBRA

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About twenty years ago Professor Alfred Tarski obtained a mathematical basis for a decision method for the class of all true sentences of "elementary algebra." After being delayed by the war his results have been published by Project Rand [**3**] and, with a few minor corrections and supplementary notes, by the University of California Press [**4**]. The present paper consists of a brief description of Tarski's method (referred to as the decision machine method), several applications of this method, and a discussion of the significance of the method.

1. The decision machine method. The decision machine method, like other logical processes, requires precise definitions. Each *variable* is an element of an infinite ordered sequence of symbols ranging over the real numbers. The variables cannot be considered over the set of integers or any other set involving "set-theoretic" notions. An *algebraic constant* is one of the symbols "1," "0," or "-1." An *algebraic term* is an expression involving only the variables, the algebraic constants, and the two operation symbols "$+$" and "$\cdot$". The *sentential connectives* are "$\sim$" meaning not, "$\wedge$" meaning and, and "$\vee$" meaning non-exclusive or. The symbol "E" is an *existential quantifier* and is used in quantifier expressions such as "(Ex)," meaning there exists an x such that, and "$(E_k x)$," meaning there exist exactly k values of x such that.

Expressions having one of the forms $\alpha=\beta$ and $\alpha>\beta$, where α and β are algebraic terms, are called *atomic formulas*. An expression built up from atomic formulas by means of sentential connectives and quantifiers, according to an obvious recursive law, is called a *formula*. Thus the expresions

$$x = y, \qquad 1 + 1 = 0 + 1, \qquad 1 + (-1) > -1$$

are atomic formulas, whereas the expressions

$$(Ey)[(x = 2) \wedge (x + y = 5) \wedge (y > 0)] \tag{1}$$

$$(Ex) \sim (Ey)[(y > 0) \wedge (x + y < 7)] \tag{2}$$

and

$$(x < 0) \vee (x = 0) \vee (x > 0) \tag{3}$$

are formulas but are not atomic formulas since each of these three formulas involves at least one sentential connective or a quantifier. Formulas that contain quantifiers such as $(E_k x)$ may be regarded as abbreviations for other formulas that contain only quantifiers such as (Ex).

A variable is called a *free variable* in a formula according to the following conventions:

(i) x is a free variable in an atomic formula if it occurs in the formula,

(ii) x is a free variable in a formula of the form $(Ey)\phi$ if and only if $y \neq x$ and x is free in the formula ϕ,
(iii) x is a free variable in $\sim\phi$ if and only if x is free in ϕ, and
(iv) x is a free variable in the formulas $\phi \wedge \theta$ and $\phi \vee \theta$ if and only if x is free in at least one of the formulas ϕ, θ. A formula in which there are no free variables is called a *sentence.*

Thus x is a free variable in the above formulas (1) and (3) but not in the sentence (2).

These definitions are used by Tarski in [4] to prove that any formula can be replaced by an equivalent formula without quantifiers and that in this process no additional free variables are introduced. As a special case of this result any sentence can be replaced by an equivalent sentence without quantifiers. The final result of [4] is based upon the presentation of a mechanical procedure by means of which any sentence without quantifiers may be replaced by one of the sentences $0=0$, $0=1$ (*i.e.*, the validity of any sentence of "elementary algebra" may be decided).

Intuitively, the decision machine method depends upon the elimination of quantifiers, *i.e.*, upon finding for each formula ϕ an equivalent formula θ without quantifiers and with no free variables except those occurring in ϕ. We may visualize this process as follows: Since all formulas are defined in a finite recursive manner, any formula ϕ that contains quantifiers is composed of a part containing all but one of the quantifiers followed by a part, say θ, composed of exactly one quantifier followed by a formula without quantifiers. The elimination of quantifiers is based upon the replacement of θ by an equivalent formula θ_1 without quantifiers and with no free variables except those in θ. In this way we may, by an elementary theorem of logic, obtain a new formula ϕ_1 that is equivalent to the given formula ϕ, contains one less quantifier than ϕ, and has no free variables except those in ϕ. Then, since any formula ϕ can contain only a finite number of quantifiers, the above process may be applied a finite number of times to obtain a formula, say ϕ_m, that is equivalent to ϕ, has no quantifiers, and has no free variables except those in ϕ.

The desired replacement of a formula, such as $(Ex)\phi$, having a single quantifier (Ex), by an equivalent formula without quantifiers may be accomplished as follows: The formula ϕ has now been assumed to have no quantifiers. By a theorem of elementary logic the formula ϕ is equivalent to a formula ϕ_1 which is a disjunction of conjunctions of atomic formulas and their negations, *i.e.*,

$$\phi \leftrightarrow \phi_1 = \phi_{11} \vee \phi_{12} \vee \cdots \vee \phi_{1n},$$

where ϕ_{ij} is a conjunction of formulas having the form $\alpha=\beta$ or $\alpha>\beta$ or their negations. The theorem used here is an analogue of the theorem of boolean algebra which asserts that any element in a subalgebra generated by n elements may be expressed as a sum of products of the elements and their complements. The original formula $(Ex)\phi$ is now equivalent to

$$(Ex)(\phi_{11} \vee \phi_{12} \vee \cdots \vee \phi_{1n})$$

and therefore, by elementary logic, to

$$(Ex)\phi_{11} \vee (Ex)\phi_{12} \vee \cdots \vee (Ex)\phi_{1n}.$$

We have sketched the dependence of the method of elimination of quantifiers upon the elimination of a single quantifier and, now, upon the elimination of a single quantifier in formulas consisting of a single quantifier and a conjunction of atomic formulas and their negations. Since

$$\sim (\alpha = \beta) \leftrightarrow (\alpha > \beta) \vee (\beta > \alpha)$$

and

$$\sim (\alpha > \beta) \leftrightarrow (\alpha = \beta) \vee (\beta > \alpha),$$

the negations of atomic formulas may be replaced by disjunctions of atomic formulas. Also, since the quantifier may, as above, be distributed over the terms of a disjunction, the elimination of quantifiers depends upon the finding of equivalent formulas without quantifiers for formulas consisting of a quantifier and a conjunction of atomic formulas.

Finally, we note that the atomic formulas are of the form $\alpha=\beta$ and $\alpha>\beta$ where α and β may be considered as polynomials in a variable x whose coefficients are polynomials in other variables with integral coefficients. Accordingly, if a formula such as $(Ex)\phi$ consists of a quantifier and a conjunction of atomic formulas, then the formula states that there exists x such that a certain set of polynomial equations and inequalities are simultaneously satisfied. The method for eliminating quantifiers is essentially a method for expressing formulas, such as $(Ex)\phi$, by means of disjunctions and conjunctions of equalities and inequalities (*i.e.*, of atomic formulas) involving the coefficients of the given polynomials. The method used is a generalization of the method of Sturm and provides the core of the decision machine method.

2. Operations used. A complete description of the decision machine method would require an amplification rather than a few comments upon Tarski's work in [4]. It is hoped that the following brief description of the operations used in the decision machine method will provide an intuitive understanding of the method for all readers and facilitate further study of the method by the readers who have a particular interest in this phase of mathematics. Eleven operators are used to establish a mechanical procedure for replacing any formula by an equivalent formula without quantifiers and without introducing additional free variables. A twelfth operator gives rise to a decision regarding the validity of sentences (*i.e.*, of formulas without free variables) obtained using the other eleven operators. Tarski defines each of the operators in "elementary algebra." We shall endeavor merely to identify the nature of or need for the operators.

Let α and β be polynomials in x with algebraic terms that do not involve x

as coefficients. In "elementary algebra" polynomials may have leading coefficient zero. Then

(i) Rd_x operates upon polynomials in x. The reductum of α, $Rd_x(\alpha)$, is obtained by deleting the term of α that is of highest degree in x. When α does not involve x, $Rd_x(\alpha)=0$. The symbol $Rd_x^k(\alpha)$, the k-th iterate of $Rd_x(\alpha)$, is defined recursively for nonnegative integral values of k.

(ii) P_x operates upon terms or formulas. For any term it indicates that the operations $+$ and $\cdot$ are to be performed as indicated and the resulting terms arranged in increasing powers of x. For any formula ϕ without quantifiers the formula $P_x(\phi)$ is defined such that it is an equivalent formula without quantifiers and without negation signs.

(iii) Q operates upon formulas without quantifiers and without negation signs. If ϕ is such a formula, then $Q(\phi)$ is an equivalent formula and is a disjunction of conjunctions of atomic formulas, *i.e.*, $Q(\phi)$ is in "disjunctive normal form."

(iv) M_x^n operates upon polynomials $\alpha(x)$ for non-negative integers n. The formula $M_x^n(\alpha)$ is valid if and only if x denotes a root of α of order n where x is said to denote a root of order 0 if and only if $\alpha(x)\neq 0$. The definition of $M_x^n(\alpha)$ involves the concept of a formal derivative of $\alpha(x)$.

(v) F_x^n operates upon ordered pairs of polynomials α, β. The integer n such that the formula $F_x^n(\alpha, \beta)$ is valid denotes a complicated relationship among certain roots of the polynomials that are right end-points of segments on which the polynomials are of the same sign.

(vi) G_x^n operates upon ordered pairs of polynomials and is defined in terms of F_x^n. The formula $G_x^n(\alpha, \beta)$ is valid if and only if for increasing x there exist exactly n more values of x at which the product $\alpha\beta$ changes from positive to zero to negative than there are values of x at which the product changes from negative to zero to positive.

(vii) R_x operates upon ordered pairs of polynomials. The polynomial $R_x(\alpha, \beta)$ is the negative of the remainder when $\alpha(x)$ is divided by $\beta(x)$.

(viii) S operates upon formulas of the form $G_x^k(\alpha, \beta)$ and gives rise to an equivalent formula that is used in the definition of the operator T.

(ix) H_x^n operates upon ordered pairs of polynomials. The integer $n=h(\alpha, \beta)$ such that the formula $H_x^n(\alpha, \beta)$ is valid denotes that there exist exactly n "numbers x such that the difference between the order of x in α and the order of x in β is an odd integer, not necessarily positive."

(x) T operates upon formulas of ten specified kinds and gives rise to equivalent formulas without quantifiers or additional free variables.

(xi) U operates upon any formula of "elementary algebra" and gives rise to an equivalent formula without quantifiers or additional free variables. "If ϕ is any sentence, then $U(\phi)$ is an equivalent sentence without any variables or quantifiers."

(xii) W operates upon sentences that involve no variables and no quantifiers. If ϕ is such a sentence, then $W(\phi)$ is equivalent to ϕ and is exactly

one of the two sentences

$$0 = 0, \qquad 0 = 1.$$

Formal definitions of the above operators in "elementary algebra" may be found in [4]. The first eleven operators serve to replace any sentence of "elementary algebra" by a sentence without variables or quantifiers. Then since each term of such a sentence is obtainable from the constants 1, 0, and -1 using the operations $+$ and $\cdot$, each term may be assigned an integer as its value. The operator W uses these integers to provide a decision method for the class of all true sentences of "elementary algebra."

Basically the decision machine method is a formalization and extension of Sturm's Theorem. It is primarily applicable to sentences of the form $(E_k x)\phi$ where k is necessarily an integer and ϕ is a "formula" that contains no free variables other than x. The most severe limitations of the method are based upon the necessity for avoiding all set-theoretic ideas. Accordingly one cannot consider integral variables, rational variables, polynomials of arbitrary degree, or the solvability of an equation in terms of radicals. This limitation is implied by the word "elementary" in the phrase "elementary algebra."

3. Applications. Consider the sentence ϕ

$$(E_2 x)(x - 5 = 0), \tag{4}$$

i.e., there exist exactly two values of x such that $x-5=0$. Tarski defines the formula $T(\phi)$ [4; 31–34] to be

$$[\sim(-5 = 0) \vee \sim(1 = 0)] \wedge SG_x^{-2}(x - 5, 1) \tag{5}$$

where the last occurrence of "1" appears because it represents the formal derivative with respect to x of the polynomial $x-5$. As mentioned in Section 2, the formula $G_x^{-2}(x-5, 1)$ denotes that for increasing values of x there exist exactly -2 more values of x at which the product $(x-5)$ (1) changes from positive to zero to negative than there are values of x at which the product changes from negative to zero to positive. The formula $SG_x^{-2}(x-5, 1)$ in the formula (5) is defined [4; 24] to be

$$\begin{aligned}[(1 = 0) \wedge SG_x^{-2}(-5, 1)] &\vee [(1 = 0) \wedge SG_x^{-2}(x - 5, 0)] \\ &\vee [(1 > 0) \wedge SG_x^{-1}(1, 0)] \vee [(0 > 1) \wedge SG_x^{-3}(1, -x + 5)].\end{aligned} \tag{6}$$

Since $1>0$, we are now concerned with $SG_x^{-1}(1, 0)$ which is defined to be the sentence $0=1$. Thus the sentence (4) is equivalent to the sentence $0=1$ and is not valid. The above statement of the formula (6) defining $SG_x^{-2}(x-5, 1)$ has been simplified—relative to the corresponding statement in [4]—by using the operators Rd_x, P_x, and R_x mentioned in Section 2. Formula (6) has been stated without explanation in order to illustrate the recursive nature of the definitions and the mass of details involved in the decision machine method without merely

repeating the intricate web of definitions given in [4].

Many familiar theorems may be expressed for calculation by the decision machine method. As in Sturm's Theorem the number of distinct real roots of a polynomial $\alpha(x)$ is k where the sentence

$$(E_k x)(\alpha = 0)$$

or the equivalent sentence

$$G_x^{-k}[\alpha, D_x(\alpha)]$$

without quantifiers is valid. The number of roots of multiplicity j on interval $a < x \leqq b$ is the integer n where the sentence

$$(E_n x)\{M_x^j(\alpha) \wedge (a < x) \wedge [(x < b) \vee (x = b)]\}$$

is valid [5]. The Cauchy-Sylvester Theorem [1] as extended in [2] may also be easily expressed in "elementary algebra."

When the system of polynomials under consideration

$$0 < f_j(x) \qquad (j = 1, 2, \cdots, s)$$

contains only inequalities, the number of values of x satisfying the system is either zero or infinite. Thus the decision machine method can test only for consistency. However, it is possible to extend Sturm's theorem in the sense of counting segments of values of x that have zeros of the polynomials as their finite end points and on which the system is satisfied. This problem has been completely solved by traditional methods [2] in the case of polynomials in one variable. The steps used in the solution of this problem involve systems of equations and inequalities and may be expressed for calculation by the decision machine method.

In addition to applications such as the above, the decision machine method may formally be used to solve decision problems and to obtain some theoretical results in the elementary algebra of complex numbers, euclidean and projective geometries, topology, and other related theories.

4. Significance. Even though the applications of theorems such as those mentioned above are often very tedious by traditional methods, they appear much shorter than those of the decision machine method. The following quotation from [3; 4] bears out this point.

> "Since a decision machine . . . requires no intelligence for its application, it is clear that whenever one can give a decision method for a class K of sentences, one can also devise a decision machine to decide whether an arbitrary sentence belongs to K. It is with this possibility in mind that the decision method for elementary algebra is presented here. . . . The decision method for elementary algebra gives the mathematician the assurance that he will be able to solve this question one way or another by working at it long enough. He still may hesitate to spend the time that would be required for such a solution. A machine of the sort suggested here would remove this hesitation . . . . "

Without detracting from the logical significance of the decision machine method, it is certainly true that most mathematicians will "hesitate to spend the time that would be required for such a solution" in the case of any problem of sufficient complexity as to be of mathematical interest. Thus the decision machine method will not supplant the usual method for the application of Sturm's Theorem. Neither can it be expected to supplant present methods of applying the other theorems mentioned above (until a machine is available) even though the present methods for calculating them may be very tedious.

The decision machine method does make possible the formal extension of many theorems from one to n variables since at each step the possible vanishing of each leading coefficient may be taken as a separate alternative. Thus formally the problem of extending many theorems becomes one of finding statements for the more general theorems in the terminology of "elementary algebra" as formulas without free variables. The consistency theorems are the easiest to formulate. In general, the system

$$\begin{aligned} 0 &= f_i(x_1, x_2, \cdots, x_n), \qquad (i = 1, 2, \cdots, s), \\ 0 &< f_j(x_1, x_2, \cdots, x_n), \qquad (j = s+1, \cdots, t), \end{aligned}$$

is consistent if and only if

$$\begin{aligned} &\sim (E_0x_1)(E_0x_2) \cdots (E_0x_n)[(0 = f_1) \wedge (0 = f_2) \wedge \cdots \wedge (0 = f_s) \\ &\quad \wedge (0 < f_{s+1}) \wedge \cdots \wedge (0 < f_t)]. \end{aligned}$$

The variables may also be restricted by relations such as $a_j < x_j < b_j$.

The decision machine method for determining the validity of sentences of "elementary algebra" is based upon an extension of Sturm's Theorem. Theorem 29 is a key theorem of [4] and is based upon the ten cases in the definition of the operator T. This definition is very compactly stated by nearly four pages of symbols involving the pyramid of definitions on the preceding twenty-five pages. When the polynomials have real coefficients, the first case of Theorem 29 is precisely Sturm's Theorem and the proof of the second case is based upon the Cauchy-Sylvester Theorem.

In conclusion, the decision machine method represents a considerable formal advance of theoretical interest to all mathematicians. Prior to the construction of a decision machine it will not supplant present procedures and should not discourage continued search for practical procedures to solve problems of elementary algebra. It has practical value in that mathematicians may first seek to formulate problems for a decision machine and then seek methods for making the decision.

The existence (past, present, or future) of a decision machine is entirely possible. Some, and perhaps all, of the operators (Section 2) used in Tarski's decision machine method may be used in our present electronic computers. Whoever succeeds in developing codes of instructions to enable one of our present machines to serve as a decision machine for "elementary algebra" will

thereby solve a class of decision problems of far greater scope than those considered in [4] since within the accuracy of the machine (for example, modulo 2^{-39} on the interval from -1 to 1 on the University of Illinois computer) all convergent power series may be considered as polynomials. Thus a decision method for all functions that may be represented by convergent power series in a finite set of real variables may now be possible within the accuracy of the machine used. Even though the result obtained is modulo 2^{-39}, for example, and therefore is not a precise theoretical result, such a decision method would have very wide application—both practical and theoretical.

Added in proof. Since the present paper was submitted, A. Seidenberg has shown that the results obtained by Tarski's decision machine may also be obtained in algebraic geometry. See "A New Decision Method for Elementary Algebra," *Annals of Mathematics*, vol. 60, 1954, pp. 365–374. Although the two methods appear equally suitable for human calculation, Seidenberg's method provides new hope for the construction of a decision machine.

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THE FIRST CONFERENCE ON TRAINING PERSONNEL FOR THE COMPUTING MACHINE FIELD*

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1. The nature of the conference. This conference, which was held in Detroit, Michigan, June 22–23, 1954, was sponsored by Wayne University with the cooperation of the Association for Computing Machinery, the Industrial Mathematics Society, and the local chapter of the Professional Group on Electronic Computers of the Institute of Radio Engineers. The details were arranged by Professor Arvid W. Jacobson, director of the Wayne University Computation Laboratory, with the help of his staff. The purpose of the conference was to discuss

(a) manpower requirements in the computer field,

* The full proceedings of this conference have been published and are available from the Wayne University Press, Detroit 1, Michigan. The cost is $5.00.

(b) educational programs for training computer personnel, and
(c) possibilities for cooperation among business, industry, government agencies, and educational institutions in solving the manpower problem.

The program consisted of four groups of invited addresses and two panels, followed in each case by extensive discussion from the floor. The speakers represented a wide range of organizations of the kinds just mentioned.

That such a conference was opportune hardly needs emphasis, for the shortage of competent candidates for positions in the computer field is becoming increasingly critical as the demand for computers grows. Not too widely recognized is the fact that this problem is of importance to the entire mathematical fraternity. However, although many mathematicians have scorned computer art, most of us recognize that it has added a whole new dimension to research in engineering and in the physical and the social sciences. Perhaps many of us have felt that, much as we might like to do so, we cannot adequately prepare our students for participation in this field for lack of a machine. However, the speakers indicated repeatedly that *the most important educational requirements for work in the computer field may be met even by those who have no expensive equipment.* It is the purpose of this report to outline the details of conclusions such as these, to list the constructive actions suggested, and to call attention to the opportunities, existing and future, for employment in this field.

2. Manpower needs in the computer field. At present, computers are most extensively used for engineering calculations, next for scientific research calculations, and only to a relatively small extent for the processing of business data. In all of these areas, and especially in the third, as machines are built that better meet existing and developing needs, and as potential users learn what these new tools for data-processing can do, the demand for the machines expands at an accelerating rate. This of course creates a corresponding demand for a variety of trained personnel. Present indications are that this demand will reach dramatic proportions within a decade.

Employees in this field who require mathematical training may be grouped into the following classifications.

(A) Employees of the computer manufacturer (who may also use computers).

(1) Machine design men, trained in electronics, mathematics, information theory, the logical design of computers, *etc.* Such men are most difficult to find. Usually they grow up in the company. (Manufacturers of telephone switching systems, industrial control systems, and other computer-like devices also need men with these qualifications for work in their research and development organizations, so that a broad basic training rather than one governed solely by computer considerations is indicated. The same observation applies to the next category of employee as well.)

(2) Electronic circuit designers. These will be trained principally by departments of electrical engineering. Many more will be needed than men of the first type.
(3) Technical salesmen, educated in special fields of application, experienced in the use of the machines they sell, and able to communicate effectively with the prospective user. Many of these will be mathematics majors or minors.
(4) Training staff for the manufacturer's own employees and for those of his customers. Again, many of these will be mathematics majors.
(5) Assembly and maintenance workers. For a large proportion of these, high school training will be adequate.

(B) Employees of the computer user (who may also build computers).

(1) Professional computer staff to develop program libraries, to do numerical analysis, to develop better methods of using machines, to reduce data and formulas to computable form, *etc.*
(2) Applied mathematicians with computer experience to assist research workers in various fields in formulating their problems properly for computation. Teams of this kind have proved to be especially effective in research.
(3) Programmers.
(4) Maintenance workers.

In the first two of these categories, advanced mathematical training is required; in the last two, a high school education is often sufficient but intelligence and reliability are essential. In all categories, *the premium is on people who are above average in personality, mental ability, and education.* The more education a candidate for employment has, the greater are his opportunities to find a responsible position with a high salary.

Currently there is a shortage in all the above classes except perhaps in the case of programmers. The shortage is in part due to the fact that potential employees are not aware of the good salaries, the opportunities to advance, the interesting work, and the pleasant associations characteristic of the field. However the shortage will continue even when this knowledge is more widely circulated, simply because there are not enough students now receiving the necessary instruction at the various levels.

A related problem is that the present number of mathematics teachers is too small to take care of the future training need. Moreover, because of the salary differential, we can expect future graduates to be attracted increasingly to positions other than teaching. In fact, teachers themselves have often been attracted to such positions, and in the long run, the problem of finding teachers will be more critical than that of finding students. The public will thus be faced increasingly with the necessity of making teaching more attractive to able candidates in order to preserve the quality of the schools.

3. Educational institutions and the manpower need. Clearly, the greatest responsibility for meeting this manpower need falls on the educational institutions. The primary considerations, as they were developed by the various speakers of the conference, appear to be these:

(1) The ultimate effect of the computer will be to increase leisure time by accelerating the increase in output per man-hour characteristic of our technology. Too much emphasis on vocationalism and too little emphasis on the liberal aspects of education would tend to destroy the benefits of this leisure, and would be a serious disservice to democracy.

(2) Even from the point of view of effective participation in computer work, specific machine training must not be allowed to replace fundamental scientific education. Neglected education is a handicap that may never be overcome, but machine operation can always be learned on the job in a few months. The rapidity with which machines become obsolete lends emphasis to this point.

(3) No extensive introduction of new mathematics courses is indicated. However, the existing courses need to recognize in a realistic way the importance of computation in the scientific and industrial activities of our time, to train students to think in terms of many variables, *etc.* Thus, significant changes in emphasis and point of view are in order.

(4) Education should be designed to challenge the student's best abilities and particular emphasis should be placed on courses and methods of teaching which require creative thinking in the learning process rather than imitation and drill alone. Indeed, what the student needs most from his schooling is a basic foundation of principles by whose aid he can continue his learning throughout his years of employment and thus keep abreast of changing technology.

(5) There should be a far-reaching re-emphasis on mathematical modes of thought and analysis at all levels of the educational process.

As mentioned previously, these primary goals require no expensive machinery for their attainment.

Comments directed to specific educational levels were also made. At the *high school level*, these included the following:

(6) Experience shows that able high school graduates can easily be trained to perform some of the more routine tasks of computing such as programming, for example, thereby releasing more highly trained staff for work in which advanced training is necessary.

(7) The interests of high school students are typically dormant and need stimulation and encouragement. This can be done by emphasizing the importance of computation in their mathematics courses, by introducing computer "literature" into high school libraries, by providing experience with desk machines, by competitions and prizes, *etc.*

(8) An emphasis on accuracy and checking is vital. "Getting the method right" isn't enough: a computing machine does just what it is told to

do, and one error in programming may cause it to make thousands of errors a minute.

(9) The vocational high schools can provide good preparation for maintenance work on computers through training in shop work, electronics, mathematics, *etc.*

Comments directed primarily at *the college level* included these:

(10) Numerical analysis, though difficult, is the most important single mathematical course needing to be introduced into the curriculum. However, the approach must be up-to-date: much of the classical material on computation is obsolete.

(11) Training in abstract mathematics (such as algebra at the Birkhoff-MacLane level, for example) is most valuable for those who will analyze problems and reduce them to mathematical form prior to computation and especially for those who will design logical machines.

(12) Consideration should be given to introducing basic logic, number systems other than decimal, and vectors and matrices, all of which are fundamental in modern computer mathematics, into the program of the freshman year, or even into the high school program. Mathematical induction, the "soul of programming," deserves a special emphasis.

(13) The subject of finite approximation procedures should be introduced wherever possible (for example, in studying integration) because such procedures are used in computing with machines, which cannot pass to the limit.

(14) Experience with desk calculators is important because it gives a "feel" for computation and should precede training with electronic computers.

(15) The student expecting to enter the computing field should develop an interest in, and obtain experience with, applications, since an essential qualification is the ability to communicate with the engineers and scientists who provide the problems.

(16) There is a wide gap between the undergraduate's mathematical training and what the graduate school or industry will require of him. More attention must be given to improving his mathematical preparation.

(17) It was repeatedly emphasized that a broad, thorough training in mathematics, appropriately oriented with respect to computation and applications, is the best training for mathematical work in the computer field, and that only a general familiarity with what the machines can do is prerequisite to such employment. Most of the specialized training may be left to graduate courses or may be obtained on the job.

Considerable regret was expressed by a number of speakers that many mathematicians view the whole science of computation as being somewhat beneath their consideration. Indeed, this attitude was felt to have constituted a serious impediment to progress in view of the influence exerted by such mathematicians on graduate students. The following comments are directed to *the graduate level* and are in part an answer to this attitude:

(18) The rewriting of analysis in a form adapted to the computer is a far from trivial problem and will require years for its completion.

(19) The mathematical analysis of an applied problem and its reduction to computable form is often not trivial either, even though only elementary mathematics may be involved.

(20) Besides numerical analysis, the qualitative theory of differential equations (as opposed to the search for closed solutions), the effect of higher order terms that no longer need to be neglected in analyzing physical problems, practicable existence theorems and tests, the theory of errors, the theory of systems, an abstract theory of digital control processes, and the development of quantitative methods of research in the social sciences, were all cited as important areas for mathematical investigation.

(21) Graduate students are attracted to staffs with lively, current problems. Thus there will be an inadequate supply of applied mathematicians until more mathematicians in the universities become active in that kind of research. In fact, the sets consisting of the mathematics which is taught, the mathematics which is the object of research, and the mathematics which the world needs, have too limited an intersection.

(22) In institutions having a computer, special lectures and special courses may be given to inform students and staff in other departments about the nature of the computer, what it can do, what it should not be expected to do, how it can aid their research, and how it should be programmed for this purpose. In this way, the computer can be used to exert a beneficial, unifying influence on the university.

Some of these observations, as well as those made earlier concerning the type of training that is essential, indicate that the interests of the pure mathematician, the applied mathematician, and the computer specialist are not so divergent as might once have appeared.

4. Non-educational employers and the manpower problem. The observation of those familiar with the recruiting problem was that "Ph.D.'s are hard to find and harder to recruit" for positions in the computing field. This is partly due to the relatively small number of possible candidates, and partly due to an unwillingness to accept nonacademic employment because of anticipated loss of freedom of research, limited vacation time, routine type of work, and so on, for which the considerably better pay does not always seem to be adequate compensation. As Ph.D.'s in such employment are increasingly provided with semiprofessional and nonprofessional aid to help with the more routine procedures, and as nonacademic research organizations take on a more academic flavor, it will become progressively easier to recruit such personnel. In fact, a real problem may ultimately be to induce enough Ph.D.'s to stay in teaching positions.

Some specific ways in which nonacademic employers can help to increase the supply of highly trained personnel are these:

(1) Funds should be given to schools for the purpose of setting up computation laboratories.
(2) Scholarship funds should be provided for promising students.
(3) Students should be given summer positions during their period of schooling and part-time work in winter where that is feasible.
(4) Consultation opportunities should be provided for teachers. This will make it easier for them to stay in teaching, where their services will benefit many employers rather than just one.
(5) Endowed professorships, fellowships for further study, funds for visiting professorships, and temporary positions in business, industry, or government, should be provided for the teachers also.
(6) Industry should allow near-by schools without computers to use its machines on a part-time basis and provide significant problem material as a means of giving practical experience to both students and staff.
(7) Interest, goodwill, and inspiration should be created by providing informative lectures, orientation programs, appropriate literature, *etc.*, in schools and colleges.
(8) Through cooperation with and support of appropriate agencies (such as this conference, for example), non-academic employers can aid in the study of the needs for training and research and in the formulation of policies and plans for handling the problems that arise.

Many forward-looking companies and certain government agencies (the National Science Foundation, for example) are already active in the above ways. Their help has been widely appreciated by students and teachers throughout the country. It is to be hoped that other companies will soon follow suit.

Clearly not all the funds for such activities can come from the computing machine manufacturers. Indeed, other companies stand to gain much in the long run from sharing in these responsibilities in order to advance computer art.

An outstanding example of this sort of shared responsibility is to be seen in the Wayne University Computation Laboratory, which derives advice and support from some twenty companies in the Detroit area. Because of this cooperation, the Laboratory is enabled to make a unique contribution to the economic life of the community as well as to give its students particularly valuable training.

There is no reason why similar laboratories could not be set up in many other universities. Indeed, this should be done, for in this area as in others, the universities must serve as the primary sources of new talent and new ideas. Moreover, theirs is also the responsibility of helping to ensure that the potentialities of these new machines will be used only for the preservation and advancement of the values of our free, democratic society.

5. Conclusion. An important feature of these meetings was the enthusiasm of the speakers for what the computer and related devices can now do and will

do in the future. The automatic office, the automatic factory, and better systems of communication, as well as better computers, were indicated as being well on the way toward realization. Moreover, in all these developments, *mathematics* will play an important role. Whether or not we as *mathematicians* exercise any significant influence thereon depends on the extent to which we are willing to cooperate in the education of workers in this field. If we are unwilling to recognize the rising importance of applied mathematics and computation, much of our present responsibility will of necessity be taken over by other departments of our universities and by industrial training programs. On the other hand, there is much to be gained, for all concerned, from mutual respect and cooperation.

AN EXPERIMENT WITH TELEVISION

MARGUERITE LEHR, Bryn Mawr College

"Of myself I say nothing, but in behalf of the business which is in hand I entreat men to believe it is not an opinion to be held but a work to be done."—Francis Bacon

1. Introduction. This is a report on a semester's experience with a mathematics program on a commercial network, concerned not so much with the program itself as with the powerful means at our disposal, for our purposes. I say "our" not only in the wide sense of national concern with an educational dilemma, if not disaster, in mathematical training but also in the special sense of mathematicians, with all that implies of the specialist's jealous concern for his own field of knowledge. Fifteen weeks convinced me that the best we can manage—best in a mature professional sense even in so supposedly esoteric a field as pure mathematics—pays off in that unknown general public at the receiving screen of a television camera in precisely the ways we want and need.

The program called *University of the Air* which has just completed its fourth year over Philadelphia's Channel 6 is a joint undertaking of WFIL and some twenty-five institutions of the Delaware Valley region, ranging from Rutgers and Lehigh through Philadelphia's varied list to Lincoln and the University of Delaware. Individual faculty members or departments submit to the studio's educational director suggested plans for a fifteen-week "course," October through January or February through May, to be developed in either a seventeen or twenty-seven minute period once a week. Ten courses are selected for each semester and the lecturers supply outlines from which the studio compiles a semester syllabus, announced at each talk and sent out for twenty-five cents. In this way, five mornings a week from 11:15 to 12:00, subjects ranging from Chemistry of the Body and Folk Art to T. S. Eliot have been presented, with sufficient response to warrant the studio's continuing the plan even with an allocated

and developing educational channel. The director had been particularly anxious to present mathematics or theoretic physics, and Bryn Mawr's committee asked me if I would consider offering a mathematics program.

The opportunity and the risk were both great. A mathematics course offered with or without credit on an educational channel can be planned at some definite level of technical competence. Without this sifting of audience, lecturers tend to stress immediate practicality or to settle, perhaps reluctantly, for the thin fare of puzzles and pretty designs. I was sure one could whittle out fifteen topics exhibiting non-trivial mathematical ideas and strategies of thought, and present them in such a way that you would hear more if you knew more. Proper ordering and careful verbal echoes of past talks could give unity even though each question must be posed, developed and signed off in twenty-seven minutes. Surest of all was my knowledge that mathematics *per se*, shown as pure and driving human activity, not as accumulated results, intrigues people of all kinds and trainings; I've never seen it fail. (What, never? Well, hardly ever!) Nevertheless I hesitated, aside from the work and the risk.

It is an ironic fact in academic fields that from outside the cry is, "Specialists can't *communicate*," while on both sides but particularly within the profession those who can or try to are suspect. They must be sugar-coating or watering down; even more likely, they are really devoted amateurs satisfied by over-simplifications which miss the root of the matter and misrepresent the activity. This is a costly prejudice, in a time when a proper diffusion of knowledge is one of our few safe-guards. No test situation however restricted can be neglected. Here, the syllabus permitted each talk to be conceived as sparking thought rather than completing some tiny piece of work. Topics with aspects being treated in one's advanced lectures would have the requisite overtones of current live activity, and close attention to verbalization could satisfy a more informed listener yet not lose too soon the less trained. But what conception of the public would justify such use of time by a member of the faculty in a small department which must carry mathematics curriculum through the Ph.D.?

The public includes people who hear a child's sums called his mathematics, and industries which classify as mathematicians little girls who punch adding machines. The public includes highly trained and able specialists who are naive about the intention of mathematics to the point of deep mistrust. This is our business, not to be dismissed with a smile even if our concern is entirely and narrowly vigorous growth of the pure field. It is a counsel of defeat to say, "Who would be watching T.V. at that hour?" (You'd be surprised at the spread that showed up even in chance encounters.) The program could be attacked only by forgetting "the public," except as people with minds to be intrigued by what had intrigued men before them, by trusting to thinking in front of them instead of instructing them. For us, the main problem is forcing that high level of attention from which ideas can spring with conviction. For many of them, reaching such a level even for a few minutes could be an exciting experience; one knows this from one's freshmen. My plan was an experiment in

"talking from saturation"; the Monday morning time was a deliberate choice, as insurance against a divided mind. My intention was never to fight shy of abstractions; rather, to highlight the abstracting process wherever possible, from the child's first fresh leaps to the articulate strategies of the working mathematician.

2. Syllabus and course content. The syllabus contained a brief introduction and a two-hundred word section for each week, with abstract and three references selected to offer wide range of treatment, and as much further bibliography as possible in the strict text or treatise sense. Any one sending for the syllabus was already interested in at least one of the ten subjects; the abstracts were therefore intended to catch attention by questions raised, while the range of reference would (I hoped) reassure and serve a more informed reader. This was one device for coping with the unknown spread of mathematical sophistication in the potential audience. To give some notion of the attack, I have reproduced the introduction and the list of topics with single relevant sentences. The abstract "On Ideas of Space" indicates style in general, and the summary serves as one statement of aims. Repeatedly referred to were:

What Is Mathematics?—Courant and Robbins
Geometry and the Imagination—Hilbert, Cohn-Vossen
An Invitation to Mathematics—Dresden
Mathematical Snapshots—Steinhaus
Current *Scientific American* articles listed were:
Crystals and the Future of Physics—Le Corbeiller, January, 1953
Mathematical Machines—Davis, April, 1949
Leonhard Euler and the Koenigsberg Bridges—(Translation), July, 1953
Greek Astronomy—de Santillana, April, 1949
Probability—Warren Weaver, October, 1950
Statistics—Warren Weaver, January, 1952
Topology—Tucker and Bailey, January, 1950
Symbolic Logic—Pfeiffer, December, 1950
Is There an Infinity?—Hans Hahn (Translation), November, 1952

Some further references are given in the condensed version as illustrative.

3. Foreword and topics from the syllabus *Invitation to Mathematics*. Most people associate mathematics with getting answers or giving proofs, and it is easy to see why. Mathematics does set up good rules for getting quick answers, and does accumulate good reasons for trusting those rules. But the life of the enterprise is *inquiry;* as we learn how to ask good questions, answers come, often more powerfully than when we strain for them. So these talks are concerned with raising questions: questions of practical import or of pure curiosity, questions about number, space, pattern, logic, questions which mathematicians know have paid off in increasing our comprehension of the world we live in.

You are wrong if you think questions of this kind are special, technical, occurring only to a few gifted people. Almost every small child does things and asks about things which touch such mathematical sides of experience. He may be hoping for a rule (*what* to do), he may be groping for a reason (*why* it works); mathematics itself swings between rules and reasons. In these talks children's casual remarks will sometimes be used, for illumination or to surprise you into an attitude of observant curiosity. Even the dictionary under "mathematics—fr. Greek" gives first mathematikos—*disposed* to learn, and only second mathemata —*things* learned.

Note: In reading mathematics, you are meant not to acquiesce but to agree or disagree. The recorded "things learned" are results of an activity that had conviction; they were recorded to avoid the waste of re-thinking, but they must stand up under repeated re-reading. To read the record with conviction is often slow and sometimes difficult. Some of the texts quoted (and some famous mathematicians) give explicit advice on *how* to read. This advice, like most oracular speech, is summed up in contradictions: *go back*; *go on*; *stop a while.* These must, of course, be mixed in judicious proportions! The suggestions for reading include as many types of approach as possible; WARNING: they often reflect not only the activity but the personality of the *man* writing about mathematics.

1. Preview: Questions, questions, everywhere . . .

Try not to ask too soon: Where does this get us? What does this apply to? Franklin flew a kite—a childish pursuit; but by letting it run free he learned something far from childish. If in these talks kites are once air-borne, don't pull them down too soon. There's electric charge in those clouds!

2. Regular Shapes, via tiles, pyramids and honeycombs.

Plato, Descartes, Euler—mathematics knows no nationality.

3. Regular Patterns and Symmetry. (Crystal groups.)

Symmetry—H. Weyl. (Not easy reading, but studded with pictures.)

4. Products and Primes: a multiplication table sets a pattern.

One, Two, Three, Infinity—G. Gamov

5. Matching and Counting, Big and Infinite.

Number, the Language of Science—T. Dantzig

6. Questions the natural numbers will not answer.

But very old questions on measurement are not answered even in this extended sense of number. The circle leads us to the oldest of those number notions which hold in themselves an entire "program of procedure."

Elementary Analysis—Kenneth O. May

7. Maps, terrestrial and otherwise.

Note: The mathematician extends the concept *map* too, and takes over the word for much wider use, just as he does with words like number and dimension.

The Round Earth on Flat Paper—National Geographic Society

8. Routes, Maps and Networks.
9. Kepler (1571–1630).

"It is no novelty to the history of Science that factors of thought which guided genius to its goal should be subsequently discarded. The names of Kepler and Maupertius at once occur in illustration." A. N. Whitehead.

10. On Measures of "How likely?"

I. From an idea, to be checked against experience.

Cardano, the Gambling Scholar—O. Ore

11. On Measures of "How likely?"

II. From what has happened, to form an idea.

"Life is the art of drawing sufficient conclusions from insufficient premises." Samuel Butler

12. On Ideas of Space, from knowledge of curves and surfaces.

"Besides, what is common sense now was abstract reasoning with earlier ages." S. T. Coleridge—*Anima Poetae.*

What kind of space do we live in? We are at a grave disadvantage just because we are *in* it! Men thought the earth was flat; how did they discover its shape? If we lived on a ring of Saturn, how would we find out the nature of our world? School geometry makes an orderly study of figures *in a plane:* of line and parallel, of circles with tangents from exterior points, of shortest distances. What happens to such theorems on a sphere? On a strip with a twist in it?

We almost automatically study the surface *and* its surrounding space; your so-called *plane* geometry probably said, "Superpose the two triangles," tacitly assuming you could turn one over! But geometry *on* a surface, stretching our imagination about flat-curved, closed-open, inside-outside, bounded-infinite, shows us many possibilities for two-dimensional worlds, each with its own geometry. What is more natural than to inquire about various kinds of three-dimensional worlds?

13. Questions on Least and Most.

Both kinds of questions: Is there a biggest? and How do I find it?—lead to and yield to the limit notion.

14. Always? Sometimes? Never?
15. What is Mathematics?

"Civilization advances by extending the number of important operations which we can perform without thinking about them." A. N. Whitehead.

If the examples of mathematical activity discussed have served their purpose, they have shown in simple ways continual interplay: between rules to save thinking and reasons why the rules are safe; between exploiting the individual problem and deliberately losing it in generality; between cultivating the intuition and strengthening the natural human ability to abstract. They have shown how sensible, necessary and powerful is the devising of notation and vocabulary. They have borne on mathematics as a logical structure, mathematics as a tool of science, mathematics as a language. Perhaps they have given some

intimation that mathematics is one of the most deeply rooted expressions of human culture.

4. In operation.

(1) There was no rehearsal; studio time is expensive. The director was on the floor for half an hour before the stand-by signal, and particular points for the camera were settled then. This had the disadvantage of lower technical standard but the advantage of live first-hand talk, not to speak of time saved for the lecturer. When the same camera crew could be used, the camera men became interested and followed well.

(2) There was a small blackboard, back of a small flat desk. Erasing a board is ugly and time-wasting; that space was kept for calculated use, to write notation while I talked sentences or to develop figures where successive stages were illuminating. A small desk easel (actually a painted magnetic bulletin board) allowed me to work with charts and mounted materials such as advertisements and plates, or to make small drawings directly in front of me; in the close-up camera, these filled the screen.

(3) Models, wall-paper and rubber tiles, a free-sphere globe (dulled with lacquer) a painted two-liter flask that served in three talks (*e.g.*, Networks; on a sphere?) in fact, some variety of objects to be handled, helped focus and hold attention. Warning: These must remain accessories, not become distractions. Note: Your audience needs the occasional slackening of attention: plan this, deliberate and controlled.

(4) The audience was warned and occasionally reminded that attention might break or they might be suddenly lost, but they were told this was a well-known phenomenon, not a judgment on them! The advice was: "Relax and stay with; the sentences may suddenly pick you up again. In any case, there is a three-minute end with some point of mathematical import stated carefully, serving to pinpoint the day's undertaking."

(5) I made no attempt to learn a strict script, though I made a record of timed careful formulations and of gadgetry in the order needed. I did learn a strictly timed three-minute end, since signals were given at 3, 2, 1 minute to go and 30 second wind up, and I did prepare one section rigorously condensed but amenable to elaboration should the early part develop quickly.

(6) To whom does one talk? I admit that had worried me. All of us who trust to thinking in front of our audiences, with preparation consisting largely of pre-thinking ourselves into the vocabulary from which they will listen but letting the idea run apparently free toward our own goals, know how important is the pull of the attentive minds. Believe it or not, the sending camera with its lens and its two red eyes (showing that it is the one on the air) is an individual to whom you talk, whom you convince, to whom you gesture and say—"You . . . , " a kind of Martian, I agree, but he serves. This alone is worth discovering.

(7) The preparation was demanding in time, in the very nature of the attempt; for me, it paid off doubly. It was of course satisfying to have a belief borne out, to find that there is a public for such attempts. An unexpected dividend showed up in the accompanying college work, which profited from the rigorous effort to state clearly and summarize soundly.

(8) To the studio, response was more than satisfactory early in the game, helped by a pure piece of luck—that the microphone did not kill the live voice. My own first reports from outside the college came from former students, undergraduate and graduate, who watching at sets in student lounges or neighbors' houses reported with almost unholy glee those who were caught to their own surprise. In January, when as an experiment two phones were announced at the end of the hour, seven of the eight calls I had time to take came from men; the studio (like us) had assumed that the audience was the so-called housewife. She is there, and don't forget she is the mother of those students we want; she occasionally writes to say she talked with her husband about this world new to her but familiar to him. Recognizing your voice at the Postoffice, she may even give you more than you hoped for: "I knew mathematics was expert; I didn't know it was profound." But the range is great; from a cerebral palsy victim, now for the first time too old to have a teacher come to his home, typing out a careful request for texts, to lawyers, bankers and radio engineers who, in the course of the morning for one of a thousand reasons, turned a knob and finding mathematics by chance, stayed with it. The news of these drifts back still. Another kind of response comes: from Tennessee and West Virginia, from Georgia, Connecticut and the West Coast, from teachers' colleges, from faculty members importuned by local stations, from young men interested in the medium itself, the scattered notes and letters come asking what was tried and how it went. And over and over, from this year's audience: "Are you going on?" It is not an opinion but a work to do, and it has a sure foundation best illustrated by another remark.

The program was called "Invitation" and I had used Webster's definition: a drawing on, allurement, enticement. The drawing on was my concern; given half a chance, good mathematics would do the rest. In June when WFIL was far from my mind, there came one of those sudden reminders: "I followed your program and I remember how you ended . . . 'that mathematics can and does entice people'; well, it did and it does."

Yes, it does. Why don't we believe that and use it for our own purposes? The television camera is a powerful ally to the working mind.

ON THE DEFINITION OF AN ANALYTIC FUNCTION

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1. Introduction. From the point of view of facility in developing the theory it is convenient to call a complex function f *analytic* on an open set Ω provided the derivative f' exists and is continuous on Ω. However, the continuity requirement on f' becomes superfluous in the light of the Cauchy-Goursat theorem, and for this reason the latter is often regarded as essential to a first course in complex function theory.

It is nevertheless true that conditions far weaker than the existence of f' will suffice to ensure analyticity. Theorems of Besicovitch, Looman-Menchoff, Maker and others* present such conditions, but their proofs draw on real function theory to such an extent that they are not generally considered germane to an introductory treatment of complex analysis.

As a possible replacement for the Cauchy-Goursat theorem we suggest the following criterion for analyticity, very closely related to results of Besicovitch [2] and Looman [3]. It appears, in fact, as a special case of the more elaborate theorem of Looman-Menchoff-Saks-Besicovitch-Maker [4, pp. 266–267]. Although the elementary nature of the proof which we submit is due largely to an observation of Meier [5] (stated explicitly in §3), the most complicated part of Meier's argument is avoided by the present hypotheses on f.

THEOREM. *Let $f=u+iv$ be a continuous complex function defined on an open subset Ω of the plane. If*

(1) $\displaystyle\limsup_{\zeta\to z}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|$

is finite for all except perhaps a countable number of points z of Ω, and

(2) *the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

hold almost everywhere,

then f is analytic on Ω.

In the proof it has not been possible to dispense altogether with material belonging properly to the domain of real function theory, but an attempt has been made to eliminate as much of this as possible. What remain are (i) the Lebesgue bounded convergence theorem and (ii) the following special case of the Baire category theorem:

LEMMA. *Let Ω be an open set and F a non-empty subset closed in Ω. If F is covered by sets F_n $(n=1, 2, \cdots)$ closed in Ω, then there exists a neighborhood $\omega(\subset\Omega)$ and an index N such that $F_N\supset F\cap\omega\neq 0$.*

* See [2], [3], [4], [5], and in particular [6, pp. 195–201].

Certainly (i) is a result which (if unfamiliar to the student) can be stated without proof, while (ii) is easily established by elementary methods.

2. The areal mean functions. It is convenient to introduce the functions u_n and v_n obtained from u and v, respectively, by averaging over squares of side $1/n$:

$$
\begin{aligned}
u_n(x, y) &= n^2 \int_0^{1/n} \int_0^{1/n} u(x + s, y + t)dsdt, \\
v_n(x, y) &= n^2 \int_0^{1/n} \int_0^{1/n} v(x + s, y + t)dsdt.
\end{aligned} \tag{1}
$$

These functions are defined on the open set Ω_n consisting of all points (x, y) for which the square $[x, x+1/n]\times[y, y+1/n]$ lies in Ω.

As is well known [6, pp. 178–179], the continuity of u and v results in continuous differentiability of u_n and v_n. In fact, a direct calculation yields

$$
\frac{\partial u_n(x, y)}{\partial x} = n^2 \int_0^{1/n} [u(x + 1/n, y + t) - u(x, y + t)]dt,
$$

$$
\frac{\partial u_n(x, y)}{\partial y} = n^2 \int_0^{1/n} [u(x + s, y + 1/n) - u(x + s, y)]ds,
$$

and $\partial v_n/\partial x$, $\partial v_n/\partial y$ are given analogously.

The motivation for considering the areal mean functions now becomes manifest with the observation that the theorem would follow at once if we could differentiate under the integral sign in (1). That is, the C' functions u_n and v_n would satisfy the Cauchy-Riemann equations, and the functions $f_n = u_n + iv_n$ would therefore be analytic; but, these functions clearly converge uniformly on compact subsets of Ω to f, so that f would necessarily be analytic on Ω.

However, there is no justification *a priori*, for differentiating under the integral sign. To remedy this deficiency, we have recourse to an argument based on the lemma of §1.

3. Proof of the theorem. Let F be the set of all points z of Ω such that f is not analytic on any neighborhood of z. Clearly, F is closed in Ω, and f is analytic on $\Omega - F$. We shall assume that F is not empty and show that this leads to a contradiction.

For positive integral n let F_n be the set of all points z of Ω such that

$$
|\zeta - z| < 1/n \quad \text{implies} \quad |f(\zeta) - f(z)| \leqq n|\zeta - z|.
$$

It is obvious from the continuity of f that each F_n is closed in Ω. Also, every point z at which $\limsup_{\zeta\to z} |f(\zeta)-f(z)|/|\zeta-z|$ is finite belongs to some F_n, and the remaining points of Ω are contained in a set of the form $\bigcup_{n=1}^{\infty} E_n$, where each E_n consists of a single point.

Inasmuch as the sets F_n and E_n in their aggregate cover Ω, and *a fortiori* F,

there exists according to the lemma a neighborhood ω intersecting F in a nonempty subset of one of the covering sets. This cannot occur for any of the covering sets E_n, in view of the fact that a continuous function analytic on a deleted neighborhood must actually be analytic on the full neighborhood. Hence, for some index N, $F_N \supset F \cap \omega$.

There is no loss of generality in supposing that the center z_0 of ω belongs to F and that the radius r_0 of ω is less than $1/N$. At every point z of $F_N \cap \omega$ the inequality

$$|f(\zeta) - f(z)| \leqq N|\zeta - z|$$

then holds for all ζ on ω. The following reasoning, due to Meier [5, p. 186], shows that this gives rise to the Lipschitz condition

$$|f(\zeta) - f(z)| \leqq 2N|\zeta - z| \tag{2}$$

for all ζ, z on the neighborhood ω' of radius $r_0/2$ about z_0.

First of all, we note that the inequality $|f'(z)| \leqq 2N$ is satisfied at all points z of $\omega' - F$. This follows from the Cauchy integral formula,* since z_0 in F implies that some point z' of F lies on the boundary β of the circle of analyticity of f about z and for ρ the radius of this circle

$$|f'(z)| = \left|\frac{1}{2\pi i}\oint_\beta \frac{f(\zeta) - f(z')}{(\zeta - z)^2} d\zeta\right| \leqq \frac{1}{2\pi}\cdot\frac{2N\rho}{\rho^2}\cdot 2\pi\rho = 2N.$$

Now, if the line segment $\overline{\zeta, z}$ joining ζ and z lies in $\omega' - F$, an integration along this segment yields (2) directly. On the other hand, if $\overline{\zeta, z}$ intersects F in a point z', then

$$\begin{aligned} |f(\zeta) - f(z)| &\leqq |f(\zeta) - f(z')| + |f(z') - f(z)| \\ &\leqq N|\zeta - z'| + N|z' - z| = N|\zeta - z|, \end{aligned}$$

and (2) is verified in all cases.

At this point we invoke the areal mean functions of §2, but for the region ω'. Taking (x, y) on ω_n' and $h(\neq 0)$ sufficiently close to 0, we find

$$\begin{aligned} &\frac{u_n(x + h, y) - u_n(x, y)}{h} - \frac{v_n(x, y + h) - v_n(x, y)}{h} \\ &\qquad = n^2 \int_0^{1/n}\int_0^{1/n}\left[\frac{u(x + h + s, y + t) - u(x + s, y + t)}{h}\right. \\ &\qquad\qquad \left. - \frac{v(x + s, y + h + t) - v(x + s, y + t)}{h}\right] dsdt. \end{aligned}$$

In the present situation the Lipschitz condition (2) ensures that the absolute value of the integrand cannot exceed $4N$, and this permits us to employ

* A simple limiting case of the weak form of the Cauchy integral formula will suffice here.

the Lebesgue bounded convergence theorem. Since the integrand, by hypothesis, converges to 0 almost everywhere on $[0, 1/n]\times[0, 1/n]$ as $h\to 0$, we obtain

$$\frac{\partial u_n}{\partial x}=\frac{\partial v_n}{\partial y} \quad \text{on} \quad \omega_n'.$$

Similarly

$$\frac{\partial u_n}{\partial y}=-\frac{\partial v_n}{\partial x} \quad \text{on} \quad \omega_n'.$$

Thus, for each n the function $f_n=u_n+iv_n$ is analytic on ω_n'. These functions, moreover, converge uniformly to f on compact subsets of ω' and thereby confer their analyticity on f. Having arrived in this fashion at an obvious contradiction to the fact that z_0 belongs to F, we conclude that f must be analytic throughout Ω.

4. Essentially the same methods can be used to establish the following variant of Theorem 1 of Besicovitch [2].

THEOREM. *Let Ω be an open subset of the plane and $E(\subset\Omega)$ the union of countably many sets of zero length closed in Ω. If $f=u+iv$ is a bounded complex function on $\Omega-E$ such that*

(1) $\limsup_{\zeta\to z}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|$ *is finite for z on $\Omega-E$, and*

(2) *the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$ hold almost everywhere on $\Omega-E$,*

then f can be extended so as to be analytic on Ω.

A discussion and proof of this result are given in [1].

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MATHEMATICAL NOTES

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A REMARK ON STIRLING'S FORMULA

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We shall prove Stirling's formula by showing that for $n=1, 2, \cdots$

$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n} \cdot e^{r_n} \tag{1}$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n} \cdot \tag{2}$$

The usual textbook proofs replace the first inequality in (2) by the weaker inequality

$$0 < r_n$$

or

$$\frac{1}{12n+6} < r_n.$$

Proof. Let

$$S_n = \log (n!) = \sum_{p=1}^{n-1} \log (p+1)$$

and write

$$\log (p+1) = A_p + b_p - \epsilon_p \tag{3}$$

where

$$A_p = \int_p^{p+1} \log x\, dx, \quad b_p = \tfrac{1}{2}[\log (p+1) - \log p],$$

$$\epsilon_p = \int_p^{p+1} \log x\, dx - \tfrac{1}{2}[\log (p+1) + \log p].$$

The partition (3) of $\log (p+1)$, regarded as the area of a rectangle with base $(p, p+1)$ and height $\log (p+1)$, into a curvilinear area, a triangle, and a small sliver* is suggested by the geometry of the curve $y=\log x$. Then

* Taken from G. Darmois, *Statistique Mathématique*, Paris, 1928, pp. 315–317. The only novelty of the present note is the inequality (7) which permits the first part of the estimate (2).

$$S_n = \sum_{p=1}^{n-1} (A_p + b_p - \epsilon_p) = \int_1^n \log x \, dx + \tfrac{1}{2} \log n - \sum_{p=1}^{n-1} \epsilon_p.$$

Since $\int \log x \, dx = x \log x - x$ we can write

$$S_n = (n + \tfrac{1}{2}) \log n - n + 1 - \sum_{p=1}^{n-1} \epsilon_p, \tag{4}$$

where

$$\epsilon_p = \frac{2p+1}{2} \log \left(\frac{p+1}{p}\right) - 1.$$

Using the well known series

$$\log \left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \frac{1}{7(2p+1)^6} + \cdots. \tag{5}$$

We can therefore bound ϵ_p above and below:

$$\epsilon_p < \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{(2p+1)^2} + \frac{1}{(2p+1)^4} + \cdots\right\} \tag{6}$$
$$= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \dfrac{1}{(2p+1)^2}} = \frac{1}{12}\left(\frac{1}{p} - \frac{1}{p+1}\right),$$

$$\epsilon_p > \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{3(2p+1)^2} + \frac{1}{[3(2p+1)^2]^2} + \cdots\right\} \tag{7}$$
$$= \frac{1}{3(2p+1)^2} \cdot \frac{1}{1 - \dfrac{1}{3(2p+1)^2}} > \frac{1}{12}\left(\frac{1}{p + \dfrac{1}{12}} - \frac{1}{p + 1 + \dfrac{1}{12}}\right).$$

Now define

$$B = \sum_{p=1}^{\infty} \epsilon_p, \qquad r_n = \sum_{p=n}^{\infty} \epsilon_p, \tag{8}$$

where from (6) and (7) we have

$$\frac{1}{13} < B < \frac{1}{12}. \tag{9}$$

Then we can write (4) in the form

$$S_n = (n + \tfrac{1}{2}) \log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = C \cdot n^{n+1/2} e^{-n} \cdot e^{r_n},$$

where r_n is defined by (8), ϵ_p by (5), and from (6) and (7) we have

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

The constant C, known from (9) to lie between $e^{11/12}$ and $e^{12/13}$, may be shown by one of the usual methods to have the value $\sqrt{2\pi}$. This completes the proof.

The preceding derivation was motivated by the geometrically suggestive partition (3). The editor has pointed out that the inequalities (6) and (7) permit the following brief proof* of (2). Let

$$u_n = n! n^{-(n+1/2)} e^n.$$

Then the series

$$\log\left(1 + \frac{1}{n}\right)^{n+1/2} = 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots$$

together with (6) and (7) yield the inequalities

$$\exp\left\{\frac{1}{12}\left(\frac{1}{n+\frac{1}{12}} - \frac{1}{n+1+\frac{1}{12}}\right)\right\} < \frac{u_n}{u_{n+1}} = e^{-1}\left(1+\frac{1}{n}\right)^{n+1/2}$$

$$< \exp\left\{\frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right\}.$$

Hence

$$v_n = u_n e^{-1/12n}$$

increases and

$$w_n = u_n e^{-1/(12n+1)}$$

decreases, while

$$v_n < w_n = v_n e^{1/12n(12n+1)}.$$

Since

$$v_1 = e^{11/12}, \qquad w_1 = e^{12/13}$$

* A modification of that attributed to Cesàro by A. Fisher, *Mathematical theory of probabilities*, New York, 1936, pp. 93–95.

it follows that

$$v_n \to C, \qquad w_n \to C, \qquad v_n < C < w_n, \qquad e^{11/12} < C < e^{12/13}.$$

Thus

$$u_n = Ce^{r_n}$$

where r_n satisfies (2).

ON RESTRICTED FUNCTIONS

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Let $f(x)$ be a real function of a real variable defined on a set E. Let P be a point property of f. The function f is said to be *peculiar* with respect to the property P if there exists a partition of E into two subsets, E_1 and E_2, each everywhere dense in E, such that the property P holds at every point of E_1 and fails to hold at every point of E_2. Functions peculiar with respect to continuity and differentiability are well known.

In a recent paper [2], H. P. Thielman investigated two generalizations of continuity—neighborliness, a concept defined by Bledsoe [1], and cliquishness. He showed that, although there are functions peculiar with respect to continuity, neighborliness, and differentiability, there exists no function which is peculiar with respect to cliquishness. It is the purpose of this note to define another point property which is similar to cliquishness in this respect.

Let $f(x)$ be defined on a set E, and let a be a limit point of E. The function $f(x)$ is said to be *restricted* at the point a if $\limsup_{x\to a} f(x)$ and $\liminf_{x\to a} f(x)$ are both finite. Otherwise, $f(x)$ is said to be *unrestricted* at a.

If a function is restricted (unrestricted) at each point of a set E, it is said to be restricted (unrestricted) on E.

A function may be restricted on a set but not bounded on that set; the function $f(x)=1/x$, defined on the open interval $(0, 1)$, is such a function.

THEOREM 1. *If $\limsup_{x\to b} f(x)=+\infty$ for all $b\in E$ and if a is a limit point of E, then $\limsup_{x\to a} f(x)=+\infty$.*

Proof: Let M and δ be arbitrary positive numbers. Let $b\in E$ be such that $0<|b-a|<\delta/2$. Choose δ_1 so that $0<\delta_1<|b-a|$. Then, since $\limsup_{x\to b} f(x)=+\infty$, there exists $c\in E$ such that $f(c)>M$ and $|c-b|<\delta_1$. But

$$|c-a| \leqq |c-b| + |b-a| < \delta.$$

Since M and δ are arbitrary, $\limsup_{x\to a} f(x)=+\infty$.

COROLLARY. *If $f(x)$ is unrestricted on E and if a is a limit point of E, then $f(x)$ is unrestricted at a.*

This follows immediately from the above and an analogous theorem for the limit inferior.

THEOREM 2. *If f is unrestricted on a set $E_1 \subset E$ and if E_1 is everywhere dense in E, then f is unrestricted on E.*

THEOREM 3. *Let f be defined on E, restricted on $E_1 \subset E$, and unrestricted on $E_2 = E - E_1$. If E_1 is everywhere dense in E, then E_2 is nowhere dense.*

The proof is similar to that of Theorem II in [2].

Hence, there is no function peculiar with respect to restrictedness. Restrictedness is also seen to be similar to cliquishness in the following theorem.

THEOREM 4. *Let $f_n(x)$ $(n=1, 2, \cdots)$ be restricted on a set E. Then $f(x) = \lim_{n\to\infty} f_n(x)$ may be unrestricted at every point of E.*

Let $f(x)$ be defined on $(0, 1)$ as follows:

$$f(x) = \begin{cases} q \text{ if } x = p/q,\ p \text{ and } q \text{ relatively prime positive integers,} \\ 0 \text{ if } x \text{ irrational.} \end{cases}$$

It is easily seen that $f(x)$ is unrestricted on $(0, 1)$.

However, consider the following sequence of functions, each defined on $(0, 1)$:

$$f_n(x) = \begin{cases} q \text{ if } x = p/q,\ q \leqq n,\ p \text{ and } q \text{ relatively prime positive integers,} \\ 0 \text{ otherwise.} \end{cases}$$

For all n, $f_n(x)$ is restricted on $(0, 1)$, and $\lim_{n\to\infty} f_n(x) = f(x)$.

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A NOTE ON EXTREME POINTS*

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Since the advent in 1940 of the Krein-Milman theorem [5], the subject of extreme points of convex sets has steadily grown in importance. We contribute here three remarks which, though easily proved, appear to be new and interesting. It is established, in particular, that every metric space can be isometrically obtained as the set of all extreme points of a convex set (in some normed linear space), and that every compact metric space can be topologically obtained as the set of all extreme points of a compact convex set. As the main tool in

* Sponsored by the Office of Ordnance Research, U. S. Army, under Contract DA-04-200-ORD-292.

it can be verified that cl K is compact and $\{$extreme points of cl $K\} \subset TM = \{$extreme points of $K\}$. However, we do not know that every point of TM is an extreme point of cl K, and hence do not know that every compact metric space is isometric with the set of all extreme points of some compact convex set. In any case, the following result is easily established.

THEOREM 3. *If M is a compact metric space and E is an infinite-dimensional Banach space, then there is a compact convex subset of E whose set of extreme points is homeomorphic with M.*

Proof. We assume without loss of generality that E is separable. It then follows from a theorem of Clarkson [3; p. 413] that E is linearly homeomorphic with a strictly convex Banach space, so we may as well assume that E itself is strictly convex, *i.e.*, that every point of the unit sphere $S = \{x: \|x\| = 1\}$ is an extreme point of S. Let H be a hyperplane in E which misses the origin. As is well-known, M must be homeomorphic with a subset of Hilbert space, and since all infinite-dimensional separable Banach spaces are of the same dimension-type [4; (1.4)], it follows that H contains a homeomorph M' of M. Now let $g = \{(x, x/\|x\|): x \epsilon M'\}$ and let K be the closed convex hull of gM'. Then g is a homeomorphism of M' into S, K is compact [2; p. 81], gM' includes all extreme points of K [2; p. 84], and $K \subset C = \{x: \|x\| \leqq 1\}$. Furthermore, $gM' \subset S$ and every point of S is an extreme point of C, so every point of gM' is an extreme point of K. The proof is complete.

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ELEMENTARY PROOF THAT *e* IS NOT QUADRATICALLY ALGEBRAIC

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We propose to indicate how the elementary means employed by Pennisi* to show that e is not rational can be applied to prove that e is not quadratically algebraic, or, in other words, that

$$ae^2 + be + c = 0 \tag{1}$$

* Pennisi, L. L., Elementary proof that e is irrational, this MONTHLY, 1953, vol. 60, p. 474.

is impossible, for a, b, and c integral but not all zero.

From the series for e and $1/e$, it appears that

$$e(n!) = I + \frac{1}{n+\theta}, \qquad \text{for all positive integers } n, \tag{2}$$

$$e^{-1}(n!) = J + \frac{1}{n+1+\phi}, \quad \text{for all odd, positive integers } n, \tag{3}$$

where I and J are integers and θ and ϕ proper fractions, all four depending on n for their actual values. It is only necessary to verify that the value of

$$\frac{1}{n+1} + \frac{1}{n+1}\,\frac{1}{n+2} + \cdots$$

is of the form $1/(n+\theta)$, while the value of

$$\frac{1}{n+1} - \frac{1}{n+1}\,\frac{1}{n+2} + \cdots$$

is of the form $1/(n+1+\phi)$. Taking n odd from here on, and substituting (2) and (3) in the equation obtained from (1) by multiplying its left side by $e^{-1}(n!)$, we see that

$$\frac{a}{n+\theta} + \frac{c}{n+1+\phi}$$

must be an integer, which integer must be zero, indeed, for n sufficiently large. This implies, in turn, that $a+c$ must be zero and that θ and $1+\phi$ must be equal, since, by hypothesis, a and c cannot be zero themselves. To say, however, that θ and $1+\phi$ are equal is absurd, seeing that unity lies between them.

CLASSROOM NOTES

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A NOTE ON NEWTON-COTES QUADRATURE FORMULAS

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It is well known that Simpson's 1/3 Rule for numerical integration, which is based on approximating the integrand by a parabola or a series of parabolic arcs, gives the exact result when the integrand is a polynomial of third degree. This somewhat curious fact is actually a special case of the more general

theorem that the Newton-Cotes quadrature formulas, *i.e.*, the integration formulas based on fitting a polynomial of nth degree to the integrand $f(x)$ by requiring that the polynomial assume exactly the values of the integrand at $(n+1)$ uniformly spaced points in the interval of integration, give exact results when $f(x)$ is a polynomial of $(n+1)$th degree (or lower) provided that n is even. Although this theorem is stated, for example, by Milne,* no *simple* proof appears available in the literature, even for the special case of $n=2$. One method, for example, of proving this theorem is by an analysis of the remainders,† but such proofs are usually rather indirect and involved, especially for general n. In the classroom the author has found, in fact, immediately after showing that the remainder in Simpson's 1/3 Rule is proportioned to $f^{(iv)}(x)$ and stating that hence Simpson's Rule must be exact when $f(x)$ is a cubic, that it is usually the better students who either raise questions as to just why this is so or actually remain skeptical regarding the truth of this theorem.

The purpose of this note is to furnish a simple proof of the general theorem stated above. The general theorem can be proven by first noting that if $\phi(x)$ is a polynomial of nth degree such that $\phi(x_0+ih)=y_i$ $(i=0, 1, 2, \cdots, n)$, then

$$\text{(1)} \qquad \begin{aligned} \int_{x_0}^{x_0+nh} \phi(x)dx = h\int_0^n \Big(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots \\ + \frac{u(u-1)(u-2)\cdots(u-n+1)\Delta^n y_0}{n!}\Big) du \end{aligned}$$

where $u=(x-x_0)/h$ and $\Delta^k y_0$ denotes the kth differences†† of the y_i. In numerical integration, the actual integrand $f(x)$ is replaced by $\phi(x)$, while $y_i=f(x_0+ih)$. Equation (1) is essentially an integration of Newton's Forward Interpolation Formula, which is derived in a simple manner in Scarborough's book.§ Suppose, now, that n is even. It is then desired to prove that if $f(x)$ is a polynomial of $(n+1)$th degree, then

$$\text{(2)} \qquad \int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+nh} \phi(x)dx$$

exactly. We note that Equation (2) would necessarily have to be satisfied if $\phi(x)$ were replaced by $\psi(x)$, a polynomial of $(n+1)$th degree satisfying the relations $\psi(x_0+ih)=f(x_0+ih)$, $i=0, 1, \cdots, (n+1)$. But from (1),

$$\int_{x_0}^{x_0+nh} \psi(x)dx = \int_{x_0}^{x_0+nh} \phi(x)dx + A\,\frac{h}{(n+1)!}\,\Delta^{n+1}y_0$$

* W. E. Milne, Numerical Calculus, Princeton University Press, 1949, p. 124.

† Cf., *e.g.*, W. E. Milne, *op. cit.*, pp. 108–124. Also, for $n=2$, J. B. Scarborough, Numerical Mathematical Analysis, Johns Hopkins Press, Second Edition, 1950, pp. 174–176.

†† $\Delta y_0=y_1-y_0$, $\Delta^2 y_0=\Delta y_1-\Delta y_0=(y_2-y_1)-(y_1-y_0)=y_2-2y_1+y_0$, $\Delta^3 y_0=y_3-3y_2+3y_1-y_0$, *etc.*

§ *Op. cit.*, pp. 61–63.

where

$$A = \int_0^n u(u-1)(u-2)\cdots(u-n)du.$$

By letting $u=n-v$, $du=-dv$, it is readily found that if n is even,* then $A=-A$. Hence, $A=0$ and consequently, Equation (2) must hold, and the proof is complete.

The author hereby expresses his thanks to Professor R. M. Foster for his valuable discussions.

NOTE ON LINEAR INTERPOLATION ERROR

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Although linear interpolation is probably the most widely used approximating procedure, bounds for error are seldom given even to engineering students. Until recently, the derivation of such bounds has been available only to experts in the field of finite differences and the information has been buried in the notation peculiar to this subject. The following derivation of error term uses only Taylor's series with remainder.

Let $y=f(x)$ and its first two derivatives be continuous in the interpolation interval $x_0 \leqq x \leqq x_1$. Also let

$$y = I(x) - E \tag{1}$$

where

$$I(x) = \frac{(x_1-x)y_0 + (x-x_0)y_1}{x_1-x_0}$$

is the interpolated value, E is the error due to interpolation, and $y_0=f(x_0)$, $y_1=f(x_1)$.

Now by Taylor's expansion

$$y_0 = y + y'(x_0-x) + R_0 \tag{2}$$

$$y_1 = y + y'(x_1-x) + R_1 \tag{3}$$

where $R_i=\frac{1}{2}f''(\theta_i)(x_i-x)^2$, $i=0,1$, and $x_0 \leqq \theta_0 \leqq x \leqq \theta_1 \leqq x_1$.

Multiplying (2) by (x_1-x), (3) by $(x-x_0)$, and summing gives

$$(x_1-x)y_0 + (x-x_0)y_1 = y(x_1-x_0) + (x_1-x)R_0 + (x-x_0)R_1. \tag{4}$$

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* If n is odd, then only the identity $A=A$ is obtained.

Dividing by (x_1-x_0) gives (1) where

$$E = \frac{(x_1 - x)R_0 + (x - x_0)R_1}{x_1 - x_0}$$

or

$$E = \tfrac{1}{2}(x_1 - x)(x - x_0)\left[f''(\theta_0)\cdot\frac{x - x_0}{x_1 - x_0} + f''(\theta_1)\,\frac{x_1 - x}{x_1 - x_0}\right]. \tag{5}$$

This formula gives slightly sharper bounds than the usual one

$$E = \tfrac{1}{2}(x_1 - x)(x - x_0)f''(\theta), \qquad x_0 \leqq \theta \leqq x_1. \tag{6}$$

Equation (6) may be obtained from (5) as follows. The bracketed expression in (5) is the weighted average of $f''(\theta_0)$ and $f''(\theta_1)$ and has a value between these two. But $f''(x)$ is continuous and must take on this value at some point θ between θ_0 and θ_1. Equation (6) follows.

A relation between the errors for direct and inverse interpolation may be obtained as follows. Equation (4) may be written $(x_1-x)y_0+(x-x_0)y_1=y(x_1-x_0)+E(x_1-x_0)$. Subtracting $x_1y_0-x_0y_1$ from both sides and regrouping terms gives

$$(y_1 - y_0)x = (y_1 - y)x_0 + (y - y_0)x_1 + E(x_1 - x_0)$$

whence

$$x = I(y) - \overline{E}, \quad \text{where} \quad \overline{E} = -E\frac{x_1 - x_0}{y_1 - y_0}.$$

It may be noted that in practice (5) is superior to (6) only if $f''(x)$ is easily bounded in the intervals $[x_0, x]$, and $[x, x_1]$ such as for log x. Otherwise (6) is preferable due to its simplicity.

Other bounds of error are given by Kowalewski* and Hummel.† The former gives

$$E(x) = \int_{x_0}^{x}\frac{(u - x_0)(x_1 - x)}{x_1 - x_0}f''(u)du + \int_{x}^{x_1}\frac{(x - x_0)(x_1 - u)}{x_1 - x_0}f''(u)du$$

from which (5) may be derived directly using the integral law of the mean. The latter gives the remarkably close bounds

$$\frac{(x_1 - x)(x - x_0)}{(x_1 - x_0)^2}[f(x_1) - f(x_0) - (x_1 - x_0)f'(x_0)]$$
$$\leqq E \leqq \frac{(x_1 - x)(x - x_0)}{(x_1 - x_0)^2}[(x_1 - x_0)f'(x_1) - f(x_1) + f(x_0)]$$

but these are not quite as easily derived as (5).

* Interpolation und Genaherte Quadratur, Kowalewski, 1932, p. 25.

† The accuracy of linear interpolation; P. M. Hummel, this MONTHLY, vol. 53, 1946, pp. 364–366.

ASYMMETRICAL DISTRIBUTIONS WITH ZERO THIRD MOMENTS

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A point about the third moment often confuses beginning students in Statistics. Every textbook in the subject mentions the fact that the third moment about the mean of a symmetrical distribution is zero. Sometimes the reader is cautioned that the converse is not necessarily true, *i.e.*, even an asymmetrical distribution may have a zero third moment (see Hoel, *Introduction to Mathematical Statistics*, p. 15, Johnson and Tetley, *Statistics*, Vol. 1, p. 71, and Mood, *Introduction to the Theory of Statistics*, p. 97, the second giving a simple discrete distribution as an example, and the third, a picture of an asymmetrical continuous distribution, but without its equation). Usually, however, the point is overlooked and the student is left to assume that the converse is necessarily true. Kendall (*Advanced Theory of Statistics*, Vol. 1, p. 81) says that "the size of μ_3 relative to $\mu_2^{3/2}$ (or $\sqrt{\beta_1}$) will indicate the extent of the departure from symmetry," which implies that if $\mu_3=0$ there is no departure. Croxton and Cowden (*Applied General Statistics*, p. 255) state explicitly that "the third moment about the mean is not zero unless the distribution is symmetrical about the mean," and a similar incorrect remark appears in Smith and Duncan (*Elementary Statistics and Applications*, p. 191).

Students to whom the subject is mentioned often express a desire to see how an asymmetrical distribution with a zero third moment can be constructed, especially a continuous one with a relatively simple expression for the probability density.* The following example will enable the student to satisfy himself that he can construct a number of them without too much difficulty.

Start with the probability density

$$f(x) = \begin{cases} 0 & \left(x \leqq -\dfrac{c_1}{c_2}\right) \\ c_1 + c_2 x & \left(-\dfrac{c_1}{c_2} \leqq x \leqq 0\right) \\ (c_1 + c_3 x)e^{-x} & (x \geqq 0) \end{cases}$$

and determine c_1, c_2, and c_3 from the conditions:

$$\text{area} = 1, \text{ mean} = 0 \text{ (for convenience), third moment} = 0.$$

Thus,

$$\frac{c_1^2}{2c_2} + c_1 + c_3 = 1$$

* It has been pointed out that I. W. Burr, Cumulative frequency functions, Ann. Math. Stat., vol. 13, 1942, pp. 215–232, gives a whole series of such densities, of which the following is typical: $f(x)=0(x<0), f(x)=[3(6.96)x^{5.96}]/(1+x^{6.96})^4(x\geqq 0)$. It would be difficult, however, to show a class that this does have a zero moment.

$$\frac{c_1^3}{6c_2^2} - c_1 - 2c_3 = 0$$

$$\frac{c_1^5}{20c_2^4} - 6c_1 - 24c_3 = 0.$$

These equations yield the solution

$$c_1 = \frac{12}{a^2 + 6a + 6}$$

$$c_2 = \frac{12}{a(a^2 + 6a + 6)}$$

$$c_3 = \frac{a^2 - 6}{a^2 + 6a + 6}$$

where

$$a = \sqrt{20 + 2\sqrt{70}}; \quad i.e., \quad c_1 = .152,\ c_2 = .025,\ c_3 = .389.$$

Not every start will prove as fruitful as this illustration. For example, if we try

$$f(x) = \begin{cases} (c_1 + c_2 x)e^{-x^2/2} & (x \leqq 0) \\ (c_1 + c_3 x)e^{-x} & (x \geqq 0) \end{cases}$$

the same procedure produces a negative value for c_3. Thus, since its right hand tail dips below the axis, $f(x)$ cannot serve as a probability density.

THE GEOMETRIC SIGNIFICANCE OF THE STANDARD DEVIATION AND COEFFICIENT OF VARIATION

P. B. JOHNSON, Occidental College and Haverford College

The following interpretation of the standard deviation of a set of readings is intended to meet a problem which has bothered the author for years. Beginning students believe that taking the sum of the absolute deviations from a mean is an easier operation than taking the sum of the squares of the deviations. They do not yet know advanced theory. The notion of a standard deviation leaves them cold, and appears like a forced and unpredictable torture used by people who only want to do things the hard way. The moment of inertia analogy from physics helps some, but also leaves them cold, since they frequently don't have a feeling for physics. While the following theorem involves n-dimensions (perhaps even harder a notion to grasp than the moment of inertia) it does show that the standard deviation is a direct measure, of tape-measure directness, of the failure of the readings $x_1, x_2, \cdots, x_n$ to be equal.

Regard $x_1, x_2, \cdots, x_n$ as the coordinates of a point X in n dimensional euclidean space with rectangular cartesian coordinates, say $y_1, y_2, \cdots, y_n$. Let $\mathbf{V}$ be the "units" vector with coordinates $(1, 1, \cdots, 1)$. Then

THEOREM 1: *The standard deviation is the distance of the point X from the line $y_1 = y_2 = \cdots = y_n$, measured in units equal to the length of* $\mathbf{V}$.

Proof. We recall that the dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ gives

$$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = |\mathbf{A}|\,|\mathbf{B}| \cos\theta \tag{1}$$

where $\mathbf{A}$ is the vector from the origin to the point $a_1, a_2, \cdots, a_n$. $|\mathbf{A}|$ is the length of $\mathbf{A}$, similarly for $|\mathbf{B}|$, and θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

Let $\mathbf{U}$ be the unit vector in the direction of the line $y_1 = y_2 = \cdots = y_n$, with coordinates $(1/\sqrt{n}, 1/\sqrt{n}, \cdots, 1/\sqrt{n})$. Let $\mathbf{X}$ be the vector with coordinates $(x_1, x_2, x_3, \cdots, x_n)$. The standard deviation, s, is given by

$$ns^2 = \sum x^2 - \frac{\left(\sum x\right)^2}{n}. \tag{2}$$

Recalling $|\mathbf{V}|^2 = n$, we may write (2) in the form

$$|\mathbf{V}|^2 s^2 = \mathbf{X}\cdot\mathbf{X} - (\mathbf{X}\cdot\mathbf{U})^2 = |\mathbf{X}|^2 - |\mathbf{X}|^2\cos^2\theta = |\mathbf{X}|^2\sin^2\theta = |\overrightarrow{XP}|^2. \tag{3}$$

In Figure 1, P is the projection of X on the line $y_1 = y_2 = \cdots = y_n$. $|\overrightarrow{XP}|$ is the distance from X to P. From (3), taking square roots,

$$s = \frac{|\overrightarrow{XP}|}{|\mathbf{V}|}, \tag{4}$$

which is the conclusion of the theorem.

THEOREM 2. *The coefficient of variation is the tangent of angle θ in* Figure 1.

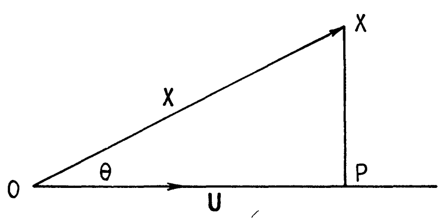

FIG. 1

Proof. The distance $|\overrightarrow{OP}| = \mathbf{X}\cdot\mathbf{U} = \sum x/\sqrt{n}$. The coefficient of variation $= ns/\sum x = |XP|/|OP| = \tan\theta$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1146. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Show that any rectangle whose edges and diagonal are measured in integers can be made the base of a rectangular parallelopiped whose three edges and main diagonal are measured in integers.

E 1147. *Proposed by E. P. Starke, Rutgers University*

If $\cos\alpha$ is rational $(0<\alpha<\pi)$, prove there are infinitely many triangles with integer sides having α as one angle. In particular, given $\cos\alpha=r/s$, find a three-parameter solution for the sides a, b, c.

E 1148. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let a, b, c be arbitrary points on the sides BC, CA, AB of triangle ABC, and let A', B', C' be the reflections of A, B, C in the midpoints of the segments bc, ca, ab. Show that triangles abc and $A'B'C'$ have equal areas.

E 1149. *Proposed by A. S. Gregory, University of Illinois*

For each $n=1, 2, \cdots$ find the least positive integer which when added to 2^n yields a perfect square.

E 1150. *Proposed by Frank Hawthorne, Hofstra College*

If three points are selected at random in a rectangle $A\times 2A$, what is the probability that the triangle so determined is obtuse?

SOLUTIONS

The Counterfeiters of Lower Slobbovia

E 1096 [1954, 46, 472]. *Proposed by L. R. Ford, Jr., The RAND Corporation, Santa Monica, Calif.*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance,

and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

II. *Solution by John Selfridge, University of California at Los Angeles.* The recent solution [1954, 473] of this problem will give the correct answer if all the dishonest coiners are among the first 6, or if it is known that there are at most 3 dishonest coiners. If the ith coiner is dishonest for $i=1$, 3 and 7, or for $i=3$, 4, 5, and 7, or for $i=1$, 2, 5, 6, and 7, then the ratio D'/D is 23; that is, in general the integer S is *not* unique.

To correct this, proceed as in the above solution for the first two weighings. Then if $D\neq 0$ take $(i-1)(i-1)!$ coins from the ith coiner, determine the total weight T', and compute $D'=T'-(N!-1)W_g$. If $D'=0$ only the first coiner is dishonest. If $D'\neq 0$ select the largest n such that $(n-1)!-(1/n)\leqq D'/D$, and then find the largest k such that $D'/D\leqq (n!-1)/k$. It follows that there are k dishonest coiners, the number of coins taken from these for the third weighing was kD'/D, and this is a unique sum of k numbers of the form $(i-1)(i-1)!$ so that we tell which coiners are dishonest.

III. *Solution by Leonard Carlitz, Duke University.* Number the coiners $1, 2, \cdots, N$ in some manner and assume the counterfeiters are numbered $i_1<\cdots<i_m$, $m\leqq N$. Let the weight of the good coin be 1 (first weighing). Second, take one coin from each coiner; if we let $1-\epsilon$ be the weight of a counterfeit coin, then this weighing gives $N-m\epsilon$ and the discrepancy is $D=m\epsilon$. Next, if $p>N$, but otherwise arbitrary, take p^i coins from the ith coiner; this weighing gives the discrepancy

$$D' = (p^{i_1}+\cdots+p^{i_m})\epsilon.$$

Now the equation

$$k = D'/D = (p^{i_1}+\cdots+p^{i_m})/m, \qquad (i_1<\cdots<i_m,\ m<p),$$

determines $m, i_1, \cdots, i_m$ uniquely for fixed k. For assume

$$(p^{i_1}+\cdots+p^{i_m})/m = (p^{j_1}+\cdots+p^{j_n})/n,$$

with $n\leqq N<p$ and $j_1<\cdots<j_n$. Then

$$n(p^{i_1}+\cdots+p^{i_m}) = m(p^{j_1}+\cdots+p^{j_n}).$$

Since the decimal representation of an integer to the base p is unique it follows that $m=n$, $i_1=j_1, \cdots, i_m=j_m$.

Concerning Pandiagonal Heterosquares

E 1116 [1954, 345]. *Proposed by C. W. Trigg, Los Angeles City College*

We define a pandiagonal heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns, and diagonals

(broken as well as straight) have the same sum. Is there any n for which these $4n$ sums are consecutive numbers?

Solution by D. C. B. Marsh, University of Colorado. In summing rows, columns, and diagonals of such an array, each integer would be counted four times; the sum of all such sums is therefore $4[n^2(n^2+1)/2]$, or $2n^2(n^2+1)$. If there were a least integral sum k such that the $4n$ sums ranged from k to $k+4n-1$ consecutively, the grand total would also be given by $2n(2k+4n-1)$. Equating these two expressions and simplifying we have $n(n^2+1)=2k+4n-1$. But the left side is always even whereas the right side is always odd. Hence k cannot exist, nor n such that the $4n$ sums are consecutive numbers.

Also solved by W. E. Briggs, S. H. Eisman, A. L. Epstein, Michael Goldberg, R. K. Guy, A. R. Hyde, J. B. Muskat, Walter Penney, and the proposer.

Construction of a Right Triangle

E 1117 [1954, 345]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a right triangle in which the legs and the altitude on the hypotenuse can be taken as the sides of another right triangle.

I. *Solution by L. V. Mead, Montana School of Mines.* Take an arbitrary line segment AB as the diameter of a semicircle. Let point D divide this diameter in mean and extreme ratio, so that $AB/AD=AD/DB$. Draw the perpendicular to AB at D intersecting the semicircle at C. Draw AC and BC, forming the inscribed right triangle ABC. Then, since AD and BC are both mean proportionals between the same two line segments AB and DB, $AD=BC$. Therefore the right triangle ADC has sides that are equal to the legs and altitude on the hypotenuse of the inscribed right triangle ABC.

II. *Solution by J. V. Pennington, Houston, Texas.* Construct right triangle ABC having legs $AB=2$ and $BC=1$. Extend AC one unit to D. With A as center and AD as radius, draw arc DE to cut BC produced in E. Then ABE is the required triangle. For, if F is the foot of the perpendicular from B on AE, $FB=\sqrt{2\sqrt{5}-2}$, $AB=2$, $EB=\sqrt{2\sqrt{5}+2}$, whence $(EB)^2=(FB)^2+(AB)^2$.

Also solved by Leon Bankoff, R. J. Beeber, Hüseyim Demir, Monte Dernham, I. A. Dodes, William Douglas, S. H. Eisman, E. I. Gale, Michael Goldberg, Louisa Grinstein, R. K. Guy, H. J. Hauer, Vern Hoggatt, R. T. Hood, J. P. Hoyt, A. R. Hyde, M. S. Klamkin, Sam Kravitz, Josef Langr, D. C. B. Marsh, George Mott, C. S. Ogilvy, M. J. Pascual, Walter Penney, L. L. Pennisi and M. C. Scholomiti (jointly), L. A. Ringenberg, Azriel Rosenfeld, N. Shklov, R. P. Tapscott, C. W. Trigg, Chih-yi Wang, and the proposer.

It can be shown that all right triangles having the desired property are similar. Evidently the triangle formed by the legs and the altitude has the same property as the original triangle.

Permutations and Permutable Transpositions

E 1118 [1954, 345]. *Proposed by Joel Brenner, State College of Washington*

Let us say a permutation is of type T if it is a product of mutually permutable transpositions. Is every permutation p a product of two permutations of type T?

Solution by the Proposer. If the permutations are of a finite set, the answer is yes. Since any pair of transpositions with all four elements distinct permute, and since any permutation of a finite set can be expressed as the product of disjoint cycles, it suffices to exhibit the following decomposition of a cycle of n elements:

$$(12 \cdots n) = R \cdot S,$$

where

$$R = (12)(3n)(4, n-1) \cdots ([(n+2)/2], [(n+5)/2]),$$

$$S = (13)(4n)(5, n-1) \cdots ([(n+3)/2], [(n+6)/2]).$$

Also solved by D. C. B. Marsh and Walter Penney.

The Tac-Locus of Two Parabolic Pencils of Circles

E 1119 [1954, 345]. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1, 1) to a line of slope m. Find the locus of the points of tangency of the two families.

Editorial Note. This is the same as problem E 1112 [1954, 259], solutions to which have already appeared. By an oversight the problem was published a second time.

Square Roots of Binomial Surds

E 1120 [1954, 345]. *Proposed by W. B. Carver, Cornell University*

(a) A student working a numerical problem gets the result

$$59\sqrt{90 - 14\sqrt{7}} + 4\sqrt{4555 + 1721\sqrt{7}},$$

but he finds the answer in the book, which happens to be correct, is

$$145\sqrt{26 + 2\sqrt{7}}.$$

Show that the student's result is correct.

(b) Find all sets of integers k, R, S, T, A, B, C, D, E, F which satisfy the equation

$$R\sqrt{A + B\sqrt{k}} + S\sqrt{C + D\sqrt{k}} + T\sqrt{E + F\sqrt{k}} = 0,$$

with $k>0$ and having no square factors, $A+B\sqrt{k}>0$, $C+D\sqrt{k}>0$, $E+F\sqrt{k}>0$, and the radicals meaning in all cases the positive square root.

Solution by Leonard Carlitz, Duke University. We shall prove the following slightly more general result. Let k be a square-free rational integer $\neq 1$ and consider the equation

$$u\sqrt{\alpha}+v\sqrt{\beta}+w\sqrt{\gamma}=0, \tag{1}$$

where u, v, w are rational and α, β, γ are numbers of the field $R(\sqrt{k})$; thus $\alpha=a_1+a_2\sqrt{k}$, a_1, a_2 rational, *etc.* Transposing the third term and squaring we get

$$u^2\alpha+v^2\beta+2uv\sqrt{\alpha\beta}=w^2\gamma.$$

Thus $\alpha\beta$ is a square of $R(\sqrt{k})$; similarly for $\beta\gamma$ and $\gamma\alpha$. But this implies

$$\alpha:\beta:\gamma=\lambda^2:\mu^2:\nu^2, \tag{2}$$

where λ, μ, ν are in $R(\sqrt{k})$. Then (1) becomes

$$u\lambda+v\mu+w\nu=0. \tag{3}$$

Conversely, if (3) is satisfied and α, β, γ satisfy (2), then (1) will hold. Hence a necessary and sufficient condition that (1) holds is that (2) be satisfied.

As for the special numerical problem, we have

$$\frac{90-14\sqrt{7}}{26+2\sqrt{7}}=\frac{45-7\sqrt{7}}{13+\sqrt{7}}=\frac{317-68\sqrt{7}}{81}=\left(\frac{17-2\sqrt{7}}{9}\right)^2,$$

$$\frac{4555+1721\sqrt{7}}{26+2\sqrt{7}}=\frac{47168+17818\sqrt{7}}{324}=\left(\frac{151+59\sqrt{7}}{18}\right)^2.$$

Since

$$\frac{59(17-2\sqrt{7})}{9}+\frac{4(151+59\sqrt{7})}{18}=\frac{1305}{9}=145,$$

the student's result is indeed correct.

Also solved by F. J. Duarte and Walter Penney.

SOLUTIONS

Distinct Elements in a Set of Sums

4466 [1951, 703]. *Proposed by Leo Moser, University of Alberta*

Let

$$0 = a_1 < a_2 < \cdots < a_n < 1, \qquad 0 = b_1 < b_2 < \cdots < b_m < 1,$$

and suppose that $a_i+b_j=1$ is not solvable. Prove that if the mn numbers a_i+b_j are reduced (mod 1) then at least $m+n-1$ of the residues will be distinct.

Solution by Peter Scherk, University of Saskatchewan. The problem can be reformulated as follows: Let G denote the additive group of real numbers (mod 1). Let A and B denote finite subsets of G. The set $A+B$ consists of all the elements $a+b$ where $a\subset A$, $b\subset B$. Let $[A]$, $\cdots$ denote the number of elements of A, $\cdots$. Suppose $0\subset A$, $0\subset B$; but $a+b=0$, $a\subset A$, $b\subset B$ only if $a=b=0$. Then

$$[A+B] \geqq [A]+[B]-1.$$

This result can be established readily by means of a method of a paper by J. H. B. Kemperman and the author (*Canadian Journal of Mathematics*, v. 6, pp. 230–237). We prove it for any (finite or infinite) abelian group G. Our statement is obvious if $[B]=1$. Suppose it proved for $[B]\leqq n-1$ and let $[B]=n>1$.

Let $b\subset B$, $b\neq 0$. Then $0\not\subset A+b$ while $0\subset A$. Since $[A+b]=[A]$, there are elements that lie in $A+b$ but not in A. Thus there exists an $a_0\subset A$ such that

$$a_1 = a_0 + b_1 \not\subset A$$

has solutions $b_1\subset B$. Let $A_1=\{a_1\}$ and $B_1=\{b_1\}$. Since $0\not\subset B_1$, we have

$$0 < [A_1] = [B_1] < [B]. \tag{1}$$

Let A_2 be the union of A with A_1 and let B_2 be the complement of B_1 in B. Thus

$$0 \subset B_2 \subset B \quad \text{and} \quad 0 \subset A \subset A_2,$$

and by (1),

$$\begin{aligned}[A_2]+[B_2] &= [A]+[A_1]+[B_2] \\ &= [A]+[B_1]+[B_2] \\ &= [A]+[B].\end{aligned} \tag{2}$$

We will verify

$$A_2 + B_2 \subset A + B \tag{3}$$

and

$$a_2 + b_2 = 0,\ a_2 \subset A_2,\ b_2 \subset B_2 \quad \text{only if} \quad a_2 = b_2 = 0. \tag{4}$$

Let $a_2 \subset A_2$, $b_2 \subset B_2$. We need consider only the case $a_2 \not\subset A$, *i.e.*, $a_2 = a_0 + b_1 \subset A_1$. Thus

$$a_2 + b_2 = (a_0 + b_1) + b_2 = (a_0 + b_2) + b_1.$$

Since $b_2 \subset B_2$, the definition of B_1 implies $a_0 + b_2 \subset A$. Thus $(a_0+b_2)+b_1$ lies in $A+B_1$, hence in $A+B$. Also $(a_0+b_2)+b_1=0$ would imply $b_1=0$ which is impossible. This yields (3) and (4).

Since $[B_2]<[B]$, our induction assumption implies

$$[A_2 + B_2] \geqq [A_2] + [B_2] - 1. \tag{5}$$

Finally, (3), (5) and (2) yield

$$[A + B] \geqq [A_2 + B_2] \geqq [A_2] + [B_2] - 1 = [A] + [B] - 1,$$

which completes the proof.

Special Case of Valiron's Theorem

4559 [1953, 631]. *Proposed by Harry Pollard, Institute for Advanced Study, and Paul Erdös, University of Notre Dame*

Let $0 < a_1 \leqq a_2 \leqq \cdots$ and put

$$F(y) = \prod_{k=1}^{\infty} \left(1 + \frac{y^2}{a_k^2}\right).$$

The necessary and sufficient condition for the convergence of

$$\int_1^{\infty} \frac{\log F(y)}{y^2} dy$$

is that $\sum_{k=1}^{\infty} 1/a_k$ converges.

Solution by R. P. Boas, Jr., Northwestern University. If $0<\alpha<1$, we have the well known relation

$$\int_0^{\infty} x^{-\alpha-1} \log (1 + x) dx = \pi/(\alpha \sin \pi\alpha).$$

(The integral can be evaluated by contour integration, or transformed into a beta function by integration by parts.) Hence

$$\begin{aligned}\int_0^{\infty} x^{-\alpha-1} \log F(x^{1/2}) dx &= \sum_{n=0}^{\infty} \int_0^{\infty} x^{-\alpha-1} \log (1 + x/a_n^2) dx \\ &= \sum_{n=0}^{\infty} a_n^{-2\alpha} \pi/(\alpha \sin \pi\alpha).\end{aligned}$$

The formal computation is justified because everything is positive and so the convergence of either side implies that of the other. Putting $x^{1/2}=y$, we see that

$$\int_0^\infty y^{-2\alpha-1}F(y)dy \quad \text{and} \quad \sum_{n=0}^\infty a_n^{-2\alpha}$$

converge or diverge together, $0<\alpha<1$. The problem is the case $\alpha=\frac{1}{2}$.

Remark. The problem is a special case of Valiron's theorem [*Bull. Sci. Math.* (2) 45, 258–270 (1921)], according to which, if $f(z)$ is an entire function of fractional order α with zeros z_n, and $M(r)=\max |f(z)|$ for $|z|=r$, then

$$\int_1^\infty M(r)r^{-\alpha-1}dr \quad \text{and} \quad \sum |z_n|^{-\alpha}$$

converge or diverge together. Here $f(z)=F(z^{1/2})$, $z_n=-a_n^2$, $M(r)=f(r)$.

Also solved by W. E. Briggs, F. A. Ficken, J. D. Hill, A. E. Livingston, R. T. C. Smith, O. E. Stanaitis, Chih-yi Wang, J. E. Wilkins, Jr., G. N. Wollan, and the Proposers.

Integral Solutions of Inequalities

4560 [1953, 631]. *Proposed by L. J. Mordell, St. John's College, Cambridge, England*

Prove that there are exactly $(\lambda+1)^{n+1}-\lambda^{n+1}$ sets of n integers, $x_1, x_2, \cdots, x_n$, which satisfy the inequalities

$$|x_r| \leqq \lambda, \qquad |x_r - x_s| \leqq \lambda, \qquad (r, s = 1, 2, \cdots, n),$$

where λ is a positive integer.

Solution by W. J. Blundon, Memorial University of Newfoundland, St. Johns. If the elements of an ordered set $x_1, x_2, \cdots, x_n$ are chosen from k distinct numbers (allowing repetitions), the number of possible distinct sets is k^n. If a specified number (say, the largest) is to be included at least once, the number of distinct sets is $k^n-(k-1)^n$.

Under the given restrictions on the x's, divide the sets into mutually exclusive classes, each class being characterized by the size of its largest x_i. Let p be a non-negative integer, and let $f(y)$ be the number of sets in the class in which the maximum x_i of every set is y. Then clearly

$$f(p) = (\lambda+1)^n - \lambda^n, \qquad p = 0, 1, 2, \cdots, \lambda;$$
$$f(-p) = (\lambda - p + 1)^n - (\lambda - p)^n, \qquad p = 1, 2, 3, \cdots, \lambda.$$

Summing for all the classes, the number of possible sets is

$$(\lambda+1)\{(\lambda+1)^n - \lambda^n\} + \sum_{p=1}^{\lambda} \{(\lambda - p + 1)^n - (\lambda - p)^n\} = (\lambda+1)^{n+1} - \lambda^{n+1}.$$

Also solved by H. L. Alder, I. N. Baker, Leonard Carlitz, A. R. Hyde, J. B. Kelly, David Loev, and Michael Skalskyj.

Editorial Note. As noted by H. F. Mattson, if in the above discussion k^n is replaced by $\binom{k}{n}$ one gets

$$\frac{\lambda!(1+\lambda+n\lambda)}{n!(\lambda-n+1)!},$$

as the number of sets of n distinct integers satisfying the inequalities. Here we agree that $1/m!=0$ whenever m is negative.

Finite Series

4561 [1953, 632]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Prove

$$S_{2,p}+\sum_{n=1}^{r}\left[n^2\binom{n+p}{p}\right]^{-1}=S_{2,r}+\sum_{n=1}^{p}\left[n^2\binom{n+r}{r}\right]^{-1}$$

where $S_{2,p}=\sum_{n=1}^{p}1/n^2$.

Solution by Leonard Carlitz, Duke University. Let p be an arbitrary, fixed integer. For $r=1$, the proposed relation reduces to

$$S_{2,p}+\frac{1}{p+1}=1+\sum_{n=1}^{p}\frac{1}{n^2(n+1)},$$

which is evidently true since

$$\sum_{n=1}^{p}\frac{1}{n^2}-\sum_{n=1}^{p}\frac{1}{n^2(n+1)}=\sum_{n=1}^{p}\frac{1}{n(n+1)}=\sum_{n=1}^{p}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{p+1}\cdot$$

Assuming now that the stated result holds for $r-1$, we have

$$\begin{aligned}S_{2,r}+\sum_{n=1}^{p}\left[n^2\binom{n+r}{r}\right]^{-1}&-S_{2,r-1}-\sum_{n=1}^{p}\left[n^2\binom{n+r-1}{r-1}\right]^{-1}\\&=\frac{1}{r^2}-\sum_{n=1}^{p}\frac{(r-1)!n!}{n^2}\frac{n}{(n+r)!}=\frac{1}{r^2}-\sum_{n=1}^{p}\frac{(r-1)!}{n(n+1)\cdots(n+r)}\\&=\frac{1}{r^2}-\frac{(r-1)!}{r}\left\{\frac{1}{r!}-\frac{1}{(p+1)\cdots(p+r)}\right\}\\&=\frac{(r-1)!}{r(p+1)\cdots(p+r)}=\left[r^2\binom{r+p}{p}\right]^{-1}.\end{aligned}$$

This evidently completes the induction proof.

Also solved by Harley Flanders, L. A. Fulk, A. E. Livingston, Ernst Trost, Chih-yi Wang, and the Proposer.

Five Mutually Orthogonal Spheres

4611 [1954, 646]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given five spheres, if one of them is orthogonal to the four others, then the centers of the four are the vertices of an orthocentric tetrahedron whose orthocenter coincides with the center of the fifth sphere.

Note by N. A. Court, University of Oklahoma. The statement of the theorem is inaccurate. It should read: *Given five spheres, if every one of them is orthogonal to the four others, then the centers of any four spheres are the vertices of an orthocentric tetrahedron whose orthocenter coincides with the center of the fifth sphere.*

But this is a well known proposition. See J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, theorem 5, p. 257. Also N. A. Court, *Modern Pure Solid Geometry*, p. 275, art. 834.

Also solved by Roscoe Woods.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Engineering Statistics and Quality Control. By I. W. Burr. New York, McGraw-Hill Book Co., Inc., 1953, xi+442 pages. $7.00.

Professor Burr's implicit thesis in this carefully written intermediate text in statistics is that engineers will use statistical training more directly in solving engineering problems than any other branch of "secular" mathematics. He defines statistics as the science which is concerned with problems involving chance variation. Chance variation becomes increasingly important as a science of measurement becomes more refined and as industry drives for closer and closer tolerances.

Professor Burr does not treat statistics as a narrow subject of application of formula and technique for the solution of particular problems. Rather he conceives statistics as a way of thinking based upon the fundamentals of science,

an observational approach to problem analysis and solution. Statistics is a vital phase of industrial life because it involves a rational scientific approach appropriate to engineering investigation combined with methods and procedures for handling the data obtained from such investigation.

Professor Burr conceives that the establishment of basic courses in statistics in the Engineering curriculum not as a narrow vocational subject as in the use of the slide rule and in the study of trigonometry, but rather as a broad, cultural, philosophical, rational background to engineering problems and their solutions.

The topics included in the book are largely those that have become part of statistical quality control literature. These include the frequency distribution, the fundamental statistical measures of average and variability, normal curve control charts, probability, acceptance sampling by attributes and acceptance sampling by measurements. These topics cover essentially what the practicing engineer needs to know about statistical quality control. These are a fundamental fabric upon which further study can be founded.

Most examples in the book are drawn from the voluminous literature of quality control applications and from the author's experience. Most of the examples aptly illustrate the theory that is presented. The examples used give the touch of reality to the statistical theory presented and illustrate more the manner of approach to similar problems rather than the solution of the problems presented. This in effect becomes a case study approach founded on adequate theory. The book includes sufficient mathematical notation and derivation to serve as a satisfactory prerequisite for advanced courses.

In the preface the author has explained his basic reasoning for using the notations followed by the American Standards Association and the American Society for Quality Control rather than the notations that have developed in the statistical literature. This is a matter of taste and judgment in which it is felt that the author has erred, but probably this will have no serious impairment on the use of the book for Engineering students.

This reviewer feels that the notation in statistical quality control should conform with the larger and ever-increasing literature of mathematical statistics, since statistical quality control has represented one of the fighting fronts upon which real problems of the world have been presented to the mathematical statisticians for solution.

Professor Burr's book is one written by a mathematician who sees in statistics a scientific approach, a vehicle for mathematical reasoning, a necessary and successful tool for practical application and an indispensable part of the training of current generations.

WILLIAM R. PABST, JR.
Bureau of Ordnance, Navy Department
Washington, D. C.

matical operations; and by other violations of what the reviewer considers good usage. On the other hand, the reviewer has pointed out that the author does an excellent job of leading the student very slowly and very carefully to each new basic concept; that for the most part the selection and arrangement of topics is good; and that the standard of rigor is satisfactory.

F. C. Smith
College of St. Thomas

Introductory Calculus with Analytic Geometry. By E. G. Begle. New York, Henry Holt and Company. 1954. x+304 pp. $4.50.

With new calculus books appearing each year, it is the rare instructor who has the time, the inclination, or the opportunity to examine all of them. Equally rare is the book which makes a real contribution to the *understanding* of calculus at the first year level. In the reviewer's opinion Begle's book does make such a contribution and should be examined carefully by all instructors of beginning calculus. This is not to say that this book is suitable for use in every institution or for all types of students. In his preface Begle states, "This text differs from most others in this field in that it treats calculus as a branch of mathematics rather than as a mere adjunct of the physical and engineering sciences. . . . We start with a list of axioms and show how the theorems of the calculus are derived from these axioms . . . . Our aim in presenting calculus in this fashion is to give the student more of an understanding of the basic concepts of the subject than is usually done in an introductory course." Such a program implicitly assumes that the student is ready to learn to think rigorously; this demand may limit the effective use of the text to classes of greater than average ability.

It is important to observe the concise style of the text. The author scrupulously avoids verbosity and repetition. The book thus serves as an excellent training ground for precise reading and logical reasoning. Definitions and theorems are clearly stated, all theorems are proved correctly, and the proofs are never intermingled with examples or discussion. Applications are used expertly, sometimes to motivate and sometimes to illustrate the basic methods and theorems of the calculus. Many results which are interesting and useful, but not essential to the chain of reasoning, are included as problems, making the problem lists an integral part of the text.

The book's most significant departure from tradition arises from Begle's careful treatment of functions and functional notation. Distinction is made between functions and functional values, and the distinction is preserved in the notations f and $f(x)$. Arithmetic combinations of functions are denoted $\{f+g\}$, $\{fg\}$, *etc.*, and a function of a function is denoted $\{f(g)\}$. The derivative of f is denoted Df or f', and the notation df/dx is carefully withheld until after the properties of derivatives are developed and the Chain Theorem proved (correctly, but with an unfortunate misprint). The definite integral is denoted

$\int_a^b f$, and again the usual notation is not introduced until the properties of the definite integral are established.

The tenor of the book is set forth unmistakably in the first chapter, entitled Foundations and designed to show that algebra, like geometry, can be derived rigorously from a relatively short list of axioms. Direct and indirect proofs are used to establish many familiar properties of numbers, including ordering. The role of definitions in mathematics is brought out with the introduction of absolute value. The Least Upper Bound Axiom is also introduced, to be used later as the basis on which the principal theorems of the calculus are proved.

Chapter 2 introduces plane analytic geometry, while Chapter 3 covers functions, limits, and continuity. Limits are defined precisely, and all necessary limit theorems are proved in detail. The student is not spared the pain of having to ponder the mysteries of δ and ϵ. Consequently those who assimilate Chapter 3 are ready to understand calculus. The others probably can learn calculus. The next three chapters concern the derivative and its usual applications to physics and geometry. The Mean Value Theorem plays a central role in the proofs.

The high point of the book comes with Begle's careful presentation of the definite integral, which he motivates most reasonably by analyzing the two problems of measuring work and area. This leads naturally to a study of upper and lower integrals and to the definition of the integral as their common value. The Existence Theorem for the definite integral of a continuous function is proved, followed by the Fundamental Theorem and the Mean Value Theorem for integrals. Finally it is proved that this definition of the integral is equivalent to the usual limit definition. Integration techniques and applications are developed further in Chapter 8. When using this text in a preliminary form, the reviewer was pleasantly surprised to find that many students enjoyed the rigor and clarity of this development in spite of its intricate details, and that the concept of the definite integral was unusually well understood.

In Chapter 9 the logarithmic function is defined and studied as a definite integral. Then inverse functions are investigated, and the exponential function is obtained as the inverse of the logarithmic function. The final chapter concerns trigonometric functions. The functions are defined, the usual identities are developed, and the graphs are examined—all in one brief section which is certainly inadequate for anyone who is not already familiar with these notions. The inverse sine function is next established in integral form, from which its derivative and the derivatives of all trigonometric functions easily follow. This approach, while interesting, appears to be artificial and less effective than the usual development.

The utility of the book to students would be increased by more generous use of figures as well as summary lists of differentiation and integration formulas at the ends of appropriate chapters. The problem lists are good, and some exercises will challenge even the best students. Answers are given for selected problems. Only a few misprints were noted.

Since this is a one year text, instructors will naturally wonder what follow-up text can be used. The reviewer feels that this question, while important, is not paramount, simply because Begle's treatment lays a firm foundation; students should have little difficulty in shifting to another text for subsequent study, whether their interest lies in engineering or analysis. Begle has set up a worthwhile goal, has carefully chosen topics which lead thereto, and has accomplished his aim with remarkable success.

D. T. FINKBEINER
Kenyon College

Differential and Integral Calculus. By Philip Franklin. New York, McGraw-Hill Book Company, Inc., 1953. 11+641 pages. $6.00.

This is a substantial text in elementary calculus. It contains all the topics ordinarily included in a first course, many of them developed more fully than usual, and also a number of additional features.

After a first chapter on limits, the derivative is introduced in connection with rates of change and applied in situations involving polynomials and other simple types of algebraic functions. Integration is brought in early, as the inverse of differentiation, but a full discussion comes much later in the book. After the first brief account of integration, derivatives of transcendental functions are obtained and applied to the usual types of problems in geometry and physics. Differentials are defined earlier but a chapter on them immediately precedes the discussion of the definite integral as the limit of a summation. Various methods of integration and applications to areas, volumes, arc lengths, centroids, moments, pressure, and work follow. Infinite series, Taylor's series, and partial differentiation are next. A chapter is devoted to vectors and three-dimensional analytic geometry in preparation for the work on multiple integrals. The book ends with an introduction to differential equations.

Due to the practice of introducing only one new idea at a time, the development of the subject proceeds in an unhurried fashion. Each chapter is introduced by a brief summary at its beginning. Throughout the book there is a well balanced emphasis on applications. Topics which are especially well done include increasing and decreasing functions and maxima and minima.

A special feature of the book, which students will find very helpful, consists of a number of sections marked with an R which contain pertinent review material in trigonometry and analytic geometry. As another feature, certain sections are set in smaller type and marked with an asterisk. According to the preface this indicates that they "either treat a topic not universally studied by beginners, or contain a proof with some details a little difficult for the average class." It is a bit puzzling to find all discussion of the definition and theory of limits confined to such sections.

The exposition is full and generally very clear; the language is precise and lucid. Much appeal is made to geometric intuition, which is right and proper in

a beginning course in calculus. The degree of rigor maintained is probably as high as is practicable at this level. There are a few places where the distinction between plausible and rigorous argument should perhaps be more sharply drawn. Also, occasionally the reader may be in doubt whether a certain statement can be inferred from the context or is simply an assertion to be accepted without proof.

The illustrative examples are well chosen and are worked out in clarifying detail. The exercises appear entirely adequate unless one insists on some applications from the social sciences. The typography and figures are above reproach and the misprints discovered are too trivial to mention.

This book is not for a streamlined course, but where time permits, it could serve as an excellent text for thorough training in elementary calculus.

E. A. CAMERON
University of North Carolina

NEW BOOKS RECEIVED

Quantum Mechanics. By F. Mandl. New York, Academic Press, Inc., 1954. viii+233 pages. $5.80.

The Microphysical World. By William Wilson. New York, Philosophical Library, 1954. vii+216 pages. $3.75.

Statistics in Research. By Bernard Ostle. Ames, Iowa, The Iowa State College Press, 1954. 487 pages. $6.95.

Decision Processes. By Thrall, Coombs and Davis. New York, John Wiley and Sons, Inc., 1954. 8+332 pages. $5.00.

Mathematical Thinking in the Social Sciences. Edited by P. F. Lazarsfeld. Glencoe, Illinois, The Free Press, 1954. 444 pages. $10.00.

Functionals of Finite Riemann Surfaces. By M. Schiffer and D. C. Spencer. Princeton, New Jersey, Princeton University Press, 1954. x+451 pages. $8.00.

Mathematics and Plausible Reasoning, Volume I. By G. Pólya, *Mathematics and Plausible Reasoning, Volume II.* By G. Pólya. Princeton, New Jersey, Princeton University Press, 1954. Vol. I, xvi+280 pages. $5.50. Vol. II, x+190 pages. $4.50. ($9.00 the set.)

Elementary Statistics. By John M. Howell and Ben K. Gold. Dubuque, Iowa, William C. Brown Company, 1954. 154 pages. $3.00.

Niels Henrik Abel. By Oystein Ore. Sweden, Gyldendal Norsk Forlag, 1954.

A Short Table for Bessel Functions of Integer Orders and Large Arguments. By L. Fox. (Royal Society Shorter Mathematical Tables, Number 3.) Published for the Royal Society at the University Press, Cambridge, 1954. 28 pages. 6s. 6d.

Royal Society Mathematical Tables, 3, Table of Binomial Coefficients. Edited by J. C. P. Miller. New York, Cambridge University Press, 1954. $5.50.

Trigonometry. By E. P. Vance. Cambridge, Mass., Addison-Wesley Publishing Company, Inc., 1954. viii+158 pages. $3.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor C. L. Clark of Oregon State College was appointed to represent the Mathematical Association of America at the inauguration of President Owen M. Wilson of the University of Oregon on October 19, 1954.

Mrs. Roberta E. Presnell of Rockford College represented the Mathematical Association of America at the inauguration of President Miller Upton of Beloit College on October 29, 1954.

Assistant Professor Josephine Mitchell, who is on leave from the University of Illinois, has been appointed a member of the Institute for Advanced Study and has been awarded the Marion Talbot Fellowship of the American Association of University Women.

Dr. M. F. Ruchte of the University of Nebraska is a National Science Foundation Post-Doctoral Fellow at Yale University.

Butler University announces the following: The Department of Mathematics is now the Department of Mathematics and Astronomy; Professor H. E. Crull is Head of the Department and Director of the new J. I. Holcomb Observatory; Dr. R. H. Oehmke of Illinois Institute of Technology has been appointed to an assistant professorship; Mrs. Theresa M. C. Oehmke, formerly a computer for Air Weapons Research, University of Chicago, has been appointed Acting Instructor.

Carnegie Institute of Technology reports: Miss Irene Harwick has been appointed to an instructorship; Dr. Frederick Ling has accepted an assistant professorship; Mr. G. B. Thorp has been appointed Lecturer.

Duquesne University reports the following: Assistant Professor A. G. Anderson has been promoted to an associate professorship and named Chairman of the Department of Mathematics; Former Acting Chairman R. E. Smith remains at the University as Associate Professor in the Department of Management, School of Business Administration; Mr. Donato DeFelice of the University of Pittsburgh has been appointed to an instructorship.

Florida State University announces: Assistant Professor R. L. Plunkett of Vanderbilt University has been appointed to an assistant professorship; Mr. J. E. Snover of Syracuse University and Dr. R. F. Williams of the University of Virginia have been appointed to assistant professorships; Associate Professor D. B. Goodner has been promoted to a professorship; Assistant Professor H. E. Taylor has been promoted to an associate professorship.

Hampton Institute reports the following: Mr. S. R. Beyma has been appointed Acting Chairman; Mr. O. R. Barker has been appointed to an assistant professorship; Miss Harriett R. Junior has been appointed to an instructorship.

The Johns Hopkins University makes the following announcements: Assistant Professor F. I. Mautner has been awarded a Guggenheim Fellowship and will be a member of the Institute for Advanced Study for the academic year 1954–55; Professor Wei-Liang Chow is on leave of absence and will be a member of the Institute for Advanced Study for the academic year 1954–55; Assistant Professor G. D. Mostow has been promoted to an associate professorship; Dr. Gerard Washnitzer has been appointed Visiting Lecturer for the academic year 1954–55; Mr. J. T. Robinson and Dr. Solomon Schwartzman have been appointed to instructorships.

At Lehigh University: Assistant Professor Theodore Hailperin has been promoted to an associate professorship; Dr. S. I. Goldberg has been promoted to an assistant professorship; Mr. Ti Yen, formerly a graduate assistant at the University of Illinois, has been appointed to an instructorship; Associate Professor Emeritus K. W. Lamson has returned from the University of Puerto Rico and will teach part-time during 1954–55; Mr. R. L. Korsch, Mr. F. C. Oglesby, Mr. R. C. Scott and Mr. C. E. Yingst have been appointed to part-time assistantships.

McMaster University announces the following: Associate Professors J. D. Bankier and D. B. Sumner have been promoted to professorships; Mr. W. J. McCallion, formerly a lecturer, has been promoted to an assistant professorship.

Montana State University reports the following: Dr. D. B. Higman, who has been National Research Fellow at McGill University, has been appointed to an associate professorship; Dr. W. R. Cowell of the University of Wisconsin has been appointed Assistant Professor; Associate Professor T. G. Ostrom has been appointed Chairman of the Department of Mathematics.

At Mount Allison University, New Brunswick, Canada: Professor W. S. H. Crawford is on leave of absence to engage in study and research at the University of London while on a Beaverbrook Overseas Fellowship; Mr. Neill Wallace, formerly with the United States Army, has been appointed Lecturer.

North Carolina State College announces: Associate Professors H. M. Nahikian and H. V. Park have been promoted to professorships; Mr. G. C. Caldwell of the Institute for Cooperative Research, Johns Hopkins University, Baltimore, Maryland, has been appointed to an instructorship.

Pacific University reports: Assistant Professor John Christopher of Knox College has been appointed to an assistant professorship; Professor H. F. Price has retired with the title Professor Emeritus.

At Pomona College: Professor C. G. Jaeger, chairman of the Department of Mathematics, has been elected Mayor of Claremont, California; Assistant Professor E. B. Tolsted has been promoted to an associate professorship.

Texas Southern University announces the following: Dr. A. H. Wardlaw has been promoted to the position of Assistant Professor and Acting Head of the

Department of Mathematics; Mr. J. E. Westberry has been promoted to an assistant professorship; Mrs. Jeanette E. Lockley has been appointed to an instructorship; Assistant Professor R. H. Parson is on a study leave of absence at the University of Iowa.

Tulane University reports: Mr. Leon Brown, Dr. D. E. Edmondson, and Dr. J. R. Isbell have been appointed to research instructorships; Dr. F. B. Wright, Jr., has been appointed to an instructorship.

The University of Buffalo reports: Mr. A. G. Fadell, formerly a teaching assistant at Ohio State University, has been appointed to an instructorship; Mr. A. H. Blessing, Mr. C. L. Gape, and Mr. G. E. Neu, formerly teaching fellows at the University, have been promoted to instructorships.

The University of Cincinnati announces: Dr. Wolfgang Jurkat of the University of Tübingen has been appointed Visiting Associate Professor; Mr. I. L. Lynn and Mr. B. A. Raymond have been appointed to instructorships; Mr. R. J. Fopma has been promoted to an assistant professorship; Associate Professor Carl Ludeke of the Department of Mathematics has been transferred to the Physics Department.

The University of Michigan reports the following: Assistant Professors C. L. Dolph and E. E. Moise have been promoted to associate professorships; Associate Professor Hans Samelson has been promoted to a professorship; Dr. J. W. Addison, Jr., Dr. Maurice Auslander, Dr. T. R. Jenkins, and Dr. P. J. Koosis have been appointed to instructorships; Instructors D. K. Kazarinoff, J. E. McLaughlin, and C. J. Titus have been promoted to assistant professorships; Dr. J. H. Giese of the Ballistics Research Laboratory, Aberdeen Proving Grounds, has been appointed Visiting Lecturer for the first semester 1954–55.

The Institute of Technology, University of Minnesota, announces the following: Dr. J. B. Serrin of Massachusetts Institute of Technology and Dr. H. Yamabe of the Institute for Advanced Study have been appointed to assistant professorships; Professor Jan Popken of the University of Utrecht, the Netherlands, has returned to the Netherlands after serving as Visiting Professor during the academic year 1953–54; Dr. L. W. Green and Dr. J. E. Thompson were promoted to assistant professorships; Professor P. C. Rosenbloom has returned after a year's leave of absence which was spent at the Institute for Advanced Study; Professor Rosenbloom has been appointed to the Advisory Panel for Mathematics of the National Science Foundation.

The University of Nebraska announces: Associate Professor W. G. Leavitt is Chairman of the Department of Mathematics and Astronomy; Dr. F. C. Andrews, formerly a research associate at Stanford University, has been appointed to an assistant professorship; Dr. Arne Magnus of the University of Kansas and Dr. F. W. Anderson, previously a graduate student at the University of Iowa, have been appointed to instructorships.

At the University of New Hampshire: Associate Professor W. L. Kichline has been promoted to a professorship; Mr. F. J. Robinson has been promoted

to an assistant professorship; Mr. C. G. Cullen, formerly a student at New York State College for Teachers, Miss Constance M. Foley, recently a student at the University, and Mr. B. C. McQuarrie, previously a student at Lafayette College, have been appointed Graduate Assistants.

The University of North Carolina announces the following: Professor J. W. Lasley, Jr., is serving as Chairman of the Department of Mathematics while Professor W. M. Whyburn is on sabbatical leave for the academic year 1954–55.

At the University of Washington: Professor R. A. Beaumont has been awarded a Ford Fellowship and is at the Institute for Advanced Study for 1954–55; Associate Professors R. A. Beaumont and Edwin Hewitt have been promoted to professorships; Assistant Professor V. L. Klee has been promoted to an associate professorship; Dr. L. A. Kokoris has been promoted to an assistant professorship; Miss Thelma M. Chaney, Dr. H. A. Forrester, Dr. R. L. Vaught, and Dr. J. H. Walter have been appointed to instructorships.

Wesleyan University reports the following: Visiting Assistant Professor Hing Tong of Barnard College has been appointed to an assistant professorship; Mr. E. W. Adams, formerly with the Bureau of Applied Social Research at Columbia University, has been appointed to an instructorship under the internship program of the Fund for the Advancement of Education.

Assistant Professor D. F. Atkins of the University of Richmond has been promoted to an associate professorship.

Acting Assistant Professor H. H. Barnett of Florida State University has a position as a mathematician with the Glenn L. Martin Corporation, Baltimore, Maryland.

Mr. G. C. Barton, formerly a teacher at Concrete Senior High School, Concrete, Washington, is teaching now at Las Lomas High School, Walnut Creek, California.

Dr. Donald Charles Benson, previously a research assistant at Applied Mathematics and Statistical Laboratory, Stanford, California, has been appointed to an instructorship at Princeton University.

Dr. I. E. Block of the Philco Radio and Television Corporation has accepted a position as staff consultant with the Burroughs Computation Laboratory, Philadelphia, Pennsylvania.

Mr. R. O. Blummer, Jr., recently associate mathematician with Vitro Corporation of America, Key West, Florida, has a position as a scientific analyst with the Division of Industrial Cooperation, Massachusetts Institute of Technology.

Associate Professor J. L. Brenner of the State College of Washington is spending the academic year at the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, under a contract between the State College of Washington and the Office of Ordnance Research.

Assistant Professor L. P. Burton of the University of California at Davis has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Assistant Professor Eugenio Calabi of Louisiana State University has accepted a position as Assistant Professor at California Institute of Technology for the academic year 1954–55.

Dr. J. H. Case, graduate assistant at Tulane University, has been appointed to an assistant professorship at the University of Utah.

Assistant Professor John Caulfield of Fairfield University has been appointed to an assistant professorship at Boston College.

Dr. H. J. Cohen, formerly a Fulbright scholar at the University of Paris, has been appointed to an instructorship at the City College of the City of New York.

Dr. E. H. Crisler of Oberlin College has been appointed to an assistant professorship at the University of Notre Dame.

Dr. A. E. Danese of the University of Rochester has accepted an assistant professorship at Western Reserve University.

Dr. M. A. Dengler of the Midwest Research Institute, Kansas City, Missouri, has accepted a position as senior aerodynamicist with Northrop Aircraft, Inc., Hawthorne, California.

Mr. J. I. Derr, formerly an aerophysics engineer at Consolidated Vultee Aircraft Corporation, Fort Worth, Texas, has a position as an assistant mathematician at the Rand Corporation, Santa Monica, California.

Assistant Professor J. T. Duprat of the University of Ottawa has been promoted to an associate professorship.

Mr. H. E. Flesner has been appointed to an instructorship at the University of Toledo.

Assistant Professor Herta T. Freitag of Hollins College has been promoted to an associate professorship.

Professor J. E. Freund of Alfred University has accepted a position as Professor of Statistics at Virginia Polytechnic Institute.

Mr. H. H. Goode has been appointed to a professorship in the Department of Electrical Engineering of the University of Michigan in addition to his duties as Director of the Willow Run Research Center and Assistant Director of the Engineering Research Institute.

Mr. J. H. Griesmer, formerly a student at the University of Notre Dame, has accepted a position as an assistant in research at Princeton University.

Emeritus Professor F. L. Griffin of Reed College has been appointed President of the College.

Associate Professor J. J. L. Hinrichsen has been named Head of the Department of Mathematics of Iowa State College.

Mr. W. B. Holmes, previously a flight test analyst for Boeing Airplane Company, Wichita, Kansas, has a position as a staff engineer for Instrumentation Laboratory, Massachusetts Institute of Technology.

Mr. N. C. Hoover, recently an instructor at the University of South Dakota, has accepted an instructorship at Fresno State College.

Dr. T. R. Horton of the University of Florida has a position as applied science representative with the International Business Machines Corporation, Atlanta, Georgia.

Reverend L. E. Isenecker, formerly a graduate student at Catholic University of America, is continuing his studies at West Baden College.

Dr. Paul Ito, assistant in mathematics at St. Louis University, was given a post-doctoral fellowship at the University of North Carolina for 1954–55 in mathematical statistics.

Associate Professor S. J. Jasper of East Tennessee State College has accepted an assistant professorship at Ohio University.

Professor Fritz John of New York University has received a Fulbright fellowship to lecture in mathematics at the Georg August Universitaet, Goettingen, Germany, during the February–July semester of 1955.

Professor Bjarni Jonsson of Brown University is on leave of absence for the year 1954–55 and is spending the year at the University of Iceland as Visiting Professor.

Mr. R. E. Krucklin of A. H. Johnson and Company, New York City, has accepted a position with North Atlantic Constructors, New York City, as Head of Materials and Procurement Control Department.

Mr. A. R. Lamontagne, formerly a graduate assistant at the University of New Hampshire, is a fellow at Brown University.

Mr. J. W. Lindsay has been appointed to an assistant professorship at Texas Technological College.

Mr. D. B. Lowdenslager, formerly with the Institute for Cooperative Research, Johns Hopkins University, has accepted an instructorship at the University of Virginia.

Dr. R. D. Luce, previously the managing director of the Behavioral Models Project, Bureau of Applied Social Research, Columbia University, is now at the Center for Advanced Study in the Behavioral Sciences, Menlo Park, California.

Dr. E. A. Maier, formerly of the University of Oregon, is now on the research staff of the Giustina Brothers Lumber Company, Eugene, Oregon.

Mrs. Dorothy C. Martin, instructor at Wood Junior College, has been appointed to an assistant professorship at Jacksonville State College.

Dr. R. A. Moore of the University of Nebraska has been appointed Instructor and Research Fellow at Yale University.

Professor P. B. Norman of the Polytechnic Institute of Brooklyn has been appointed to a professorship at Long Island University.

Dr. C. S. Ogilvy of Hamilton College has been promoted to an assistant professorship.

Dr. E. J. Pellicciaro of the University of Delaware has accepted a research instructorship at Duke University.

Professor H. R. Phalen has been appointed Head of the Department of Mathematics of the College of William and Mary.

Associate Professor F. M. Pulliam of the U. S. Naval Postgraduate School has been promoted to a professorship.

Dr. G. N. Raney of Columbia University has been appointed to an instructorship at Brooklyn College.

Mr. John Rausen of the University of Connecticut has been appointed a lecturer at Columbia University.

Mr. P. C. Rogers, formerly a mathematician with the United States Air Force, Washington, D. C., has accepted an instructorship at St. Joseph's College, Philadelphia, Pennsylvania.

Assistant Professor Louise J. Rosenbaum of Reed College has been appointed to an assistant professorship at St. Joseph College, West Hartford, Connecticut.

Dr. A. L. Shields of Tulane University has been promoted to an assistant professorship.

Mr. R. L. Shively of Western Reserve University has been promoted to an assistant professorship.

Assistant Professor Abraham Spitzbart of the University of Wisconsin in Milwaukee has been promoted to an associate professorship.

Dr. Robert Stanley of the University of British Columbia has been appointed to an assistant professorship at the University of South Dakota.

Assistant Professor T. T. Tanimoto of Allegheny College has accepted a position as a mathematician with International Business Machines Corporation, New York City.

Mr. C. W. Thomson of the University of Utah has been appointed to an assistant professorship at Tennessee Polytechnic Institute.

Assistant Professor R. N. Tompson of Florida State University is now with the Bell Telephone Company, New York City.

Mr. D. T. Walker, previously an instructor at the University of South Carolina, has accepted a graduate assistantship at the University of Georgia.

Mr. C. R. Wampole, recently head of the Department of Mathematics of Eastern Military Academy, Cold Spring Harbor, Long Island, New York, is teaching at Hauppauge Union Free School, Hauppauge, Long Island, New York.

Mr. G. T. Williams, formerly with the Brookhaven National Laboratory, Upton, New York, is now Chief of the Warheads Analysis Section, Aberdeen Proving Ground, Maryland.

Col. R. C. Yates, previously a professor at the United States Military Academy, has been appointed to a professorship at Virginia Polytechnic Institute.

Associate Professor J. D. Burk of the University of Toronto died on July 7, 1954.

Assistant Professor Mary E. Decherd of the University of Texas died in February, 1954. She was a charter member of the Association.

Professor Emeritus J. J. Hayes of the University of Utah died on February 27, 1954.

Mr. G. O. Peirce, instructor at Marquette University, died on August 14, 1954.

Dr. T. E. Raiford of the University of Michigan died on July 14, 1954.

Dr. R. W. Schmied, research mathematician of the Military Physics Research Laboratory of the University of Texas, died in January 1954.

Reverend J. P. Smith of Georgetown University died on March 5, 1954. He had been a member of the Association for thirty-four years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 88 persons have been elected to membership by the Board of Governors on applications duly certified.

P. L. ALGER, M.S.E.E. (Union) Consulting Engr., General Electric Co., Schenectady, N. Y.

MRS. MARY L. C. ANDREWS, M.S. (Michigan S. C.) Oakmont, Pa.

M. T. BATTLES, JR., Student, University of Notre Dame.

S. D. BECK, B.S. (Tufts) Grad. Asst., Iowa State College.

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R. M. MCLEOD, M.A. (Rice) Assistant, Rice Institute.

H. F. MCWILLIAMS, B.B.A. (Southern Methodist U.) General Motors Corporation, Arlington, Texas.

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A. C. MEWBORN, A.B. (North Carolina) Ensign, U.S.N.R.

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DONALD SCHMIDT, B.A. (Bethel) Grad. Asst., Iowa State College.
R. H. J. SCHMIDT, B.S. in M.E. (Washington) Instr., Seattle University.
S. J. SCOTT, M.A. (East Carolina T.C.) Grad. Student, University of North Carolina.
HAROLD SHULMAN, M.A. (Johns Hopkins) Mathematician, Institute of Mathematical Sciences, New York University.
E. C SPRENGLE, Student, Rutgers University.
R. G. STAPLES, III, Student, Rutgers University.
VIRGINIA R. SWANN, A.B. (East Carolina T.C.) Grad. Student, University of North Carolina.
MRS. EUNICE-GAIL TEICHMANN, M.A. (Northwestern) Teacher, Milwaukee Public Schools.
MARY V. TERHUNE, M.A. (Michigan) Mathematics Teacher, Proviso Township High School, Maywood, Ill.; Lecturer, Evening Division, Northwestern University.
J. C. THOMPSON, M.S. (North Dakota) Instr., Dickinson State Teachers College.
S. M. TUCKER, Student, Rutgers University.
H. S. VALK, M.S. (George Washington) Assistant in Physics, Washington University.
A. B. WALTERS, Electronic Engr., Cook Electric Company, Chicago, Ill.
J. E. WESTBERRY, M.S.(Atlanta), M.A. (Michigan) Asst. Professor, Texas Southern University.
EDITH F. WHITMER, Ed.D. (Missouri) Chairman, Division of Mathematics, Henderson State Teachers College.
A. G. J. WILFORD, B.A. (Queens U.) Meteorologist, Dept. of Transport, Toronto, Ontario, Canada.

THE MARCH MEETING OF THE KANSAS SECTION

The thirty-ninth annual meeting of the Kansas Section of the Mathematical Association of America was held at Baker University, Baldwin City, Kansas, March 27, 1954. Professor W. C. Foreman, Chairman of the Section, presided.

One hundred sixty-four persons attended the meeting including the following fifty members of the Association:

L. W. Akers, R. W. Babcock, Wealthy Babcock, Florence L. Black, G. L. Crumley, E. L. Dubowsky, Paul Eberhart, A. M. Feyerherm, W. C. Foreman, J. W. Forman, L. E. Fuller, W. H Garrett, F. C. German, Laura Z. Greene, J. D. Haggard, J. R. Hanna, Sabrina M. Hecht, A. J. Hoare, H. V. Huneke, Emma Hyde, H. E. Jordan, Helen F. Kriegsman, L. E. Laird, C. F. Lewis, J. M. Marr, Margaret E. Martinson, S. T. Parker, O. J. Peterson, P. S. Pretz, C. B. Read, L. M. Reagan, R. G. Sanger, W. R. Scott, A. J. Silverman, Sister Jeanette, F. B. Sloat, G. W. Smith, R. G. Smith, R. P. Smith, W. L. Stamey, E. C. Stopher, E. B. Stouffer, Ruth Kjersti Swanson, Robert H. Thompson, Wilmont Toalson, C. B. Tucker, Gilbert Ulmer, A. E. White, Ferna E. Wrestler, and P. M. Young.

The following officers were elected for the year 1954–55: Chairman, Professor E. C. Stopher, Fort Hays State College; Vice-Chairman, Professor W. R.

Scott, University of Kansas; Secretary, Professor Laura Z. Greene, Washburn University of Topeka.

In the business session a report of the Boulder Conference was given by Professor Paul Eberhart, Washburn University, and Reverend W. C. Doyle, Rockhurst College.

The following papers were presented at the morning and afternoon sessions:

1. *Archimedes and integration*, by Professor S. H. Gould, University of Kansas, introduced by the Chairman.

Newton and his contemporaries expressed great admiration of the virtuosity displayed by Archimedes in his various integrations and felt sure he must have had some unifying "method," lost in the intervening centuries. In 1906 a manuscript of Archimedes was discovered in Constantinople with an article entitled "A Mechanical Method for Geometrical Problems." The present talk illustrates, with models and an actual lever, how the volume of various solids, especially of the sphere, can be deduced by the law of the lever.

2. *Trends in applied mathematics*, by Dr. Martin Goland, Midwest Research Institute. (By invitation.)

Modern computer technology is causing significant change in both the scope and philosophy of applied mathematics. High-speed calculation permits practical numerical study of the solutions of differential and integral equations even when these are not known in closed, or readily calculable form. In addition to expanding the utility of the linear theories of engineering, physics, and other scientific disciplines, modern computers enable the scientist to cope with many important nonlinear problems.

"Computer mathematics" places emphasis on numerical methods. The generation of approximate solutions, and the analysis of their existence, convergence and accuracy are questions of immediate importance to modern analysts.

A second branch of mathematics which is finding increased importance in technical fields is that of statistics and probability. Quantum mechanics, biophysics, information theory, physics of materials, and aeronautical engineering are but examples of scientific branches which are finding probabilistic thinking of growing importance.

Finally, mention should be made of the increasing interest in topological analysis. Solutions to nonlinear equations are often clarified by topological argument. Also, the relatively new disciplines of linear and nonlinear programming can be expected to draw heavily on this branch of mathematics.

3. *Fifty years: then and now*, by Emeritus Professor W. H. Garrett, Baker University.

It was fifty years ago that the Kansas Association of Teachers of Mathematics was organized. In observance of the anniversary a special program followed the joint luncheon with the Kansas Section of the Mathematical Association of America. Professor Garrett, who was one of the founders of the K. A. T. M., introduced the other charter members present: Emeritus Professor Emma Hyde, of Kansas State College, and Emeritus Professor M. E. Rice of the University of Kansas. He then gave a brief history of the first year of the Association followed by a comparison of the progress in certain other activities during the past fifty years, such as transportation and athletics, with the progress in a liberal arts education.

4. *On consecutive integers*, by Mr. Stanley Gale, University of Kansas, introduced by the Chairman.

It has been a long standing conjecture that there is no solution in integers of the following

Diophantine equation:

$$x(x+1)\cdots(x+n-1)=y^k, \quad \text{where} \quad x>0, n>1, y<0 \quad \text{and} \quad k>1.$$

Many special cases have been contributed since the early 1800's. In particular, Pillai has proved the cases when $n \leqq 16$, $k \geqq (n+3)/2$. Oblath [*Journal Indian Math. Soc.*, (N.S.), 15, 1951, pp. 135–9, 1952] stated that there are no solutions for $n \leqq 16$, but did not prove the cases unproved by Pillai. The author fills this gap by treating the remaining cases individually.

5. *Mathematical analysis of music*, by Professor Herman Reichenbach, Sterling College, introduced by the Chairman.

A musical form is a function of time between silences. Since the climax is nearer the end and a relaxation in the middle, the total intensity curve follows half a wave of

$$f(t)=\sum_{k=1}^{4}\frac{(-1)^{k+1}}{k}\sin kwt \qquad \text{Frequency } f = 1/2t \text{ final. (The unit of } t \text{ is one meas.)}$$

This describes the changes in pitch (melody) as well as in harmony, rhythm, dynamics, technical difficulty. Thus the total form has the same structure as a single tone with its overtones. Coefficients vary according to composer and style.

Melody: Such a function holds good for Gregorian Psalmody or simple folk songs. More elaborate melodies superimpose part time waves of similar structure of reversed order or part time damped waves damping amplitude as well as wave length; or else they are built on two simultaneous waves, sounding like two melodic lines in one.

Rhythm: $\phi(t)$ is the pattern of the t-th measure, naming the beats that are not used, *e.g.*: .234 is a whole note. Since any odd measure is Arsis to the following even measure as thesis $\phi(2t) \geqq \phi(2t-1)$. The progressive increase in pattern and in use of $\frac{1}{8}$ and $\frac{1}{16}$ notes follows the total curve above.

Counterpoint: There are simple differential and other equations between $F(t)$ and $G(t)$ prescribing all the intricate rules of counterpoint, *e.g.*,

Forbidden Parallels: If $F(k)-G(k)=C=$ perfect consonance

$$F^1(k)\times G^1(k) \leqq \phi \text{ required.}$$

Complementary Rhythm recommended: $\phi(k)=-\psi(k)$.

Harmony: Cadence is harmonic oscillation: $T\ S\ T\ D\ T=-\sin wt$. The total form as cadence of cadences produces a curve similar to the total curve above.

Aesthetic gestalt requires both closeness and variety to be strong:

$$\sum R+\sum C \qquad \text{where } R = \text{relations}, \quad C = \text{contrasts}.$$

Organic forms require deviation from exactness: $t+\Delta t$ *etc.* keeping closely at the borderline of forbidden tracks.

6. *Some indeterminate forms*, by Professor S. T. Parker, Kansas State College.

Most texts dealing with indeterminate forms give an injudicious selection of problems for the so-called 0^0 and ∞^0 types. In nearly all the problems of these types in about thirty calculus books checked, the limits are unity. The speaker pointed out that this state of affairs is misleading, and suggested that teachers of the calculus make a supplementary assignment, which need include but a very few problems, for which the limits are different from unity. He outlined some procedures whereby any number of such problems could be constructed.

LAURA Z. GREENE, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirtieth annual meeting of the Nebraska Section of the Mathematical Association of America was held at Creighton University, Omaha, Nebraska, on April 24, 1954. Professor C. C. Camp, Chairman of the Section, presided at the sessions.

There were thirty-four persons present, including the following twenty-one members of the Association:

M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Camp, H. M. Cox, Morris Dansky, H. W. Doss, Jr., J. M. Earl, Edwin Halfar, L. K. Jackson, M. L. Keedy, W. G. Leavitt, E. J. Lowry, R. L. Moenter, T. A. Newton, C. R. Perisho, H. B. Ribeiro, Harry L. Rice, D. D. Rippe, Lulu L. Runge.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor L. K. Jackson, University of Nebraska; Vice-Chairman, Professor C. C. Camp, University of Nebraska; Secretary-Treasurer, Professor Edwin Halfar, University of Nebraska.

The following papers were presented:

1. *Mathematics behind the calculation of joint-annuities*, by Professor C. C. Camp, University of Nebraska.

2. *A curve of navigational interest*, by Professor O. C. Collins, University of Nebraska.

3. *The operations analysis function in the Strategic Air Command*, by Mr. T. W. Chapelle, Offutt Air Force Base, Omaha, Nebraska. (By invitation).

The Strategic Air Command Operations Analysis Office consists of a group of civilian scientists reporting directly to General LeMay, Commander SAC. The primary responsibility of the office is to provide scientific advice to General LeMay and his staff. As an integral part of the activities involved in fulfilling this responsibility, the office performs what has come to be known as Operations Research; that is, the application of scientific methods to the analysis of operational problems. This presentation outlined the organization and functions of the Strategic Air Command Operations Analysis Office and, in unclassified terms, briefly described some typical aspects of its work.

4. *Minimum variance estimates of space location of points*, by Dr. D. D. Rippe, Offutt Air Force Base, Omaha, Nebraska. (By invitation).

The rectangular coordinates of a point in space were estimated on the basis of independent and unbiased estimating points which are each subject to variability that is known. A criterion was set up such that the estimated point was an unbiased estimate of the true point with a minimum variability about the true point. The criterion was applied, as an example, to the adjusting on the basis of bombing experience of indirect aiming points which were initially considered accurate but subject to a computable variability.

5. *The analysis of performance versus reliability in a bounded system*, by Dr. W. A. Dwyer, Offutt Air Force Base, Omaha, Nebraska. (By invitation).

The potential rate of output of a certain system is dependent upon the reliability of the system and the quality of performance of the system, both of which are functions of several variables common to each. An optimization of the expected value of the end product as a function of the

common variables, whose ranges of variation are bounded, is desired. Desired also is a measure of the effect of changing the bounds. An approach to a specific problem was presented.

6. *Rings of words*, by Professor W. G. Leavitt, University of Nebraska.

The motive of an algebraist in constructing examples of algebraic structures is often analogous to that of the analyst in constructing the so-called "pathological functions." As an illustration, the ring of "words" was constructed, together with certain of its factor rings. A discussion was given of some of the interesting properties of these rings, and of modules over them.

7. *Reconstruction of Fermat's lost proof* (?), by Mr. W. H. Becker, Omaha, Nebraska.

In Dickson's *History*, V. II, appear general formulas for Pythagorean triangles (P-triangles) with hypotenuse, odd or even leg an mth power: Volpicelli p. 168, Vieta p. 226; Euler, Drach p. 768; Pocklington p. 772. Equating any pair of these, assuming P-triangles with two sides equal to two of the quantities, z^m, x^m, y^m, it is shown that no corresponding pair of z, x, y can be sides of a P-triangle. This contradicts Carmichael's problem 9, p. 103 (and its Case II corollary) *Diophantine Analysis*, thus proving the even power case of Fermat's Last Theorem, and (in Case I) that only one side of a P-triangle can be an mth power.

8. *On the teaching of mathematics in universities*, by Professor H. B. Ribeiro, University of Nebraska.

EDWIN HALFAR, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Texas Technological College, Lubbock, Texas, on April 23–24, 1954. Professor L. A. Colquitt, Chairman of the Section, presided at the sessions.

There were seventy persons in attendance, including the following forty-eight members of the Association:

J. C. Bradford, Ina M. Bramblett, H. E. Bray, J. E. Burnam, D. R. Clutterham, M. L. Coffman, L. A. Colquitt, Don Cude, J. R. Foote, Frances Freese, Gordon Fuller, Blanche B. Grover, W. T. Guy, Jr., J. O. Hassler, E. A. Hazlewood, E. R. Heineman, J. A. Hummel, B. W. Jones, E. C. Klipple, W. I. Layton, C. W. Long, H. A. Luther, Lida B. May, V. A. Miculka, P. D. Minton, R. A. Moreland, Jr., M. E. Mullings, C. A. Murray, T. K. Pan, Bob Parker, G. M. Petersen, C. J. Pipes, C. L. Riggs, Virginia B. Roberts, Marabeth Rollins, R. Q. Seale, C. R. Sherer, F. W. Sparks, D. W. Starr, W. W. Taylor, J. I. Tracey, F. E. Ulrich, R. S. Underwood, Patricia A. Ward, Margaret M. Welch, Mabel Williams, H. E. Woodward, C. B. Wright.

By invitation, Dr. S. M. Ulam delivered an address at the afternoon session. The title of his address was "Mathematical Games." At the banquet Professor B. W. Jones gave the principal address. The title of his address was "New Emphases in Mathematics."

At the business meeting the following officers were elected for the coming year: Chairman, Professor E. A. Hazlewood, Texas Technological College; Vice-Chairman, Professor M. E. Mullings, Abilene Christian College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *On the general solution of certain second order linear partial differential equations*, by Dr. M. L. Coffman, Agricultural and Mechanical College of Texas.

A method of obtaining the general solution of certain second order linear partial differential equations is presented. For example, the general solution of Laplace's equation in two dimensional polar coordinates is found.

2. *Certain pulsations and pressures in an ideal fluid*, by Professor J. R. Foote, University of Oklahoma.

Two potential functions are written, one for a source of variable strength to simulate a pulsating sphere in a fluid which moves uniformly past, and another to simulate a pulsating, expanding sphere in a fluid at rest. Using several frequencies, sinusoidal pulsations can be created in the fluid. Expressions are written for dynamic pressure, static pressure, and total head at various points upstream, downstream, and abeam of the sphere. The various pressure amplitudes at these points are computed and compared to assist in selection of the proper measuring instruments to detect pressure fluctuations in a moving stream.

3. *"Parallax" in mathematics*, by Professor R. S. Underwood, Texas Technological College.

In "extended analytic geometry," which deals with equations in n variables, the most often encountered, or "normal," loci of equations may be considered to be the silhouettes on a plane of "solid" figures. If then two such plane loci do not overlap, the corresponding solids do not touch, and the equations have no common real solutions. If the loci do overlap the equations may still be inconsistent, as if one solid is merely in front of the other. In this case it is often possible, by shifting axes, to get in effect another point of view from which the plane loci are separated, thus demonstrating that the equations are inconsistent.

4. *Extensors of non-integral order*, by Professor W. T. Guy, Jr., The University of Texas.

Extensors of integral order are extended by means of a fractional differentiation process to those of non-integral order.

5. *Some remarks concerning commutative reflexive Banach algebras*, by Professor E. R. Keown, Agricultural and Mechanical College of Texas, introduced by the Secretary.

A commutative, reflexive Banach algebra is defined to be a Banach algebra satisfying the following conditions: (a) the underlying space of the algebra is a reflexive Banach space; (b) the conjugate space is required to form a Banach algebra in its natural norm; (c) the multiplication and the metric are connected by requiring an ideal in either algebra to have as its annihilator in the other space an ideal and the norm to have a certain additive property. These conditions imply that the algebra in question is essentially an l_p-space with the multiplication corresponding to co-ordinate multiplication.

6. *Mathematical games*, by Dr. S. M. Ulam, Los Alamos, New Mexico. (By invitation).

7. *Regions of flatness of analytic functions*, by Mr. Guy Johnson, Jr., The Rice Institute.

Let S denote an infinite set of simply connected domains D_α of the z-plane contained in a domain in which a function $f(z)$ is holomorphic and non-zero. Let $\omega_\alpha(w)$ denote the function which

maps the circle $|w| < 1$ conformally on D_α such that $\omega_\alpha(0) = \alpha$ and $\omega_\alpha'(0) > 0$. If the family of functions $F(w) = f[\omega_\alpha(w)]$ is normal in $|w| < 1$ then the set S will be called regions of flatness for $f(z)$. The concept of regions of flatness was introduced by J. M. Whitaker (*Proc. Edin. Math. Soc.*, 1930–31) as regions in which the maximum and minimum modulus are in some sense of the same order. F. E. Ulrich and S. Mandelbrojt (*Duke Math. Journal*, June 1951) gave a precise definition analogous to that above in the special case where the domains D_α are circles. Conditions are stated under which a set S are regions of flatness for $f'(z)$ whenever they are for $f(z)$. If the set of regions is denumerable and $f(z)$ tends to infinity in the regions, then under given conditions the fact that they are regions of flatness for $f(z)$ implies that they are regions of flatness for its successive derivatives.

8. *Block designs*, by Professor B. W. Jones, University of Colorado.

Balanced incomplete block designs are defined and their origins and applications discussed. Methods are sketched by which it has been shown that for certain values of the parameters no designs exist.

9. *Some aspects of Dirichlet series*, by Mr. R. W. Randall, Jr., The Rice Institute, introduced by the Secretary.

10. *High-speed digital computation*, by Dr. D. R. Clutterham, Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

The main divisions of a high-speed digital computer are described: arithmetic unit, memory, control, and input-output. The binary number system and the computer's representation of numbers are outlined together with the basic operations which a general purpose computer should be able to perform. An example in preparing a problem is given. The use of Boolean algebra in logical design of computers is indicated.

11. *Requirements in mathematics by the states for graduation from high school*, by Professor W. I. Layton, Stephen F. Austin State College.

This paper attempts to analyze the kind and amount of mathematics required for graduation from high school by forty-seven states and the District of Columbia. The data are based upon material made available by state departments of education.

Means were computed for the last four years of the various high school programs for certain subjects required of all students for graduation. In these four year programs, state requirements for graduation provide the following means in units: mathematics .6, English 2.8, social studies 1.7, and science .7 of a unit. Thus, much emphasis is being given to English and social studies while mathematics ranks at the bottom of the list of subjects investigated. Also in these four year programs, 57 per cent of the states require some mathematical training for graduation, 83 per cent require English, 91 per cent social studies, and 57 per cent require credit in science.

A study of elective units allowed during the last four years of high school gives a mean of 9.2 units. With approximately 58 per cent of these four years available for electives, it seems that more mathematics could be required by the states for graduation.

12. *An analysis of the functional method of teaching geometric topics*, by Sister Claude Marie, Incarnate Word College, introduced by the Secretary.

The underlying aim of functional teaching of geometry is to enrich the course by providing means of thinking in terms of relationships which will increase the ability of the student to recognize and comprehend related situations. The study revealed important considerations of the following aspects of a functional program: aims of instruction; principles governing selection of subject matter; treatment of subject matter; and effective instructional devices. A list of criteria based

upon the above findings was drawn up which could serve as a guide for analyzing functional situations in textbooks and as a guide and stimulus for teaching the concept of "functionality."

13. *Common objectives in the teaching of high school and college mathematics*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler listed five common goals of high school and college teachers of mathematics, namely: to utilize, stimulate and develop to the fullest possible extent the students' powers of imagination, discrimination, interpretation and generalization and to secure the greatest possible degree of proficiency in the manipulation of mathematical symbols. The meaning of each goal was illustrated by teaching situations in both high school and college mathematics and numerous suggestions made on methods of teaching necessary to reach the objectives.

14. *Recent trends toward coordination of higher education in Texas*, by Professor E. A. Hazlewood, Texas Technological College.

C. R. SHERER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 30, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

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SOME MATHEMATICAL ASPECTS OF SWITCHING

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1. Introduction. In electronic digital computers, telephone switching systems, control systems for automatic factories, and other systems involving the communication or processing of data, one finds many examples of electric circuits which employ what are known as *two-state* or *bi-stable devices.* The simplest example of such a device is a switch or *contact.* When a contact is operated with the aid of an electromagnet, the combination is a called a *relay.* A switch or relay is called a *bilateral circuit element* since it permits the passage of current in either direction when it is closed.

Some commonly used symbols for contacts are shown in Figure 1. In this paper we shall often use the first type of symbol, but usually without drawing in the electromagnet which operates the springs of the contact. (Note that the contacts illustrated here have two *springs*—the one attracted by the electromagnet presses against the other.) A normally open contact is called a "make" or "front" contact whereas a normally closed one is called a "break" or "back" contact.

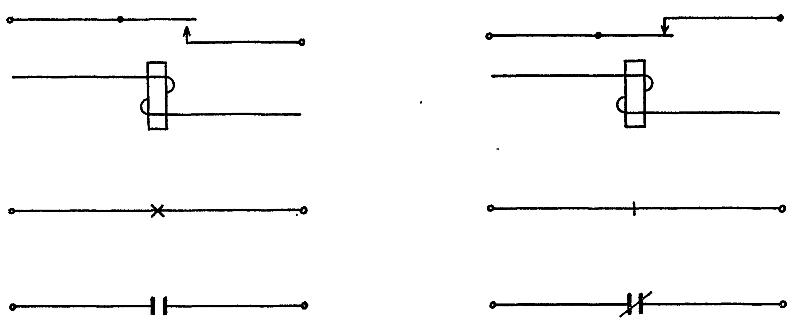


FIG. 1

There are various other two-state devices in use and in the process of development. These include rectifying diodes, magnetic cores, transistors, various types of electron tubes, and others. Magnetic drums and magnetic tape may be considered to be assemblages of two-state devices. The nature of the two "states" varies from one device to another, and includes conducting *vs.* non-conducting, closed *vs.* open, charged *vs.* discharged, magnetized *vs.* non-magnetized, high-potential *vs.* low-potential, and other conditions.

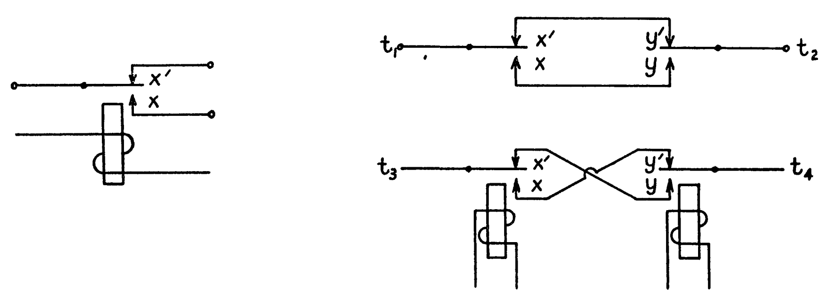
The methods and results of Boolean algebra and related subjects have been found useful in discussing circuits employing two-state devices. In this paper we use contact networks to illustrate how this is done. Naturally, there are many aspects of the design, development, and use of two-state devices which are the special responsibility of the electrical engineer. However, there are other aspects of their use which are essentially abstract in nature, and are therefore

legitimately of interest to the mathematician. The latter provide a neat and novel example of applied mathematics which it is the purpose of this paper to present to the reader in an informal way. We shall also indicate some problems which are worthy of attention and give selected references for a further study of the subject. As far as theoretical results are concerned, the paper presents nothing new—most of what is presented here may also be found in Shannon [3], [4], and Keister, Ritchie, Washburn [2].

2. A mathematical model of a two-terminal relay switching circuit. We introduce two symbols as the first step in constructing the desired model. With an open contact or path in a circuit we associate the symbol "0" and with a closed contact or path we associate the symbol "1." Although we call these symbols "zero" and "one" respectively, they are not the zero and one of ordinary arithmetic, as will presently be clear.

When the condition of a contact is variable in a problem, we call it a *circuit variable* and denote it by a literal symbol such as $a, b, x, y, \cdots$. This symbol takes on the value 0 when the contact is open, the value 1 when it is closed. With each symbol x, we associate a symbol x' to denote a contact which is open when x is closed and closed when x is open.

When two or more contacts always open and close simultaneously, we denote them by the same symbol throughout the circuit. For example, we ordinarily assume that two "make" (or two "break") contacts operated by the same electromagnet open and close simultaneously and hence we denote them by the same symbol. If a make contact is denoted by x, then a break contact operated by the same electromagnet is denoted by x'. Frequently, in order to save one spring, a make and a break are combined in one device called a *transfer*. The points just mentioned are illustrated in Figure 2A.



Transfer Contact
A

B

FIG. 2

Employing the notation above defined, the reader may verify that in Figure 2B, there is a path from t_1 to t_2 if and only if $x'=y'=1$ or $x=y=1$, and a path from t_3 to t_4 if and only if $x=0$, $y=1$, or $x=1$, $y=0$.

Figure 2A reveals one respect in which our mathematical model is a fiction. When the coil of such a transfer contact is energized, there is a brief instant when neither contact is closed. Thus the assumption that we may call one contact x and the other x' is not strictly justified. For many purposes, this matter is not serious. When it is, one must use design techniques which will take care of any resulting difficulties. These techniques include the use of fast or slow acting relays, make-before-break contacts, *etc.*

Another assumption which we make is that when the coil of a relay is energized, its contacts are instantaneously operated. This assumption too at times causes difficulties which may be taken care of by treating the coil as one two-state device (energized or not energized) and a contact on it as another (operated or not operated). In this paper, we shall restrict our attention to circuits such that these difficulties may be ignored.

Next we introduce two operations on the symbols 0, 1, x, $\cdots$. The operation of "addition," denoted by "$+$," corresponds to connecting contacts or combinations thereof in parallel. (In the figure of Postulate 1a, contacts a and b are shown connected in parallel.) The operation of "multiplication," denoted by "$\cdot$" or juxtaposition, corresponds to connecting contacts or combinations thereof in series. (In the figure of Postulate 1b, contacts a and b are shown connected in series.) The postulates governing the use of the operations $+$, $\cdot$, $'$, are summarized in Table 1, in which two circuits are indicated as *equivalent* ($\sim$) if and only if they are open and closed under the same conditions of the circuit variables. The circuit diagrams are intended to indicate how the postulates are suggested by physical considerations. The notation is similar to that used by Shannon [**3**], but the meanings of $+$ and $\cdot$ and also of 0 and 1 are interchanged. Our notation conforms to the widest current practice and is the most convenient when it comes to adapting these ideas to multi-terminal and electronic circuits [**15**], [**10**].

From this table of postulates, it is clear that the operation of addition has the meaning "or." In fact, the circuits of (1a) provide a path if a is closed or b is closed (or both are closed). On the other hand, the operation of multiplication has the meaning "and." Thus the circuits of (1b) provide a path if and only if both *a and b* are closed.

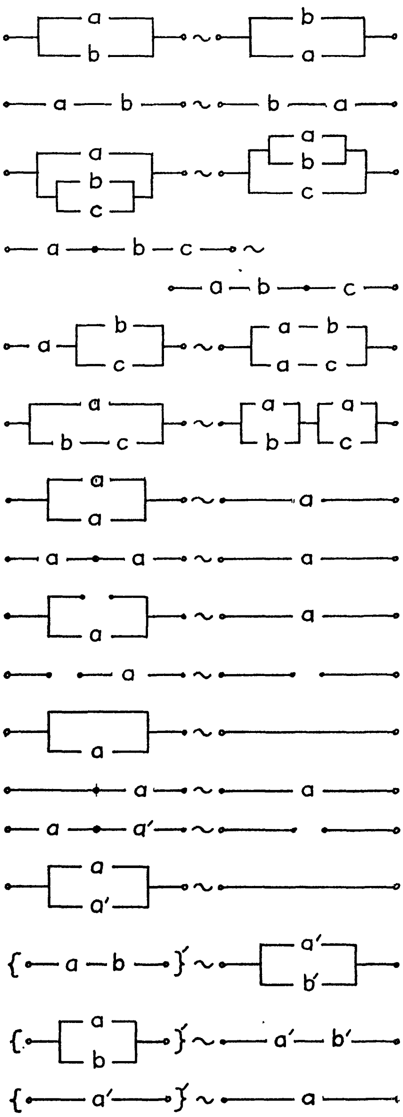
Another important fact to note is that these laws, except for (9), appear in *dual pairs*, each member of a pair being obtainable from the other by a simple interchange of the operations $+$ and $\cdot$. In these simple cases, this interchange corresponds to the interchange of series and parallel connections.

Next we observe that there is a large class of two-terminal switching circuits with which we can associate a function f whose characteristic property is that it takes on the value 1 for those combinations of values of the circuit variables which correspond to the circuit's being closed, but is 0 otherwise. This function is called the *switching function* of the circuit.

In certain simple cases the switching function has in effect already been defined: this is shown in Table 2. Now consider any two-terminal switching

TABLE 1

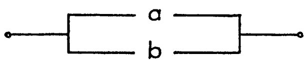
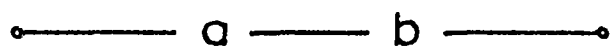
THE POSTULATES OF SWITCHING ALGEBRA



	Circuit Diagram	Symbolic Equivalent	Verbal Statement
(1a)		$a+b=b+a$	The commutative laws.
(1b)		$ab=ba$	
(2a)		$a+(b+c)=(a+b)+c$	The associative laws.
(2b)		$a(bc)=(ab)c$	
(3a)		$a(b+c)=ab+ac$	The distributive laws.
(3b)		$a+bc=(a+b)(a+c)$	
(4a)		$a+a=a$	The idempotent laws.
(4b)		$a \cdot a=a$	
(5a)		$0+a=a$	The laws of operation with 0.
(5b)		$0 \cdot a=0$	
(6a)		$1+a=1$	The laws of operation with 1.
(6b)		$1 \cdot a=a$	
(7a)		$a \cdot a'=0$	The laws of complementarity.
(7b)		$a+a'=1$	
(8a)		$(ab)'=a'+b'$	The dualization laws.
(8b)		$(a+b)'=a'b'$	
(9)		$(a')'=a$	The law of involution.

circuit built up solely of series-parallel connected contacts. (The circuits shown in Table 2 are all series-parallel. So is any circuit obtainable by substituting a known series-parallel circuit for any contact of a known series-parallel circuit,

TABLE 2

Circuit	Switching Function
	0
	1
	a
	$a+b$
	ab

or by a series of such substitutions. The symbols used to name the contacts are of course arbitrary. Non-series-parallel circuits are called "bridges.") With each contact we associate a symbol as before, designating identically behaving contacts by the same symbol and oppositely behaving contacts by a symbol and its

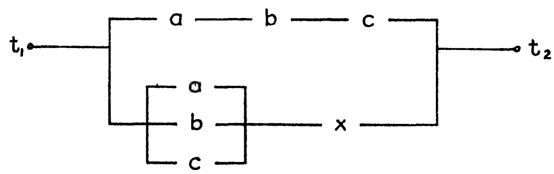


FIG. 3

prime, but using distinct symbols otherwise. Then, using the operations + and •, we can associate with the circuit a polynomial in the circuit variables which expresses formally the various series and parallel connections. For example, with the circuit of Figure 3, we can associate the switching function

$$= (a + b + c)\,(abc + x) = abc + (a + b + c)x.$$

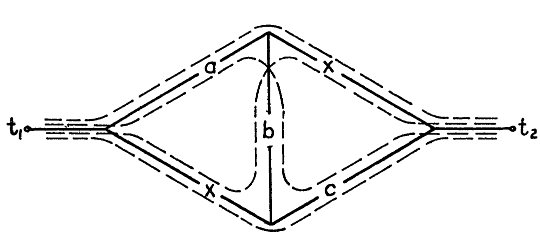


FIG. 4

We can also associate such a function with a bridge-type circuit by first determining, in effect, an equivalent series-parallel circuit. For example, the bridge circuit of Figure 4 provides the possible paths ax, $xbx = bx$, cx, and abc

between the terminals t_1 and t_2. These are just the same paths as are provided by the first-given series-parallel circuit, so that the function f may also be considered as the switching function of this bridge. It should be observed that the above bridge realizes the same switching function as the original series-parallel network, but with two fewer contacts.

With this extension of our notation to bridge circuits, the meanings of "$\cdot$" and "$+$" are no longer restricted to simple series and parallel connections, respectively. Indeed, in the function f, these operations are now to be given the meanings "and" and "or" mentioned earlier. In the example just given, there is a path through the circuit if

$$a \textit{ and } b \textit{ and } c \textit{ are } 1$$

or if

$$a \textit{ or } b \textit{ or } c \textit{ is } 1 \textit{ and } x \textit{ is } 1.$$

Thus the switching function now describes the *electrical behavior* but not necessarily the *geometry* of the network.

This procedure may be applied to any two-terminal series-parallel or bridge network which may be represented diagrammatically as in Figure 5. Here the box is assumed to contain all the contacts of the circuit, which we denote by $x_1, \cdots, x_n$ and their primes. These contacts are assumed to be operated by the corresponding electromagnets $X_1, X_2, \cdots, X_n$, which are controlled from

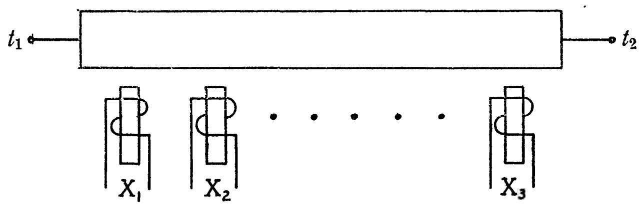


FIG. 5

outside the box. In this case the open or closed condition of the path between t_1 and t_2 depends only on the combination of values assumed by the circuit variables $x_1, \cdots, x_n$, in terms of which the switching function is expressed. Such a circuit is called a *combinational* two-terminal circuit.

It is important to note that the switching functions in n variables $x_1, x_2, \cdots, x_n$, *satisfy the laws* (1)–(9). This follows from the fact that they assume only the values 0 or 1, for which these laws have been postulated to hold.

We have one last definition to make, and then our model is complete. Let f_1 and f_2 be functions associated in the above manner with two switching circuits s_1 and s_2. If s_1 is never closed unless s_2 is also closed, then whenever f_1 is 1, f_2 is 1

also (although when $f_1=0$, we may have $f_2=1$). When f_1 and f_2 are thus related, we write $f_1 \leqq f_2$ or $f_2 \geqq f_1$, reading the symbols in the usual manner. Then for any switching functions, f, g, h we have the following rules:

(10)	$0 \leqq f \leqq 1$	The universal bounds property.
(11)	$f \leqq f$	The reflexive property.
(12)	If $f \leqq g$ and $g \leqq f$, then $f = g$	The anti-symmetric property.
(13)	If $f \leqq g$ and $g \leqq h$, then $f \leqq h$	The transitive property.
(14a)	$f \leqq g$ if and only if $fg = f$	The consistency principle.
(14b)	$f \leqq g$ if and only if $f + g = g$	

The fact that the switching functions in n variables satisfy the rules (1)–(14) demonstrates that these functions as a class constitute what is known as a *Boolean algebra*. Actually, a much shorter list of rules would suffice to establish this fact. However, the list given here is extensive enough to include the most fundamental properties. It is precisely the list given in Birkhoff and MacLane, *A Survey of Modern Algebra*, Chapter XI, except for some changes in notation which conform to the most common engineering usage.

In discussions of Boolean algebra, the functions of n variables and their primes built up with the aid of the operations $+$, $\cdot$, and $'$ are called *Boolean functions*. We shall continue to call them "switching functions" here.

3. Some useful identities. It is possible to establish an unlimited number of formal identities in Boolean algebra, starting with the laws (1) to (9). These may be obtained by algebraic methods or by "complete induction," that is, by checking the correctness of the identity for all possible combinations of values of the variables. Often such identities are suggested by network considerations.

We illustrate with two identities which are extremely useful in simplifying switching functions. The first of these is

$$x + xy = x. \tag{15a}$$

This is proved with the aid of (6b), (3a), and (6a) as follows:

$$x + xy = x\cdot 1 + xy = x(1 + y) = x\cdot 1 = x.$$

Alternatively, we have the readily computed table of values:

x	y	$x + xy$
0	0	0
0	1	0
1	0	1
1	1	1

Since the first and third columns are the same for all combinations of the variables, we conclude that $x+xy=x$. Finally, the circuits in Figure 6 are both closed if and only if x is 1. Hence $x+xy=x$.

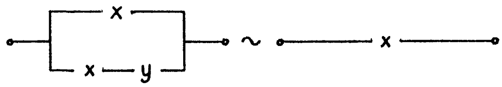


FIG. 6

The dual of (15a) is

(15b) $$x(x + y) = x.$$

Another extremely useful identity of this type is

(16a) $$x + x'y = x + y.$$

Here we have, by (3b) and (7b),

$$x + x'y = (x + x')(x + y) = 1\cdot(x + y) = x + y.$$

Circuitwise, this also makes sense, as Figure 7 indicates. In fact, if $x=1$, each circuit is closed, regardless of the value of y. If $x=0$, then $x'=1$, so that each

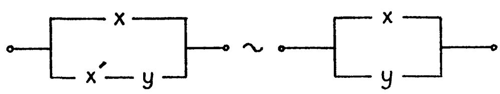


FIG. 7

circuit is then closed only when $y=1$. Thus the circuits are closed under the same conditions and hence are equivalent.

The dual of (16a) is

(16b) $$x(x' + y) = xy.$$

Other useful identities are

(17a) $$(x + y)(x' + z)(y + z) = (x + y)(x' + z)$$

(17b) $$xz + x'y + yz = xz + x'y$$

(18) $$(x + y)(x' + z) = xz + x'y.$$

The reader may enjoy deriving these.

To see how these identities are used, consider the simplification of the series-parallel circuit whose switching function is

$$f = abc + ab'c + a'b'c.$$

We have

$$\begin{aligned} f &= c(ab + ab' + a'b') \\ &= c[a(b + b') + a'b'] \\ &= c(a + a'b') \\ &= c(a + b'). \end{aligned}$$

The circuits are given in Figure 8. The simplification saves 6 contacts. An important thing to notice is that circuitwise, $c(a+b')$ is simpler than $ca+cb'$

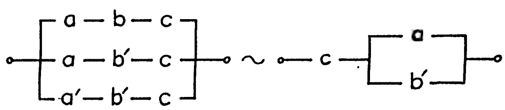


FIG. 8

since one less contact is indicated. In the case of a complicated function, it is not always easy to determine the factorization which is the simplest from the point of view of "hardware." (See, however, [16].)

4. The canonical expansions. Of particular interest among the switching functions of n variables are the products containing all n of the variables as factors, either primed or not (but not both primed and unprimed in any one case, of course). When $n=1$, we consider x_1 and x_1' to be these "products." When $n=2$, they are $x_1'x_2'$, $x_1'x_2$, x_1x_2', and x_1x_2. In general, since each of the variables is chosen in primed or unprimed form, the number of such products is 2^n. We call these the "fundamental products" of the n variables. The important property of a fundamental product is that it takes on the value 1 for exactly one set of values of x_1 to x_n, namely that combination which makes each factor of the product equal to 1.

If we define $x_j^0 = x_j'$ and $x_j^1 = x_j$, then a fundamental product may be represented in the form $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where each α is either 0 or 1. It is convenient to interpret the sequences of superscripts $\alpha_1\ \alpha_2 \cdots \alpha_n$ as integers in binary notation. Then we use the corresponding decimal integers to number the fundamental products as follows:

$$p_i = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where

$$(i)_{\text{decimal}} = (\alpha_1\alpha_2 \cdots \alpha_n)_{\text{binary}}.$$

We may employ the same scheme to number the *fundamental sums*:

$$s_i = x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n}.$$

The characteristic property of such a sum is that it vanishes for exactly one combination of values of the x's, namely the one which makes each term of the sum equal 0.

The points made thus far are illustrated for $n=3$ in Table 3.

TABLE 3

i	x_1	x_2	x_3	Non-Vanishing Product, p_i	Vanishing Sum, s_{7-i}
0	0	0	0	$x_1' x_2' x_3'$	$x_1 + x_2 + x_3$
1	0	0	1	$x_1' x_2' x_3$	$x_1 + x_2 + x_3'$
2	0	1	0	$x_1' x_2 x_3'$	$x_1 + x_2' + x_3$
3	0	1	1	$x_1' x_2 x_3$	$x_1 + x_2' + x_3'$
4	1	0	0	$x_1 x_2' x_3'$	$x_1' + x_2 + x_3$
5	1	0	1	$x_1 x_2' x_3$	$x_1' + x_2 + x_3'$
6	1	1	0	$x_1 x_2 x_3'$	$x_1' + x_2' + x_3$
7	1	1	1	$x_1 x_2 x_3$	$x_1' + x_2' + x_3'$

It is a simple matter to express a given switching function as a sum of fundamental products. For example, suppose $n=3$ and

$$f = (x_1' x_2)'(x_1 + x_3).$$

First we express f as a sum of products, not necessarily fundamental:

$$f = (x_1 + x_2')(x_1 + x_3) = x_1 + x_2' x_3.$$

Then by (7b) we have

$$f = x_1(x_2' + x_2)(x_3' + x_3) + (x_1 + x_1')x_2' x_3.$$

Now expanding and removing duplicate terms by (4a), we obtain

$$f = x_1x_2' x_3' + x_1x_2' x_3 + x_1x_2x_3' + x_1x_2x_3 + x_1' x_2' x_3,$$

which contains only distinct fundamental products. The method is obviously perfectly general. Once we have expanded a switching function into such a sum of fundamental products, we can read off by inspection of the terms present the combinations of values of the n variables for which the function is 1.

Conversely, suppose we know all the combinations of values of the n x's for which a switching function is 1. Using the fact that a fundamental product takes on the value 1 for exactly one such combination one can easily prove the following theorem ([1], p. 344):

Every switching function f of n variables may be expressed uniquely as a sum of fundamental products of these n variables:

$$f = \sum_{i=0}^{2^n-1} f_i p_i$$

where $f_i=0$ if $f=0$ for the combination that makes $p_i=1$, but $f_i=1$ if $f=1$ for this combination.

In other words, f may be written as the sum of the fundamental products corresponding to those combinations of values of the x_j's for which $f=1$. This sum is called the *canonical expansion of f.*

From this result, since each of the 2^n fundamental products has a coefficient 0 or 1 in the canonical expansion, we may conclude as a corollary:

There are 2^{2^n} *switching functions of n variables.*

The canonical expansion is useful for translating verbally stated conditions into algebraic form. Suppose, for example, we wish to design a two-terminal circuit using contacts on three relays and satisfying the following requirements:

"The network is to be closed whenever the relay x is released unless y or z (but not both) is operated, in which case it is open. It is also to be closed when x is operated unless y and z are both operated or both released, in which case it is open."

These conditions are summarized in column "f", Table 4. (The columns "g" and "h" are for later illustrations.) We have, from the table, with the aid of the above theorem,

$$f = x'y'z' + x'yz + xy'z + xyz'.$$

That is, in this case, $f_0=f_3=f_5=f_6=1$ and $f_1=f_2=f_4=f_7=0$. This is an example of a *symmetric function.* It is unchanged by any permutation of x, y, z. (The primes are not permuted along with the variables.) The series-parallel realization of this function requires 12 contacts unless some simplification is effected. If we write

TABLE 4

i	x	y	z	f	g	h
0	0	0	0	1	0	0
1	0	0	1	0	1	1
2	0	1	0	0	0	0
3	0	1	1	1	1	1
4	1	0	0	0	1	d
5	1	0	1	1	1	1
6	1	1	0	1	1	d
7	1	1	1	0	0	d

$$f = x'(y'z' + yz) + x(y'z + yz'),$$

we can obtain a series-parallel realization with only 10 contacts, as shown in Figure 9A. However, the use of a bridge circuit allows us to realize the function with only 8 contacts. (Figure 9B).

Unfortunately, there seems to be no way of looking at the switching function itself and seeing that a bridge circuit is most economical of contacts.

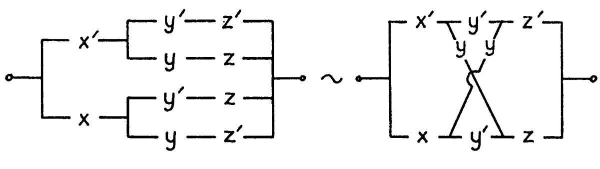


FIG. 9

In the actual construction of this circuit, transfer contacts would be employed in order to save springs. (Figure 10.)

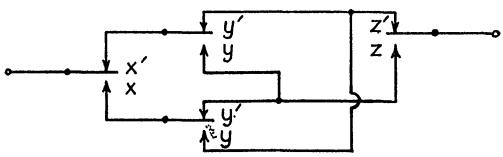


FIG. 10

Symmetric circuits, of which this is an example, are discussed in detail by Shannon [3] and in the book by Keister, Ritchie, and Washburn [2].

It may be shown that in addition to the canonical expansion, a switching function has a *dual canonical expansion*:

Every switching function g of n variables may be expressed uniquely as a product of fundamental sums:

$$g = \prod_{i=0}^{2^n-1} (g_i + s_i),$$

where $g_i = 0$ *if* $g = 0$ *for the combination of values which makes* $s_i = 0$, *but* $g_i = 1$ *if* $g = 1$ *for this combination.*

That is, g may be written as the product of those vanishing sums corresponding to the combinations of values of the x's for which $g = 0$ since when $g_i = 1$, $g_i + s_i = 1$ also.

The dual canonical expansion is sometimes more convenient than the canonical expansion for expressing a function in algebraic form. For example, from Table 4, column "g", and the preceding theorem we have

$$g = (x_1 + x_2 + x_3)(x_1 + x_2' + x_3)(x_1' + x_2' + x_3'),$$

which reduces to

$$g = x_1(x_2' + x_3') + x_1' x_3$$

with the aid of (3b), (7a), and (17b).

Finally, consider Table 4, column "h". The entries "d" in this column denote the fact that we don't care whether 0's or 1's appear in those positions. (In practice this situation arises when it follows from the conditions the circuit is to be designed to satisfy that the combinations of values of the x's labelled "d" do not occur in the use of the circuit in question.) We should therefore choose values for the d's in such a way as to render the function h as simple as possible. In this case, if we make 0's of the first two d's and 1 of the last, the h-column coincides with the x_3-column, so that

$$h = x_3.$$

Thus the circuit requires the use of only one contact.

From the preceding examples, it is clear that one method of synthesizing a two-terminal relay contact network is this:

(a) Determine the requirements the circuit is to satisfy.

(b) Express the requirements algebraically in the form of a switching function.

(c) Simplify the switching function as much as possible, using relations (1)–(18) as they apply, and any "don't-care" conditions which may exist.

(d) Draw the circuit and see what further simplifications can be made.

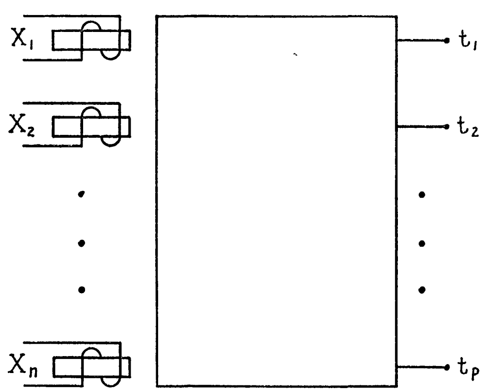


FIG. 11

5. Combinational circuits. We have been dealing up to this point exclusively with two-terminal circuits. Frequently, however, more terminals are required. We might, for example, need to design a circuit which establishes prescribed connections between terminals $t_1, t_2, \cdots, t_p$ for specified combinations of values of the variables $x_1, x_2, \cdots, x_n$, the corresponding contacts being operated respectively by coils $X_1, X_2, \cdots, X_n$, whose control is independent of the circuit in question. A circuit of this kind may be represented as in Figure 11. As before,

the box is presumed to contain all the contacts of the circuit, these being operated solely by the electromagnets $X_1, X_2, \cdots, X_n$, which are controlled from outside the box. Such a circuit is again called a *combinational circuit.*

The variables $x_1, x_2, \cdots, x_n$ are called the *inputs* of the circuit. Any combination of values of these variables is called an *input combination.* The set of functions f_{ij} of $x_1, x_2, \cdots, x_n$, which are 1 for exactly those states of the input such that t_i and t_j are connected, may be called the *outputs* of the circuit. (We define $f_{ii}=1$ for all i since any terminal may be regarded as always connected to itself.) Any particular set of values of these functions will be called an *output combination.* Thus a combinational circuit is one in which the output combination depends only on the input combination. All the two-terminal circuits previously discussed are special cases of this more general class of circuits.

In many cases, a multi-terminal (or *multiple-output*) combinational circuit may be designed first as a collection of two-terminal networks, after which one attempts to combine these networks so as to have them share contacts so far as possible. Details of the available techniques are given in [**2**], Chapter 6. Such a multi-terminal circuit may also be studied with the aid of the corresponding "switching matrix" $[f_{ij}]$. (See [**15**].)

The problems to be stated below apply in large measure to multi-terminal combinational circuits as well as to the two-terminal kind.

6. Problems. Certain problems arise naturally out of the preceding discussion. The most obvious ones relate to the issue of economical design:

(a) Is there a general way of designing a contact network satisfying given requirements and employing a minimum number of contacts? In the case of a two-terminal combinational network, this is equivalent to asking whether there is a corresponding way of defining and computing a "simplest" form for a switching function.

(b) Is there a general way of reducing a switching function to a form corresponding to the *minimum series-parallel* realization?

(c) Is there a general way of designing a minimum circuit when there are certain input combinations which never occur in the use of the circuit?

(d) Is there a general way of determining whether or not a given two-terminal circuit constitutes the minimum-contact realization of the corresponding switching function?

(e) Is there a general way of determining whether the minimum-contact realization of a given switching function is of series-parallel or of bridge type?

One could state other such mathematical problems, but these serve to illustrate the very practical type of question which may be asked. Not only is a minimal or near-minimal circuit neater and apt to be more reliable in operation, but also, when the same circuit is duplicated hundreds of times in an installation, the elimination of a few contacts from it may result in substantial savings.

There is thus an economic motivation as well as an intellectual one for solving these minimization problems.

A significant fact, from a mathematician's point of view, is that attempts to solve these problems have revealed many interesting algebraic and geometric properties of Boolean functions. In the course of these efforts, a wide variety of mathematical tools, such as group theory [**5**], lattice theory [**6**], and a geometrical approach [**7**], [**8**], [**9**], have been used. Some of the basic counting problems connected with this theory have been solved, by implication, in [**14**]. It seems clear that any further progress toward solution of the above listed problems will require the use of even more of the points of view and techniques of modern abstract mathematics.

There are also problems of an entirely different nature involved in the theory of switching. Certain of these arise in the design of large switching systems, involving perhaps hundreds of thousands of contacts. The complexities encountered in the application of the methods discussed in this paper increase at a rate which is at least of the order of 2^{2^n}, where n is the number of circuit variables, so that any hope of rigorous minimization rapidly disappears. In fact, for large systems, the real question is "*What types* of basic switching circuits should be employed, and *how* should these be employed, in order to provide the most economical systems possible to perform the given functions?" This is a problem which is, in large measure, still unsolved, and which will require keen mathematical thinking for its solution. Long since a basic problem in the construction of telephone switching systems, the problem is rapidly becoming important in other connections as well. The digital computer and the automatic factory provide striking examples. An important point is that conclusions valid for systems based on relay circuitry may be invalid for systems based on electronic circuitry, and *vice versa*, because of fundamental differences in the components employed.

7. Conclusion. Since this has been only a very brief discussion of relay switching theory, we have not treated the use of Boolean algebra in the design and analysis of circuits employing electronic components. Nor have we treated the subject of sequential circuits, namely those in which the output combination depends not alone on the input combination, but also on the past history of the circuit. Discussions of these matters will be found in references [**2**], [**10**], [**11**], [**12**], [**13**] listed below.

The fact that these topics were not treated here should not be taken to reflect on their importance. Indeed, with a large variety of electronic switching systems already in use or under development, it is apparent that such systems are destined to supersede relay systems in importance. Again, in both computers and in telephone systems, the output combination typically depends on the past history of the circuit as well as on the input combination. Thus sequential circuits are the kind that are ultimately required. Their design is often extremely

difficult, being complicated by the necessity of incorporating in them certain "memory elements" able to take account of the past history of the circuit. Once this task is accomplished, however, the problem reduces to the design of suitable combinational circuits, which are thus seen to be of fundamental importance.

We hope that the introduction to the basic concepts of combinational circuits which has been given here has been sufficiently clear and inviting to encourage the reader to investigate the further literature on his own, for switching theory—or the theory of digital control processes, as it is sometimes called—is steadily growing in importance and has need of able researchers. Indeed, the fact that abstract problems exist at all suggests that it may be possible to create an inclusive, abstract theory of digital control processes largely transcending the particular technologies used to realize the circuits, and largely unifying the many scattered results now in existence. In this possibility lies a real challenge to the mathematician.

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A GENERALIZATION OF JENSEN'S THEOREM ON THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL

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The object of this note is to present a proof of the following theorem concerning the position of points in the plane of the complex variable:

THEOREM 1. *Let $p(z)$ be a real polynomial not identically constant all of whose real zeros lie in the interval $\alpha \leqq x \leqq \beta$ of the axis of reals, and let γ be a real point not in the interval $\alpha < x < \beta$. Let each conjugate pair z_k, $\bar{z}_k$ of non-real zeros of $p(z)$ have real part in the open interval $\alpha < x < \beta$, and let $\Gamma(z_k)$ denote the circle through z_k and $\bar{z}_k$ tangent to the line γz_k at z_k. Then all non-real zeros of the derivative $p'(z)$ lie in the closed interiors of the circles $\Gamma(z_k)$.*

Theorem 1 admits only finite points γ, but if γ is allowed to become positively or negatively infinite, we obtain as a limiting case of Theorem 1 a well known result:

JENSEN'S THEOREM. *Let $p(z)$ be a real polynomial not identically constant, and consider the circles having as diameters the line segments $z_k \bar{z}_k$ joining conjugate pairs of non-real zeros of $p(z)$. Then all non-real zeros of $p'(z)$ lie in the closed interiors of these circles.*

Jensen's Theorem represents for real polynomials a sharpening of Lucas's Theorem, that *for an arbitrary polynomial not identically constant, the zeros of the derivative lie in the smallest convex polygon containing the zeros of the original polynomial.*

Both Lucas's Theorem and Jensen's Theorem are ordinarily proved* by use of Gauss's Thorem, that *the zeros of the derivative $p'(z)$ of a polynomial $p(z)$ which are not multiple zeros of $p(z)$ are the positions of equilibrium in the field of force due to unit particles situated at the zeros of $p(z)$, where each particle repels with a force equal to the inverse distance.*

We shall use Gauss's Theorem to prove Theorem 1, but it is convenient first to establish a preliminary result:

LEMMA. *Let Γ denote the circle through the points $+i$ and $-i$, tangent at $+i$ to the line joining $+i$ and $\gamma(>0)$. At an arbitrary non-real point $z=x+iy$, $x<\gamma$, interior [exterior] to Γ the force due to unit particles at $+i$ and $-i$ has a non-vanishing component perpendicular to the line γz sensed toward [away from] the axis of reals.*

The force at a point z due to a unit particle at z_0 is in magnitude, direction,

* See for instance J. L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, Colloquium Publications of the American Mathematical Sociery, vol. 34, 1950.

The Lemma of the present note is a sharpening of Theorem 1 (*loc. cit.*, p. 28).

and sense represented by the vector $1/(\bar{z}-\bar{z}_0)$, so in the field of force of the Lemma the force at z is

$$\frac{2\bar{z}}{\bar{z}^2+1}=\frac{2(x-iy)[(x+iy)^2+1]}{(x^2-y^2+1)^2+(2xy)^2},$$

whose slope is

$$\frac{y(x^2+y^2-1)}{x(x^2+y^2+1)}.$$

The locus of points (x, y) at which the line of action of the force passes through the point $(\gamma, 0)$ is

$$\frac{y}{x-\gamma}=\frac{y(x^2+y^2-1)}{x(x^2+y^2+1)}, \tag{1}$$

and for non-real z (*i.e.*, $y\neq 0$) this locus is

$$\Gamma:\quad x^2+2x/\gamma+y^2-1=0, \qquad \gamma\neq 0$$

namely the circle through $+i$ and $-i$ whose tangent at $+i$ passes through the point γ. The total force at z is zero when and only when z is zero. It is geometrically obvious that at a point $z=x+iy$, $x<\gamma$, interior [exterior] to Γ near $+i$ or $-i$ the total force has a non-vanishing component perpendicular to the line γz sensed toward [away from] the axis of reals. At a non-real point z the force component perpendicular to γz vanishes when and only when z lies on Γ, and this component varies continuously with z, $z\neq \pm i$. Consequently this component is different from zero and sensed toward [away from] the axis of reals at every non-real point $z=x+iy$, $x<\gamma$, interior [exterior] to Γ.

We are now in a position to apply Gauss's Theorem, and to prove Theorem 1. The case that $p(z)$ has only the single root β is trivial and henceforth excluded. By the possibility of interchanging left and right, and by Lucas's Theorem we need consider merely values $z=x+iy$ with $x<\gamma$. At a non-real point $z=x+iy$ with $x<\gamma$ exterior to all the circles $\Gamma(z_k)$, it follows from the Lemma that the force due to each pair z_k, $\bar{z}_k$ of particles has a non-vanishing component perpendicular to the line γz sensed away from the axis of reals; the force at z due to a single particle at a real zero of $p(z)$ to the left of γ likewise has a non-vanishing component perpendicular to γz sensed away from the axis of reals, so the total force at z is not zero, and z cannot be a position of equilibrium. Of course z cannot be a multiple zero of $p(z)$, so z cannot be a zero of $p'(z)$, and Theorem 1 is established.

In the proof of Theorem 1 we have used the Lemma only insofar as it deals with points z exterior to Γ. A similar consideration of points interior to Γ (details are left to the reader) yields the

COROLLARY. *Let $p(z)$ be a real polynomial all of whose real zeros lie in the interval $\gamma \leqq x < +\infty$ of the axis of reals. Suppose each conjugate pair z_k, $\bar{z}_k$ of non-real zeros of $p(z)$ has real part less than γ, and let $\Gamma(z_k)$ denote the circle through z_k and $\bar{z}_k$ tangent to the line γz_k at z_k. Then no non-real point z interior to all the circles $\Gamma(z_k)$ can be a zero of $p'(z)$.*

As an application of both Theorem 1 and the Corollary, we may choose $p(z)$ as real and of form $(z-z_1)^{m_1}(z-\bar{z}_1)^{m_1}(z-z_2)^{m_2}(z-\bar{z}_2)^{m_2}$, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, $y_1y_2>0$, and choose for instance γ as the intersection (supposed finite) of the line z_1z_2 with the axis of reals. It follows from Theorem 1 that a non-real zero of $p'(z)$ cannot lie exterior to both $\Gamma(z_1)$ and $\Gamma(z_2)$, and follows from the Corollary that such a non-real zero cannot lie interior to both circles $\Gamma(z_1)$ and $\Gamma(z_2)$. Thus every non-real zero of $p'(z)$ other than a multiple zero of $p(z)$ must lie either on both circles or interior to one and exterior to the other.

Some comments are in order regarding zeros of $p'(z)$ *on the circles* $\Gamma(z_k)$ in Theorem 1. If a non-real point z lies interior to no circle $\Gamma(z_k)$ and either z is exterior to at least one such circle or there exist some real zeros of $p(z)$ not at γ, then z cannot be a position of equilibrium, and can be a zero of $p'(z)$ only if it is a multiple zero of $p(z)$. On the other hand, if $p(z)$ has no real zeros other than perhaps at γ, and if all the circles $\Gamma(z_k)$ pass through a non-real point z' interior to the Lucas polygon for $p(z)$, then it may occur that z' is a zero of $p'(z)$.

It is essential in Theorem 1 to exclude a pair z_k, $\bar{z}_k$ whose real part equals γ, for if the real part equals γ there is no proper circle $\Gamma(z_k)$, and it follows from (1) that except on the line $x=\gamma$ the force at every non-real point z due to the particles at z_k and $\bar{z}_k$ has a component perpendicular to the line γz in sense toward the axis of reals. Nevertheless, in the Corollary non-real zeros of $p(z)$ with real part γ may be admitted; no non-real $z=x+iy$ with $x<\gamma$ interior to all the proper circles $\Gamma(z_k)$ can be a zero of $p'(z)$.

In the Corollary, if $p(z)$ has no real zero except perhaps at γ, then a non-real point z interior to the Lucas polygon may conceivably be a zero of $p'(z)$ if it lies on all the circles $\Gamma(z_k)$; but a non-real point exterior to no circle $\Gamma(z_k)$ and interior to at least one such circle cannot be a zero of $p'(z)$ unless it is a multiple zero of $p(z)$.

If $p(z)$ has but one pair of non-real zeros, Theorem 1 and the Corollary combine to determine a lens-shaped region (*loc. cit.*, p. 39, Theorem 1) containing all non-real zeros of $p'(z)$.

Theorem 1 may be complemented by a result on the number of zeros of $p'(z)$, wholly analogous in content and proof to a complement to Jensen's Theorem (*loc. cit.*, p. 31, Theorem 1).

ACCESSORY LINKAGES WHICH HAVE CERTAIN STABILIZING PROPERTIES

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An example of a remarkably versatile linkage is provided by the crossed parallelogram (the Hart cell). In its symmetrical configuration the midpoints of the longer sides are coincident, while the perpendiculars from these coincident points on the shorter sides are collinear, and constitute an axis of symmetry. If the four sides are now divided proportionally from the vertices which are on the same side of this axis, these four points will not only be collinear: but for every deformation any three consecutive points will act as an inversor, with the outermost point at the pole of the inversion.

While these facts are well known, it is perhaps not so generally known that the Hart cell, with simple adaptations, can be made to describe the cissoids, the lemniscates and the limaçons. These curves, with the one exception of the non-symmetrical lemniscatoids, are the inverses of the conics about their vertices, centers and foci respectively.

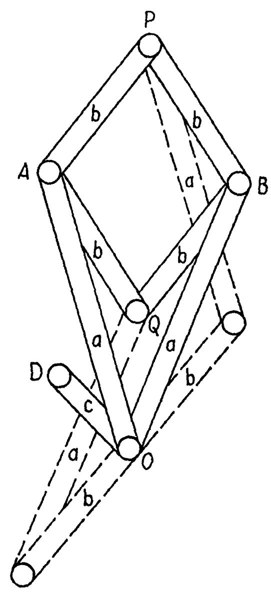


FIG. 1

Peaucellier's cell can also replace Hart's cell and, when it does, stabilization becomes necessary. We shall say a linkage or inversor is *unstable* if its predetermined tracer point is able to depart at any point from its anticipated locus or motion and follow an alternative locus or take up a contrary motion.

It is not the purpose of this paper to develop the equations of the curves mentioned in paragraph two as they appear when developed directly from the

linkage mechanisms. These are adequately dealt with by R. C. Yates in his *Geometrical Tools*, pages 172–185. In what follows, it will be understood that a term such as *elliptic cissoid* will mean that particular form of cissoid whose inverse is an ellipse. With reference to Figure 1, Yates demonstrates that when a^2-b^2 is greater than, equal to, or less than $4c^2$, the point P describes respectively the elliptic, parabolic and hyperbolic cissoids.

If the *dotted* arms a were each extended upward their own lengths and joined at their upper extremities by a bar $2b$ units in length, we would have a Hart cell. The auxiliary link DO from the fixed point D will impart to the point P of the Hart cell cissoidal motion provided the point Q is attached to a fixed pivot. This point Q will be located at the pole of the cissoid, and the whole assemblage will require no stabilization since P and Q (the inversor points) pass and repass each other with perfect smoothness and precision. On the other hand, were the cissoidigraph constructed of the bars represented by the *solid* bars of Figure 1 (*i.e.*, Peaucellier's cell plus DO), P and Q would become unstable in their position of coincidence (at the cusp in the case of the parabolic cissoid and the node in the case of the hyperbolic cissoid). In 1950 the writer stabilized Peaucellier's cell by adding the three bars a, a and $2b$ as shown in the dotted lines. At that time he failed to observe that the addition of exactly one half of a Hart cell was sufficient to stabilize a Peaucellier cell.

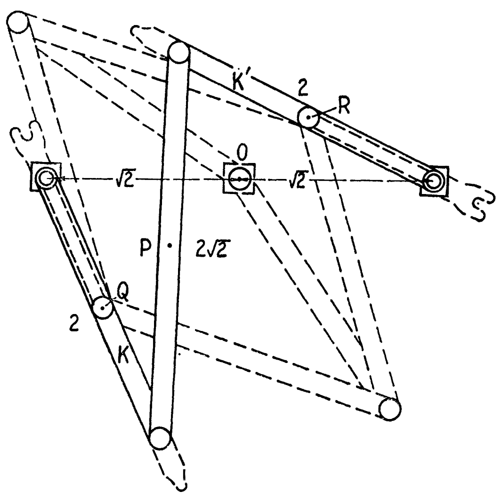


FIG. 2

In 1953, while preparing a linkage demonstration for the Upper New York Section of the Mathematical Association, the writer unwittingly performed a reverse stabilization operation on Bernoulli's lemniscatograph. This instrument is essentially a Hart cell having its opposite vertices pivoted at fixed points. Any point P on the intermediate traversing bar will describe a lemniscatoid and, in particular for the dimensions indicated in Figure 2, the mid-point will de-

scribe the lemniscate of Bernoulli; its inverse with respect to the point O is the equiangular hyperbola having its center also at O.

It is easily seen that the three movable bars become unstable whenever the configuration forces them to lie in the same straight line. The writer is indebted to Michael Goldberg for a non-linkage stabilizer as a means of correcting this situation. Essentially his solution consists of attaching two elliptical gears to the radial arms in such a manner that the extremities of these arms are made coincident with the foci of the pitch ellipses of the gears. The eccentricities of these gears equal the numerical length of a radial arm divided by the numerical length of the traversing arm. It is necessary to use only six strategic teeth from these hypothetical gears to effect stabilization. They are shown in Figure 2.

A true linkage solution to this problem was obtained by making the pivots about which the arms K and K' rotate integral with these links and extending the respective shafts through the supporting plane where they were again made integral with arms just one half the length of the original arms and positioned as shown in Figure 2. To the extremities of these half-length arms the six-bar negative Peaucellier cell was attached with its fixed point joined to O, the point midway between the two rotatable shafts. Now since QOR forms a straight line in both cases and $QO \cdot QR$ equals a constant, the two systems will work together harmoniously and will not require stabilization, since the negative Peaucellier inversor is stable in all positions.

This inverse scriber of the conics possesses complete versatility. Upon altering the dimensions of the radial arms and, of course, those of the half arms at the back of the instrument, a whole range of hyperbolic lemniscates can be drawn

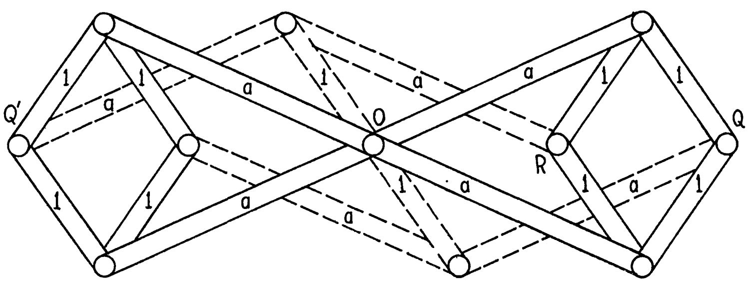


FIG. 3

having their poles at O. Attaching a novel form of Peaucellier's inversor (Fig. 3) to this point, both branches of the hyperbola can be described simultaneously. This last mentioned double inversor was the logical by-product of the stabilized inversor described in connection with Figure 1.

Now while all other intermediate points on the traversing bar of Figure 2 also describe lemniscatoids, the locations of the pole on which the inversor must be set and the center of the resultant conic must be especially calculated. The following simple formulas are quoted without proof for those inversors which generate conics whose vertices are coincident with those lemniscatoidal loops

or vertices which have the larger curvatures. Length b is the length of the shorter of the two radial arms, $2a$ the length of the traversing bar and c the distance of the tracer from the mid-point of this bar. The pole of the inversor is located $(a-c)$ units from that bearing which is more remote from the above mentioned point of osculation and lies on the line joining the two bearings. The focus of the parabola is located $32b^2/(4a^2-5b^2+4)$ units from its vertex. The eccentricity of the conic is $\sqrt{(a^2-c^2)/(a^2-b^2)}$, while the distance between the vertices is given by $4(bc+b^2)/|b-c|$.

The classic among the linkages is the one depicted in heavy lines in Figure 4, and while it has little practical interest as far as the parabola is concerned, it

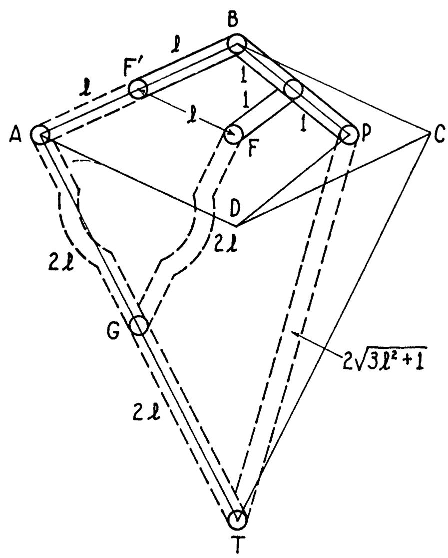


FIG. 4

has the advantage over the linkage just described in the fact that the pole of the inversor as well as the center of the generated conic is invariably found at the fixed point F. Both the points F and F' are fixed in position and are e units apart, where e is the eccentricity of the generated conic. The point P describes the elliptic or hyperbolic lemniscate according as e is less than or greater than 1. Whenever the three points F, F' and P become collinear it will be apparent the linkage will become unstable. The writer was successful in stabilizing this linkage also by means of the following simple method. The left hand link $F'B$ was extended backward its own length e beyond the pivot F' and three additional links were added of lengths $2e$, $4e$ and $2\sqrt{3e^2+1}$ as shown in the dotted lines of Figure 4.

J. J. Sylvester has a simple proof that when two jointed equal-armed V-shaped units (not necessarily congruent) are attached at their free extremities to the opposite angles of a rhombus (a pair of crossed Peaucellier cells), the dis-

tance between the vertices of these V-units is constant. [See Proceedings of the Royal Institute VII (1874).] Accordingly, the calculated distance $2\sqrt{3e^2+1}$ remains constant for every deformation of the double cell. Linkage $ABCDPT$ represents a phantom double cell. It will be observed that the angles BFP and AFT are both right angles with arms lying on the diagonals of $ABCD$. In the case of the elliptic lemniscate it will be found necessary to notch the sections AG and FG at their mid-points. These notches are cut from the inside to enable them to encircle the pivot F'. Now if this pivot were replaced by a stout pin carrying at its upper extremity a stationary arm of length e and having a hole at its outer extremity positioned immediately above F, it would be possible to attach the center O of the double Peaucellier inversor (Fig. 3) to this hole. This inversor is dynamically balanced so that if the point P of Figure 4 were joined to an inner vertex R of this cell, the outer-vertices Q and Q' would describe simultaneously either the two finite branches of the ellipse, drawing it in its entirety, or the two branches of the hyperbola each passing through a vertex.

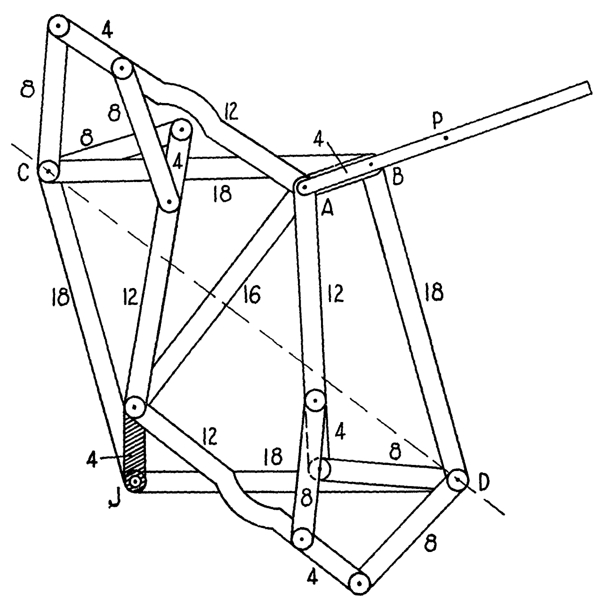


FIG. 5

Finally we come to a description of the limaçonigraph. A single auxiliary bar will convert a Hart cell into a linkage capable of drawing the hyperbolic and elliptic limaçons including the cardioid (parabolic limaçon). This versatile linkage is found described in *Geometrical Tools*. Despite the fact that the writer succeeded in also finding a known stabilizing linkage for this classic linkage, he feels it might be more interesting to describe one of his own design. The actual model works with great smoothness and precision. In the diagram of the instrument (Fig. 5) the short shaded bar 4 is attached rigidly to a stout dowel pin J.

This pin passes through the supporting base and is integral with it. The corresponding short bar AB is cranked outwardly through bar 16, which connects the shaded bar with the bar AB. It also goes through a leg belonging to each of two Hart straight line scribers, which line is the perpendicular bisector of the bar 16.

On top of the shaft which goes through these three parts is a rigidly positioned crank parallel to AB and pointing in the same direction. This latter part is the uppermost of all the links and carries the scriber, which can be set at variable distances from A. To visualize the action of the linkage let us suppose that bar 16 is held fixed instead of the shaded bar 4. It can then be easily seen that as the shaded bar is rotated, say clockwise with uniform angular velocity, the bar AB would rotate counter-clockwise with uniform velocity also, since vertex C and vertex D of the rhombus of sides 18 are constrained to move along the diagonal due to the two "A-Linkages" (Hart's) which are positioned on the common base 16. Now giving the shaded bar angular velocity zero as in the original set-up, it is apparent that the angular velocities of bars 16 and AB relative to this fixed bar are in the exact ratio of 1:2 and in the same sense.

The setting of the adjustable tracer P governs the character of the limaçon. When the tracer is set at 8 units from A it describes the cardioid, at lesser distances the elliptic limaçon (loopless), and at a greater distance than 8 from A, the hyperbolic limaçon. It is to be hoped that the principles set forth in the foregoing may be of value in mechanical situations which require the elimination of dead-centers.

A CARD GAME WITH BLUFFING

RUFUS ISAACS, The Rand Corporation

The present game—named by our daughter "Guess It"—has the following features:

A play involves not a paucity of moves but a sequence, possibly of considerable length.

It is a game of incomplete information with a kind of bluffing possible for almost all positions.

Unlike most of the examples in expositions of game theory, it has merit as a recreation. At least we have been so told by persons who have played it. The decisions as to bluffing and bluff detection appear stimulating especially to those with a flair for gambling, psychological games.

It is a rare thing for a game with such properties to be amenable to analysis. This one is; fortunately the information structure that a player acquires as he

plays is not cumulative. That is, positions recur which are tantamount to starting positions and we can solve the game by a recurrence or "programming" technique.

The optimal strategies are, of course, mixed.* Such have been the subject in recent years of much lucubration and little action. Remedially, we furnish two dials. The reader can manufacture spinners for their centers and thus possess the chance device needed for actual ventures into optimal playing.

The game. A deck of consecutively numbered cards is randomly dealt to two players, but one card is placed face down on the table. (The objective is to identify this hidden card.) For his move, a player may either *call* or *ask.*

If he calls, he announces a possible name for the hidden card. The game is now over. If correct, the caller wins; if not, he loses.

If he asks, he inquires of his opponent, "Have you such-and-such a card?" and the opponent must reply truthfully. It is then the other player's turn to move; he is faced with the same two choices. The same card may not be asked about twice.

Thus the art of the game is to acquire information with as little disbursement of it as possible until we have enough knowledge—or margin thereof—of the hidden card to warrant a call.

When we speak of a player bluffing we mean that he is asking about a card in his own hand. If a player decides never to bluff, and so informs his opponent, it is clear that, if he happens to ask about the hidden card, he loses the game. Some bluffing is thus essential for defense; offensively it may be used to decoy the opponent into a false call.

The various positions in a play differ only in their information structure. But from an alternative viewpoint we may regard the difference as essentially a tangible one with most of the residual information nugatory. Consider, for example, the opening move.

If it is a call, the play is over.

If it is an ask with a "yes" reply, then both players know the location of one card. From now on it cannot be asked for and will not be called. Thus nothing is changed if we think of this card as deleted. The original type of position has recurred with one card less and a change in turn. (In actual play, such effectively deleted cards may be placed face up on the table.)

For the remaining alternative suppose that Player I asks Player II about card x and is answered "no." The latter player is now in a quandary about card x, but his dilemma will not long persist. If he does not call, and if x was the hidden card, Player I most certainly will. Hence if Player I asks on his next turn, he must have x in his hand. Both players know it, and it may be deleted.

* A player employs a *mixed* strategy when he selects his alternatives at each move with certain probabilities rather than making definite decisions. This concept, due to J. von Neumann, is expounded in any text on game theory.

The analysis. We operate under the following assumptions, which eliminate some absurd, dominated strategies.

(1) If a player knows the hidden card with certainty and has the opportunity, he will call and win.

(2) A player never will call a card he knows is not the hidden one.

(3) A player will make a choice from a set of indifferent cards equiprobably. For example, if he holds ten cards and on his first move has decided to bluff, he will select each of his cards with a probability of one-tenth.

Let $P(m, n)$ be the win probability of the player with first move if, at the opening of a game, he holds m cards and his opponent n, under the assumption that both players will play optimally. Then $P(m, n)$ is the value of the game if we choose the payoff as 1 and 0 according as the first move player wins or loses.

Let $A(m, n)$ be same quantity, except that now the rules have suffered the amendment: the opening move must be an ask.

Inasmuch as a first move call has a probability of success equal to $1/(n+1)$, we have

$$P(m, n) = \max\left[\frac{1}{n+1}, A(m, n)\right]. \tag{1}$$

Assumption (1) shows that

$$P(m, 0) = 1,$$

and thus the half-a-loaf adage, that

$$P(0, n) = \frac{1}{n+1}. \tag{2}$$

Also

$$A(m, 0) = \frac{m}{m+1}, \tag{3}$$

for a player can always ask without betraying information by choosing his card equiprobably from the entire deck (a policy hereafter referred to as the *safe move*). The first player with a monopoly of cards and under compulsion to ask, should act so, thus putting his opponent in the position governed by (2).

We turn to mixed strategies, supposing now $m > 0$, $n > 0$. Let the first move —made by, say, Player I—be an ask and let b be the probability that he shall bluff. Let Player II's strategy be in part:

If the reply is "no" he will with probability

c_1, call the card x that was asked about. (α)

c_2, call some other card. (β)

$1-c_1-c_2$, ask. (γ)

Let us suppose I bluffs and see what happens under these three responses.

(α) Player I wins; hence the outcome has the value 1.

(β) Player II has probability $1/m$ of calling correctly; hence the outcome is $(m-1)/m$.

(γ) If Player II asks he must make his decisions as to bluffing on the premise that card x lies in Player I's hand. For if x were the hidden card, Player I would know it and so call on his move; in this case the nature of II's ask would be immaterial. In the contrary case—where the nature of the ask is effective—II's premise will be verified.

Now if we hypothesize that I is bluffing, card x will be on the verge of deletion. As we have just seen, II will act as if it already were and with I's response it will be. Thus the situation is as if II had the opening move and n cards and I had $m-1$ cards. The value is then

$$1 - A(n, m - 1).$$

Now suppose I had not bluffed but has received a "no." He has hit upon the hidden card; the probability for this situation is $(1-b)/(n-1)$. The three outcomes are:

(α) II wins; outcome $=0$.

(β) II loses; outcome $=1$.

(γ) As I will call on the next move and win, the outcome $=1$.

The probability that I receives a "yes" reply is $(1-b)\cdot n/(n+1)$. The outcome, being tantamount to a fresh start, is $1-P(n-1, m)$. We may then write

$$\begin{aligned} A(m, n) = \max_b \min_{c_1, c_2} \Bigl\{ & b\left[c_1 + c_2 \frac{m-1}{m} + (1 - c_1 - c_2)(1 - A')\right] \\ & + \frac{1-b}{n+1}[1 - c_1] + \frac{n(1-b)}{n+1}(1 - P')\Bigr\} \quad (m > 0, n > 0), \end{aligned} \tag{4}$$

where $A' = A(n, m-1)$, $P' = P(n-1, m)$.

We should note that in (4) the order of taking the max and min is immaterial. This follows from the central theorem of game theory if we can show that the right side of (4) is derivable from the pay-off matrix of some game. Such a game is obtained from ours through the following modifications:

Assumptions (1) and (2) are embodied as rules. In place of assumption (3) we put corresponding equiprobable chance moves. At the end of one round of moves the players are paid off, A' and P' being taken as any prescribed numbers.

The reader can easily verify that (1), (2), (3), (4) determine uniquely $P(m, n)$ for all m, n.

As (c_1, c_2) is restricted to the triangle $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 \leqq 1$ and the brace of (4) is linear in c_1, c_2, its minimum will be assumed at one of the vertices of the triangle. Thus

$$A(m, n) = \max_b \min_i \left[\frac{n}{n+1}(1 - P') + Z_i\right], \qquad (i = 0, 1, 2),$$

where

$$Z_1 = b\left(1 - \frac{n}{n+1}(1 - P')\right) = b\left(\frac{1}{n+1} + \frac{n}{n+1}P'\right), \qquad (c_1 = 1, c_2 = 0),$$

$$Z_2 = b\left(\frac{m-1}{m} - \frac{1}{n+1} - \frac{n}{n+1}(1 - P')\right) + \frac{1}{n+1}$$

$$= b\left(-\frac{1}{m} + \frac{n}{n+1}P'\right) + \frac{1}{n+1} \qquad (c_1 = 0, c_2 = 1),$$

$$Z_0 = b\left(1 - A' - \frac{1}{n+1} - \frac{n}{n+1}(1 - P') + \frac{1}{n+1}\right)$$

$$= b\left(-A' + \frac{n}{n+1}P'\right) + \frac{1}{n+1}, \qquad (c_1 = c_2 = 0).$$

We evaluate the minimax in two cases only. Then we will prove that no other possibilities can arise.

Case 1. $m=1$.

Here $A'=A(n, 0)=n/(n+1)<1=1/m$. Thus $Z_0>Z_2$ and we need compare only Z_1 and Z_2. The coefficient of b is positive in the former and negative in the latter. Then clearly the maximum occurs when $Z_1=Z_2$ or when

(5) $$b = \frac{1}{n+2} \text{ (the safe move!)}.$$

Then, after a short reduction,

(6) $$A(1, n) = \frac{1 + n(1 - P')}{n+2}.$$

Case 2.

$$A' \geqq \frac{1}{m}, \qquad A' \geqq \frac{n}{n+1}P'.$$

Here $Z_0 \leqq Z_2$ so that the race is between Z_0 and Z_1. The former having a negative coefficient of b, the maximum again occurs at the equality point. Thus brief calculations give for the maximum:

(7) $$b = \frac{1}{1 + (n+1)A'}$$

and

(8) $$A(m, n) = \frac{1 + nA'(1 - P')}{1 + (n+1)A'}.$$

LEMMA 1. *If* $m>0$, $n>0$, *then*

$$A(m, n) \geqq \frac{1}{n+1}, \tag{9}$$

and consequently $A(m, n)=P(m, n)$.

*Proof:** Let Player I adopt the strategy: his first move is safe; if II does not call in response, I does so on his next turn. Then if II calls his probability of correctness $\leqq 1/(m+1)$ and I attains at least $m/(m+1) \geqq \frac{1}{2} \geqq 1/(n+1)$. If I calls, his correctness probability also $\geqq 1/(n+1)$.

LEMMA 2. *If* $m>0$, $n>0$,

$$A(m, n) \geqq P(m-1, n+1). \tag{10}$$

Remark. These two lemmas ensure Case 2 if $m>1$. The latter does so more strongly than necessary, for it shows

$$A' \geqq P' > \frac{n}{n+1} P'.$$

Proof: We work inductively on $k=m+n$. Its smallest value is 2 and here

$$A(1, 1) = \frac{1+(1-P(0, 1))}{1+2} = \frac{1+\frac{1}{2}}{3} = \frac{1}{2} > \frac{1}{3} = P(0, 2).$$

We now assume $k>2$ and

$$\text{(10) holds for } m+n<k, m>0, n>0. \tag{11}$$

(a) $m=1$. From (6) and (2)

$$A(1, n) = \frac{1+n(1-P')}{n+2} \geqq \frac{1}{n+2} = P(0, n+1).$$

(b) $m=2$. We may apply (8) for, as $A'=A(n, m-1)$, (9) and (11) assure us of Case 2. Thus

$$A(m, n) = \frac{1+nA'(1-P')}{1+(n+1)A'} \geqq \frac{1+nA'(1-A')}{1+(n+1)A'} \tag{12}$$

while

$$P(1, n+1) = \frac{1+(n+1)(1-P(n, 1))}{n+3} = \frac{1+(n+1)(1-A')}{n+3}. \tag{13}$$

We attain our result by forming the difference of the rightmost terms of (12) and (13). The numerator turns out to be

* Suggested by S. Karlin.

$$1 + (n-1)A'(1-A') > 0.$$

(c) $m > 2$. Now (12) stands, but (13) is to be replaced by an instance of (8):

$$P(m-1, n+1) = \frac{1 + (n+1)A''(1-P'')}{1 + (n+2)A''}, \tag{14}$$

where $A'' = A(n+1, m-2)$, (9) and (10) applying for these arguments, and $P'' = P(n, m-1) = A'$.

A subtraction of the right sides of (14) and (12) yields for the numerator, after some rearrangement,

$$[A'' - A'] + n[A''P'' - A'P'] + A''[P'' - A'] \\ + A'A''[(n+1)^2P'' - n(n+2)P'].$$

This expression cannot be negative as (11) implies

$$A'' \geqq P'' = A' \geqq P'.$$

We may now eliminate the symbol A from our results and embody them in a

THEOREM. *The function $P(m, n)$ satisfies and is determined by*

$$P(m, 0) = 1, \qquad P(0, n) = \frac{1}{n+1}$$

$$P(1, n) = \frac{1 + n(1 - P(n-1, 1))}{n+2} \qquad (n \geqq 1),$$

$$P(m, 1) = \frac{1 + P(1, m-1)\dfrac{m}{m+1}}{1 + 2P(1, m-1)} \qquad (m \geqq 1),$$

$$P(m, n) = \frac{1 + nP(n, m-1)[1 - P(n-1, m)]}{1 + (n+1)P(n, m-1)}, \qquad (m > 1, n > 1).$$

We shall append a brief table of this function.

The optimal strategies for a player asking are given by (5) and (7). We have still to calculate the best values of c_1, c_2.

Let us return to (4) and rewrite the brace:

$$b\left[c_1\left(A' + \frac{1}{n+1}\right) + c_2\left(A' - \frac{1}{m}\right) - \left(A' - \frac{n}{n+1}P'\right)\right] \\ + 1 - \frac{n}{n+1}P' - \frac{c_1}{n+1}. \tag{15}$$

In both our cases the optimal b is between 0 and 1. It follows that the optimal c_1, c_2 must be such as to annul the bracket in (15). Thus in the (c_1, c_2)-plane, we are confined to a line and also to the triangular set: $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 \leqq 1$. A

glance at the remaining terms of (15) shows that we must select the point on the intersection of the line and triangle with the largest possible c_1. Let the equation of the line be

$$a_1c_1 + a_2c_2 = a_3. \tag{16}$$

Case 1. Here $a_1>0$, $a_2<0$ and, as

$$A' = A(n, 0) = \frac{n}{n+1},$$

$a_3>0$. The line thus has positive slope and meets the c_1-axis at a_3/a_1, which is between 0 and 1. Thus the sought point is on the hypotenuse

$$c_1 + c_2 = 1$$

which must be solved with (16). In this case the latter is

$$c_1 - \frac{1}{n+1}c_2 = \frac{n}{n+1}(1 - P'),$$

and the solution is

$$\begin{aligned} c_1 &= \frac{(n+1) - nP'}{n+2} \\ c_2 &= \frac{1+nP'}{n+2}, \qquad (P' = P(n-1, 1)). \end{aligned} \tag{17}$$

Case 2. Now $a_1>0$, $a_2>0$, $a_3>0$. The line has negative slope and, as before, meets the c_1-axis at a point between (0, 0) and (1, 0). Thus

$$\begin{aligned} c_1 &= \frac{(n+1)A' - nP'}{1+(n+1)A'} \\ c_2 &= 0. \end{aligned} \tag{18}$$

Description of a typical play. We suppose optimal strategy on both sides. Suppose first that Player II has but one card.

The play will expire quickly. Let Player I ask. If the reply is "yes," he has ascertained the hidden card; II calls in self-defense. If "no," (18) shows that II should likewise call.

Now let m and n be both large and fairly close to equality. It may be expected that P will be somewhere near $\frac{1}{2}$. At the opening position or any other where deletions have cleared the air of information, there will be an ask. The bluff probability being roughly (see (7)) $2/(n+3)$ and thus small, most moves will entail "yes" answers and so alternate deletions from each hand. A chance hit on the hidden card will or a bluff may terminate the play abruptly at any time. If such does not happen the pattern of alternate deletions will be disturbed

by an occasional detected bluff which will remove a card from the *asker's* hand. Thus there will be a random deviation from equality of hands even if m equaled n at the outset. Things come to an end, as we have seen, when one player is reduced to a single card.

The Dials. The b dial instructs Player I (whose move it is) when to bluff. There is a circle for each value of m with marks on it to show n.* The player should bluff when the pointer lands in the sector extending from the appropriate mark clockwise to the horizontal line.

The c_1 dial is used similarly by Player II when his response is "no." He calls the card just asked when the pointer falls in the sector from mark to line. When it does not, he calls the other unknown card when $m=1$ and asks if $m>1$. He should note that on *this* dial, m is the number of cards in his *opponent's* hand. If on his preceding move he bluffed (and the opponent did not call) he should consider the bluffed card as deleted when reckoning n.

The largest deck for which the dials can suffice contains 11 cards.

Some variations. There are numerous possibilities of deriving games, interesting for recreation if difficult for analysis, by altering the rules of the present one.

There might be several hidden cards. The deck may be divided into suits with one hidden card from each. An ask may concern several cards, but a full response is not obligatory. It may state, for example, that one of the cards is held but not specify which; there is no deletion here, of course. The response may be a full reply about any one asked card in the responder's hand; deletion is restored.

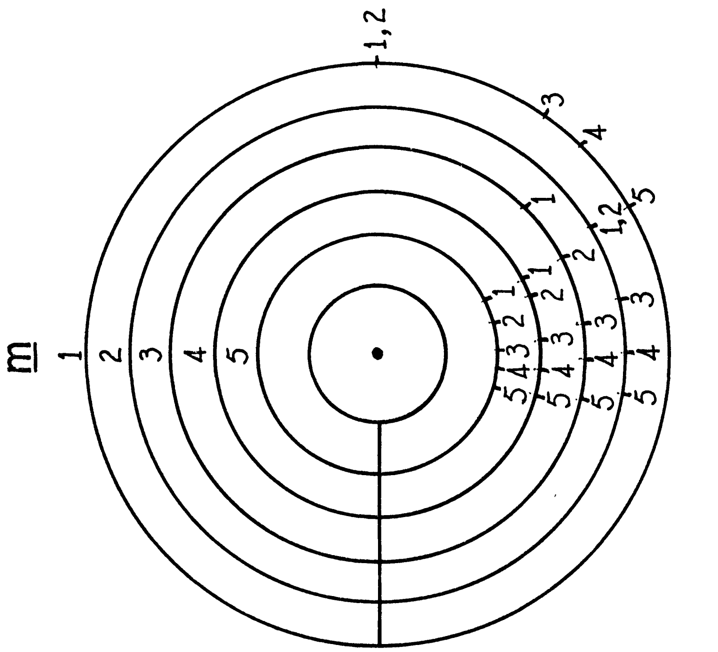
Stimulating variants arise when there are more than two players.† The rules should specify whom a player may ask—such as his left neighbor, his choice, or the outcome of a chance move. The rules also declare how much of the query or reply is privy to the parties concerned and how much is made public. The novelty is the possibility of deductive inference from this partial knowledge of the asks and replies of other players. It should be possible to devise rules with a great gamut of scope for ingenuity.

A TABLE OF $P(m, n)$

$P(m, n)$		m = 1	2	3	4	5
n	1	.5	.667	.688	.733	.75
	2	.5	.556	.625	.648	.680
	3	.4	.512	.548	.597	.619
	4	.375	.450	.513	.543	.581
	5	.333	.423	.467	.512	.538

* Of course m and n are the number of remaining cards after the proper deletions.

† We advance no ideas as to the meaning of an optimal strategy in these cases.



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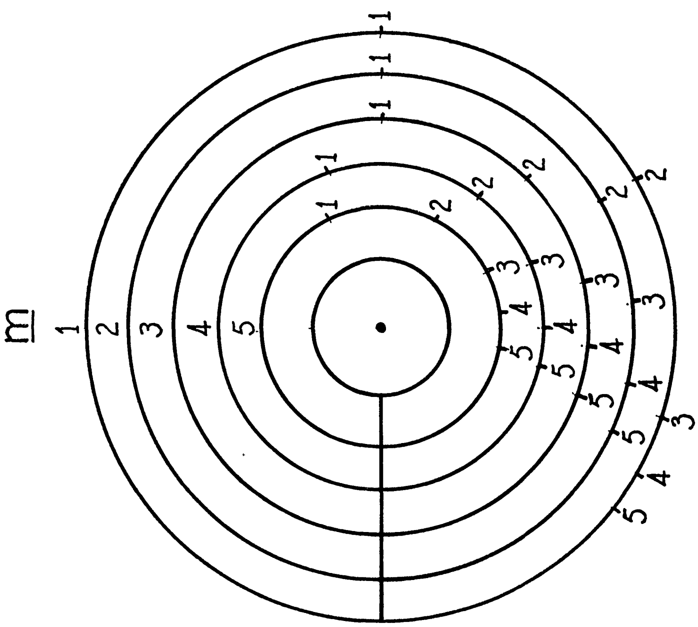

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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTE ON UNIVALENT FUNCTIONS

VIKTORS LINIS, University of Saskatchewan and University of Ottawa

Friedman [2] has proved the following theorem which is a case of Salem's theorem on univalent functions:

Let $f(z)$ be analytic and univalent in the unit circle $|z|<1$ and let its expansion be $z+\sum_{n=2}^{\infty}a_nz^n$. If all the coefficients a_n are rational integers then $f(z)$ is a rational function and one of the forms:

$$z, \quad z/(1\pm z), \quad z/(1\pm z^2), \quad z/(1\pm z)^2, \quad z/(1\pm z+z^2).$$

The purpose of this note is to give a short proof of this theorem. Let

$$z/f(z) = \sum_{n=0}^{\infty} b_nz^n; \tag{1}$$

then following Prawitz [3] we have

$$\sum_{n=1}^{\infty}(n-1)\,|\,b_n\,|^2 \leqq 1. \tag{2}$$

Since $b_0=1$, $b_1=-a_2$ and $|a_2|\leqq 2$ (Nehari [1]) it follows that $|b_1|\leqq 2$ and from (2) $|b_n|\leqq 1$ for $n\geqq 2$. The coefficients $b_n (n\geqq 1)$ can be computed from the relation

$$(-1)^nb_n = \begin{vmatrix} a_2 & 1 & & \cdots 0 \\ a_3 & a_2 & 1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1} & a_n & & \cdots a_2 \end{vmatrix}$$

and thus are rational integers. The possible values for b_n are:

$$b_0=1; \quad b_1 = 0,\pm 1,\pm 2; \quad b_2 = 0,\pm 1; \quad b_n = 0 \quad \text{for } n\geqq 3.$$

From the combination of these values we obtain 15 functions; however, the following 6 must be rejected as having zeros in the unit circle:

$$1\pm 2z, \qquad 1\pm 2z - z^2, \qquad 1\pm z - z^2.$$

The remaining 9 functions prove the theorem.

Remark. The theorem remains true if the "rational integers" are replaced by the "Gaussian integers." The possible values for b_n are:

$b_0 = 1;\quad b_1 = 0,\ \pm i,\ \pm 1,\ 1 \pm i,\ -1 \pm i,\ \pm 2i,\ \pm 2;\quad b_2 = 0,\ \pm i,\ \pm 1;$
$b_n = 0 \quad \text{for} \quad n \geqq 3.$

From the 65 possible combinations of these values only 15 satisfy the conditions of the theorem. In addition to the 9 functions listed above we obtain also the following:

$$z/(1 \pm iz), \qquad z/(1 \pm iz)^2, \qquad z/(1 \pm iz - z^2).$$

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A PROOF OF MORLEY'S THEOREM

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Morley's Theorem arose as a byproduct of some deeper geometrical results which F. Morley [1] obtained about fifty years ago. It can be stated as follows: *The points of intersection of the adjacent trisectors of the interior angles of a triangle form the vertices of an equilateral triangle.* Most of the many different proofs of this theorem which have appeared are either geometrical [2] or trigonometrical [3]. The procedure followed here, on the other hand, finds the coordinates of the points of intersection of the angle trisectors as complex numbers [4].

To carry out the proof we refer to the figure, where ABC may be any triangle. Draw the circumscribed circle of ABC, denote its radius by r, let the origin be at the center of the circle, and let the positive real axis pass through the vertex C of the triangle. Let the vertices A and B be represented by the complex numbers $re^{i\alpha}$ and $re^{i\beta}$, respectively, where α and β are positive and less than 2π. Then the trisectors of the interior angles of the triangle will intersect the circle in the points with the complex numbers indicated in the figure.

In order to find the points of intersection of the angle trisectors, we use the following result [5], which can be established readily: *If two intersecting lines cut the circle $|z| = r$ in the points given by the complex numbers a, b and c, d, respectively, the point of intersection of the lines is given by*

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

Application of this formula to the two adjacent trisectors intersecting at N gives for N the complex number

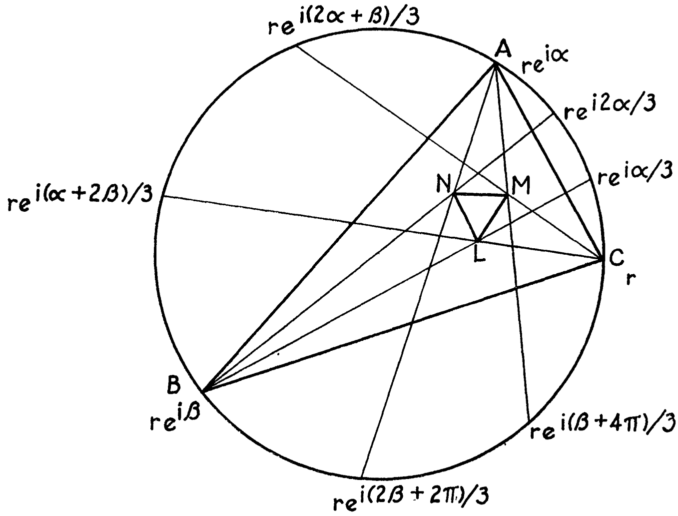
$$z_N = r\,\frac{e^{-i\beta} + e^{-i2\alpha/3} - e^{-i(2\beta+2\pi)/3} - e^{-i\alpha}}{e^{-i(\beta+2\alpha/3)} - e^{-i[\alpha+(2\beta+2\pi)/3]}}.$$

With the notation

$$p = e^{i\alpha/3}, \qquad q = e^{i\beta/3}, \qquad \epsilon = e^{i\pi/3},$$

the expression for z_N is easily reduced to

$$z_N = r(\epsilon p^2 q - \epsilon^2 q^2 p + p^2 - pq\epsilon + q^2\epsilon^2).$$



Similarly, for the points of intersection L and M, we find

$$z_L = r(q^2 p - q^2 + qp - q + p)$$

and

$$z_M = r(p^2 q + \epsilon^2 pq - \epsilon^2 p^2 + p\epsilon - \epsilon q).$$

The vector $\overrightarrow{LN}$ is given by the complex number

$$z_N - z_L = r[\epsilon p^2 q - (\epsilon^2 + 1)q^2 p + p^2 + (\epsilon^2 + 1)q^2 - pq(\epsilon + 1) + q - p]$$

which can be factored and written as

$$\overrightarrow{LN} = -r(q - p)(\epsilon^2 - q)(1 - p)\epsilon.$$

In a similar way

$$\overrightarrow{LM} = -r(q - p)(\epsilon^2 - q)(1 - p),$$

$$\overrightarrow{MN} = -r(q - p)(\epsilon^2 - q)(1 - p)\epsilon^2.$$

Consequently

$$|\overrightarrow{LM}| = |\overrightarrow{MN}| = |\overrightarrow{NL}|,$$

and we conclude that the triangle LMN is equilateral.

The length of side of this equilateral triangle is given by

$$\begin{aligned} |\overrightarrow{LN}| &= r|q - p||\epsilon^2 - q||1 - p| \\ &= r \cdot 2 \sin [(\beta - \alpha)/6]\, 2 \sin [(2\pi - \beta)/6]\, 2 \sin (\alpha/6). \end{aligned}$$

In terms of the angles A, B, C of the triangle ABC, this becomes

$$8r \sin (A/3) \sin (B/3) \sin (C/3),$$

a formula given by Kowaleski [**3**].

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FREDHOLM INTEGRAL EQUATIONS, THE RECIPROCALS OF WHOSE SOLUTIONS ARE ALSO SOLUTIONS

L. L. PENNISI, University of Illinois

We wish to prove the following:

THEOREM. *Let C be a bounded measurable set in n dimensional Euclidean space. Let $K(x, y)$ be a real-valued function in $L_2(C, C)$ and consider the Fredholm integral equation:*

$$u(x) = \int_C K(x, y)u(y)dy. \tag{1}$$

Suppose that this equation has the following properties:

(2) *There exists a real solution $u_0(x)$ of* (1) *in $L_2(C)$ whose reciprocal is in $L_2(C)$;*

(3) *If the reciprocal of a real solution $u(x)$ of* (1) *in $L_2(C)$ is in $L_2(C)$, then it is itself a solution;*

(4) *If $u(x)$ is a real solution of* (1) *in $L_2(C)$, then there exists a constant b such that $[b+u(x)]^{-1}$ is in $L_2(C)$.*

Then there is a positive integer m and a decomposition

$$C = \sum_{j=0}^{m} C_i \tag{5}$$

of C into measurable sets C_j such that C_0 and C_iC_j have measure zero if $i \neq j$, and

$$\int_C K(x, y)\chi(y, C_j)dy = \chi(x, C_j) \qquad (j \neq 0), \tag{6}$$

in which the characteristic functions

$$\chi(x, C_j) = \begin{cases} 1 & \text{on } C_j \\ 0 & \text{elsewhere} \end{cases}$$

are a complete set of linearly independent solutions of (1).

Proof. If $k > 0$, then $u_0 + ku_0^{-1}$ is a real solution whose reciprocal $u_0/(k+u_0^2)$ is bounded by $2^{-1}k^{-1/2}$ and is therefore a solution of (1). Since $K(x, y)$ is in $L_2(C, C)$, equation (1) has only a finite number m of linearly independent solutions [1]. Select $m+1$ distinct positive numbers $k_i (i=0, 1, \cdots, m)$. Then the solutions $u_0/(k_i+u_0^2)$ are linearly dependent, so that there exist constants a_i not all zero and a set E_0 of measure zero such that

$$\sum_{i=0}^{m} \frac{a_i}{k_i + u_0^2} = 0 \qquad (x \text{ not in } E_0). \tag{7}$$

Since the numbers k_i are distinct and the numbers a_i are not all zero, this is an equation in u_0^2 which has at most m solutions, and so u_0 takes on only a finite number of distinct values α_h on $C - E_0$:

$$(8) \qquad u_0 = \sum_{h=1}^{r} \alpha_h \chi(x, E_h) \qquad (r \leqq 2m),$$

in which the sets E_h are measurable subsets of C whose sum is $C - E_0$ and whose pairwise intersections are of measure zero.

The assertion that $u_0/(k+u_0^2)$ is a solution of (1) now becomes

$$(9) \qquad \sum_{h=1}^{r} \frac{\alpha_h}{k + \alpha_h^2} \int_{E_h} K(x, y) dy = \frac{\alpha_j}{k + \alpha_j^2} \qquad (x \text{ in } E_j).$$

Since this identity holds for all positive k,

$$(10) \qquad \int_{E_h} K(x, y) dy = \delta_{hj} \qquad (x \text{ in } E_j),$$

so that the functions $\chi(x, E_h)$ are linearly independent solutions of (1). Therefore $r \leqq m$, and

$$(11) \qquad 1 = \sum_{h=1}^{r} \chi(x, E_h)$$

is a solution of (1).

Let $u(x)$ be any solution of (1) in $L_2(C)$. Then by virtue of (11), $b+u$ is also a solution of (1), and by hypotheses (3) and (4), there is a value b for which $(b+u)^{-1}$ is a solution of (1). Repeating the argument of the last paragraphs with u_0 replaced by $b+u$, we infer that there are distinct values β_j and measurable sets $F_j (j = 1, \cdots, s \leqq m)$ such that

$$(12) \qquad u = \sum_{j=1}^{s} \beta_j \chi(x, F_j) \qquad (x \text{ in } C - F_0),$$

$$(13) \qquad \int_C K(x, y) \chi(y, F_j) dy = \chi(x, F_j) \qquad (j \neq 0),$$

$$(14) \qquad C = F_0 + \sum_{j=1}^{s} F_j, \quad \text{meas } F_0 = \text{meas } F_k F_j = 0 \qquad \text{if } k \neq j.$$

If the number r of sets E_h associated with u_0 is equal to m, the proof of our result is complete. If $r < m$, there is a solution $u(x)$ of (1) in $L_2(C)$ which is linearly independent of the functions $\chi(x, E_h)$. This solution may be represented as in equation (12). Pick numbers α_h $(h = 1, \cdots, r)$ so that the values $\alpha_h + \beta_j$ are all distinct, and consider

$$\begin{aligned} U &= \sum \alpha_h \chi(x, E_h) + \sum \beta_j \chi(x, F_j) \\ &= \sum (\alpha_h + \beta_j) \chi(x, E_h F_j). \end{aligned}$$

The function U is a solution of (1) in $L_2(C)$ which is constant on each of the disjoint sets $E_h F_j$ of the partition of $(C - E_0)(C - F_0)$. At least $r+1$ of these sets

have positive measure, for otherwise $u(x)$ would be linearly dependent on the functions $\chi(x, E_h)$. The characteristic functions $\chi(x, E_hF_j)$ of those sets E_hF_j of positive measure can then be shown to be linearly independent solutions of (1) by repeating the argument of the above three paragraphs. In this manner we increase the number of sets until we have a decomposition into exactly $m+1$ sets as prescribed in equation (5).

Since the function $u_0/(k+u_0^2)$ is bounded when $k>0$, it is clear that our argument is equally valid for integral equations satisfying a modified version of (2), (3) and (4) in which $L_2(C)$ is replaced the second time it occurs by any linear subset of $L_2(C)$ which contains the set $M(C)$ of bounded measurable functions on C.

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CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON PROVING THE CHAIN RULE

DAVID GANS, New York University

Choose an introductory textbook in calculus at random and examine its proof of the chain rule for differentiating a composite function. The proof will fall into one of three categories according to the amount of care taken, but regardless of the category the proof will be found to be less than satisfactory for the beginning student.

Assuming y to be a differentiable function of u, and u a differentiable function of x, a proof in what we may call category A consists in writing the identity $\Delta y/\Delta x = (\Delta y/\Delta u)(\Delta u/\Delta x)$, letting Δx approach zero, noting that Δu then approaches zero, and inferring that $dy/dx = (dy/du)(du/dx)$. This proof, of course, is defective in that it overlooks the possibility that, regardless of the restriction placed on Δx, Δu in approaching zero may attain the value zero "on the way," in which case $\Delta y/\Delta u$ cannot approach dy/du.

A proof in category B is like the above, but contains the additional remark that Δu must not be zero. Since this remark is always made with the utmost conciseness, often in a brief footnote, unaccompanied by any kind of clarifying discussion, this proof can be regarded as an improvement over the preceding only

if it is granted that it is better to leave a student mystified by a difficulty than ignorant of it.

The proofs in category C are rigorous and complete, disposing of the difficulty noted above by an ϵ-procedure of some sort. However, it is hard to see how these comparatively sophisticated proofs can be understood by any but a very small minority of students when it is realized that the latter are usually in contact with the calculus for only about a month when the chain rule is presented to them.

How, then, should we prove the rule to such students? Admittedly, the proof cited above is delightfully simple in outline. That it has not been presented better in our introductory textbooks must be charged, it seems, to the paralyzing force of tradition, *i.e.*, what is good enough for fifty calculus books is good enough for the fifty-first. The fact is, however, that this proof can easily be made rigorous and also understandable, provided that we are willing, as we should be, to make a slight sacrifice in the generality of the theorem. But some simple preliminaries are necessary. First, examples should be given to show that when $u(x)$ is an elementary function of the type familiar to the student Δu can be zero when Δx is not zero, but it cannot be zero if Δx is sufficiently small. Second, it should be shown graphically that $u(x)$ would have to be a most unusual type of function (unless it is a constant) if Δu could be zero regardless how small Δx may be. Third, it should be shown why Δu must not be permitted to equal zero if the limit of $\Delta y/\Delta u$ is to exist and equal dy/du.

With these preliminaries out of the way we can proceed to the proof. If x_1 is the value of x under consideration, and u_1 the corresponding value of u, all that is now necessary beyond what is presented in textbooks is to agree to consider only functions $u(x)$ for which Δu is not zero if x is sufficiently close to x_1 but unequal to it. That is, we state and prove the chain rule only for such functions. Thus, we only exclude functions taking on the value u_1 infinitely many times in the neighborhood of x_1. Except for the trivial case when $u(x)$ is constant in this neighborhood, such excluded functions never occur in first year work, and but rarely in the following few years. The student should be told that the chain rule also applies to these functions but that the proof of this fact is best delayed until such time as he works with them.

BRIEF REMARK ON TEACHING DIFFERENCE EQUATIONS

V. C. Harris, San Diego State College

The student of difference equations who has studied differential equations notices the many analogous features of the two. However, the substitutions used in solving for the complementary functions of certain equations, namely, of e^{mx} in one case and of a^x in the other, may *look* different. If the student is not fully convinced of the similarity in the procedures by the statement that $e^{mx}=(e^m)^x=a^x$, then go through the steps showing how the substitution of a^x can be used in solving differential equations and e^{mx} in solving difference equations.

A USEFUL REDUCTION INTEGRAL

J. W. CAMPBELL, University of Alberta

The following integral is of frequent occurrence in Calculus:

If p, q, m, n are integers, and in addition m and n are >1, then

$$\int_{p\pi/2}^{q\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \cdot \int_{p\pi/2}^{q\pi/2} f(\theta)\, d\theta$$

where
$$f(\theta) = \begin{cases} \sin\theta\cos\theta, & \text{if } m \text{ and } n \text{ are both odd,} \\ \sin\theta, & \text{if } m \text{ is odd and } n \text{ even,} \\ \cos\theta, & \text{if } m \text{ is even and } n \text{ odd,} \\ 1, & \text{if } m \text{ and } n \text{ are both even,} \end{cases}$$

and in the numerical coefficient the number of factors in the denominator and the number of factors in the numerator are equal.

The proof is simple, and the form given has several advantages.

1. The proof is based on the integration by parts

$$\int \sin^m \theta \cos^n \theta \, d\theta = \frac{-\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n \theta \, d\theta.$$

This equation implies that

$$\int_{p\pi/2}^{q\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n}\int_{p\pi/2}^{q\pi/2} \sin^{m-2}\theta \cos^n \theta \, d\theta,$$

and there is a corresponding form for reducing the exponent on $\cos\theta$.

2. It is easily remembered, for the form of statement keeps in evidence the principle that is being used.

In application one can write down the coefficient as the quotient of two products, followed by the integral for that case.

Thus

$$\int_{\pi/2}^{2\pi} \sin^7 \theta \cos^5 \theta \, d\theta = \frac{6.4.2.4.2}{12.10.8.6.4}\int_{\pi/2}^{2\pi} \sin\theta\cos\theta\, d\theta$$
$$= \frac{6.4.2.4.2}{12.10.8.6.4}\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{2\pi}$$
$$= -\frac{1}{120}.$$

Another advantage of the form is that the interval of integration $p\pi/2$ to $q\pi/2$ is general, whereas in the forms given in many Calculus texts as Wallis's formulae the interval of integration is 0 to $\pi/2$. The form here given, however,

covers all three of Wallis's forms as special cases, and in addition it applies, without modification, to whatever interval of integration turns up in the application being made.

ON A CHAINOMATIC ANALYTICAL BALANCE

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In the type of chainomatic analytical balance shown in Figure 1, one end of a flexible gold chain is attached to one end of the cross beam and the other end is attached to a slider on a vertical bar which can be moved up and down the bar by means of a crank. The scale on the vertical bar is linear and consequently there is a vernier attachment.

On asking many chemists why the scale was linear, the most frequent response was "it's obvious." The purpose of this note is to determine the character of the vertical scale.

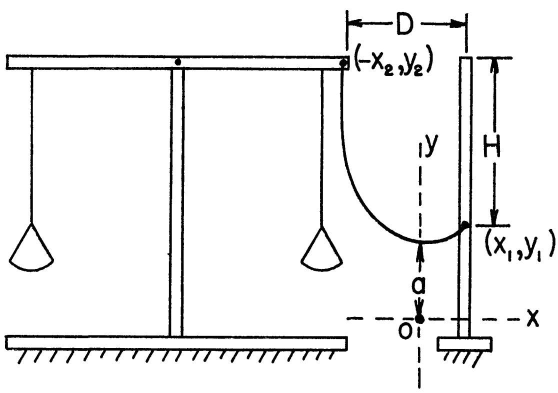

FIG. 1

Let the length and density of the chain be denoted by L and ρ, respectively Also, let D denote the distance that the vertical bar is from the end of the cross beam, and let H denote the distance that the moving end of the chain is below the cross beam. Then since the chain forms a catenary, the following equations must hold:

$$a \sinh \frac{x_2}{a} + a \sinh \frac{x_1}{a} = L, \tag{1}$$

$$a \cosh \frac{x_2}{a} - a \cosh \frac{x_1}{a} = H, \tag{2}$$

$$x_2 + x_1 = D, \tag{3}$$

$$W = \rho a \sinh \frac{x_2}{a} \cdot \tag{4}$$

Here, the equation of the catenary is taken in the form $y=a \cosh x/a$, and the coordinates of the fixed end and the movable end of the chain are taken as $(-x_2, y_2)$, and (x_1, y_1), respectively. W represents the weight added to the reading of the balance due to the chain.

The problem now is to eliminate x_1, x_2, and a from the four equations. If equations (1) and (2) are rewritten in the form

$$2a \sinh\left[\frac{x_2 + x_1}{2a}\right] \cosh\left[\frac{x_2 - x_1}{2a}\right] = L, \tag{5}$$

$$2a \sinh\left[\frac{x_2 + x_1}{2a}\right] \sinh\left[\frac{x_2 - x_1}{2a}\right] = H, \tag{6}$$

it then follows that

$$\tanh\left[\frac{x_2 - x_1}{2a}\right] = \frac{H}{L}, \tag{7}$$

and

$$2a \sinh\left[\frac{D}{2a}\right] = \sqrt{L^2 - H^2}. \tag{8}$$

Consequently,

$$W = \rho a \sinh\left[\frac{D}{2a} + \tanh^{-1}\frac{H}{L}\right] = \frac{\rho L}{2} + \frac{\rho H}{2} \coth\left[\frac{D}{2a}\right]. \tag{9}$$

It does not follow from (9) that W is linear in H, since a is a function of H as given by (8). However, it will be shown that by suitably restricting the values of L, H, and D, W can be made to be approximately linear in H, the approximation being sufficiently good for the purpose of the balance (*i.e.*, to weigh to a tenth of a milligram).

To find $D/2a$ from equation (8) we plot

$$y = \sinh x, \quad \text{and} \quad y = x\frac{\sqrt{L^2 - H^2}}{D}. \tag{10}$$

Here $x=D/2a$. If H is restricted to the range $0 \leqq H \leqq L/2$ and $D \ll L$, then it follows from the plot of (10) that $a \ll D$ (the solution $a=\infty$ does not apply here). Now

$$\coth\left[\frac{D}{2a}\right] = 1 + \frac{2e^{-D/a}}{1 - e^{-D/a}}, \tag{11}$$

and since $1 \gg 2e^{-D/a}[1-e^{-D/a}]^{-1}$, it follows that W is approximately linear in H.

If for example $L=10D$, then $D/2a > 4.3$ (for $H=L/2$), and

$$e^{-D/a}[1 - e^{-D/a}]^{-1} < \frac{1}{5000}\cdot$$

Since the chain is designed to weigh from 0.0000 to 0.1000 grams (by adjusting ρL), the error in assuming a linear scale (which allows the use of a vernier) is less than 0.00004 grams which is negligible for the purpose of the balance.

Remark. This example tends to indicate that chemists as well as mathematicians are prone to over-use the word "obvious".

A PROOF OF LEGENDRE'S DUPLICATION FORMULA

S. K. LAKSHMANA RAO, Indian Institute of Science

The following proof of the formula

$$\Gamma(2s) = (2^{2s-1}/\sqrt{\pi})\Gamma(s)\Gamma(s+\tfrac{1}{2})$$

by the use of the Mellin Transform is interesting.

The Mellin Transform of a function $f(x)$ of one variable, defined for positive values of the argument, is by definition $Mf(x) = \int_0^\infty f(x)x^{s-1}dx$. We have the Faltung theorem,*

$$Mf_1(x)\cdot Mf_2(x) = Mg(x) \tag{1}$$

where

$$g(x) = \int_0^\infty f_1\left(\frac{x}{u}\right)f_2(u)\,\frac{du}{u}\cdot$$

Now take

$$f_1(x) = e^{-x}, \qquad f_2(x) = e^{-x}x^{1/2}.$$

Then

$$Mf_1(x) = \Gamma(s), \qquad Mf_2(x) = \Gamma(s+\tfrac{1}{2}) \tag{2}$$

and

$$g(x) = \int_0^\infty e^{-x/u}e^{-u}u^{1/2}\,\frac{du}{u} = \int_0^\infty e^{-(x/u+u)}\,\frac{du}{u^{1/2}},$$

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* Sneddon, I. N., Fourier Transforms, McGraw-Hill, 1951, p. 43.

Readers interested in Laplace, Fourier, Hankel, Mellin, and other transforms are also referred to the series of books currently being prepared by the Staff of the Bateman Manuscript Project at the California Institute of Technology and published by McGraw-Hill; in particular, Tables of Integral Transforms, 1954.

the value of which is easily seen to be $\sqrt{\pi}\, e^{-2\sqrt{x}}$. For

$$g(x) = 2\int_0^\infty e^{-(x/t^2+t^2)}dt = 2\int_0^\infty e^{-(x/t^2+t^2)} \frac{x^{1/2}}{t^2}\, dt,$$

where the last integral is derived from the previous one by changing t to $x^{1/2}/t$. Therefore

$$g(x) = \int_0^\infty e^{-(x/t^2+t^2)}\left(1 + \frac{x^{1/2}}{t^2}\right) dt.$$

Now put $t - x^{1/2}/t = u$. Then

$$g(x) = \int_{-\infty}^\infty e^{-(u^2+2x^{1/2})}du = e^{-2x^{1/2}}\int_{-\infty}^\infty e^{-u^2}du = \sqrt{\pi}e^{-2x^{1/2}},$$

$$Mg(x) = \sqrt{\pi}\int_0^\infty e^{-2x^{1/2}}x^{s-1}dx.$$

Finally set $2x^{1/2} = y$, $dx = y\,dy/2$, and obtain

$$Mg(x) = \sqrt{\pi}\int_0^\infty e^{-y}\left(\frac{y}{2}\right)^{2s-1} dy = \frac{\sqrt{\pi}\Gamma(2s)}{2^{2s-1}}.$$

Therefore from (1) and (2)

$$\Gamma(2s) = (2^{2s-1}/\sqrt{\pi})\Gamma(s)\Gamma(s + \tfrac{1}{2}).$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1151. *Proposed by C. S. Ogilvy, Hamilton College*

Resolve the following paradox. The area between the curve $y = 1/x$ and the x-axis, to the right of the line $x = 1$, is infinite. Yet the volume generated by rotating this area about the x-axis is π. Thus it would require an infinite quantity of paint to cover the area; yet the volume, which completely contains and surrounds the area, could be filled with π cubic units of paint!

E 1152. *Proposed by J. J. Bowers, Student, Wesleyan University*

If, in the postulates of a field, the distributivity of multiplication over addition is replaced by distributivity of addition over multiplication, is the resulting set of postulates consistent?

E 1153. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For any angle θ, show that arbitrarily small constructible angles ϕ exist such that $(\theta-\phi)$ can be trisected.

E 1154. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The distance from the midpoint of side AB of a regular convex heptagon $ABCDEFG$, inscribed in a circle, to the midpoint of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle.

E 1155. *Proposed by Donald Bratton, Cowles Commission, Chicago*

There is a group of people who habitually receive parcels. When one of these persons x receives a parcel addressed to y, he relays it to $f(x, y)$. The function f is such that a parcel will eventually reach the person to whom it is addressed, no matter who first receives it.

One sees that a person x for which $f(x, y)$ takes on only a single value as y runs over all the values $\neq x$, is not very important. So let us call the *valence* of x the number of values assumed by $f(x, y)$ for $y \neq x$, less one. The unimportant people are thus those with zero valence. Denote by V the sum of all the valences in the group.

It is easy to show that, if $V=0$, a parcel can be addressed and started out in such a way that each person in the group will handle it. On the other hand, can you prove that, if $V\neq 0$, $V+1$ parcels can be addressed and started out so that each person in the group will handle at least *two* of these parcels?

SOLUTIONS

A Property of Any Collection of Integers

E 1121 [1954, 423]. *Proposed by W. D. Serbyn, Carnegie Institute of Technology*

Let E be any collection of n integers, not necessarily distinct. Show that there exists a non-empty subcollection $F \subseteq E$ such that the sum of the integers contained in F is divisible by n.

Solution by L. R. Ford, Illinois Institute of Technology. Let $a_1, \cdots, a_n$ be the integers of E. Of the $n+1$ integers $0, a_1, a_1+a_2, a_1+a_2+a_3, \cdots, a_1+\cdots+a_n$ two at least are congruent modulo n. Their difference, which is the sum of the integers in a subset of E, is divisible by n.

Also solved by J. T. Ahlin, J. L. Botsford, W. E. Briggs, A. L. Epstein, S. H. Gould, D. S. Greenstein, B. A. Hausmann, A. R. Hyde, P. B. Johnson,

W. S. Loud, D. C. B. Marsh, Leo Moser, D. B. Mumford, J. H. Oppenheim, F. D. Parker, J. V. Pennington, R. R. Phelps and J. L. Selfridge (jointly), L. L. Scott, R. P. Tapscott, and the proposer. Late solution by T. F. Mulcrone.

Moser pointed out that this problem is a very special case of his problem 4300. See the remark, following the solution, by J. B. Kelly [1950, 47]. Problem 4300 states that every set of n (not necessarily distinct) elements of a finite group of order n contains a subset of elements whose product is the unit element.

Stacking Cards

E 1122 [1954, 423]. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Perfectly rigid playing cards are piled on the edge of a table with the pile slanting up away from the table. How far from the edge of the table can the pile be made to extend without falling to the floor?

I. *Solution by Michael Goldberg, Washington, D. C.* Let a_i represent the distance the ith card from the top extends beyond the edge of the table. The limiting position which a succession of cards may take is one in which the center of gravity of any top set is directly above the edge of the card next below. Hence

$$(a_1 - 1/2) + (a_2 - 1/2) + \cdots + (a_k - 1/2) = ka_{k+1}.$$

From this equation it follows that

$$a_1 = a_2 + 1/2,\ a_2 = a_3 + 1/4,\ \cdots,\ a_k = a_{k+1} + 1/2k.$$

Therefore, for n cards, the distance a_1 is given by

$$a_1 = (1/2)(1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n).$$

As n increases without limit, a_1 increases without limit, because the harmonic series is divergent.

II. *Solution by Albert Wilansky, Lehigh University.* The cards can extend any distance. We see this thus: Place one card on top of a vertical stack so that its edge extends out over the top; the vertical stack is assumed to have so many cards in it that the center of gravity of the system is a certain small distance ϵ, measured horizontally, from the center of the vertical stack. Now place the whole system on top of a vertical stack with the bottom of this system projecting over the edge of this lowest vertical stack; the lowest stack is assumed to have so many cards in it that the center of gravity of the whole (new) system is horizontally within ϵ of the center of the bottom card. This may be kept up indefinitely and eventually the edge of the highest card will be any desired distance from the edge of the lowest, and the center of gravity of the system will be horizontally within ϵ of the center of the lowest card.

If the cards are of zero thickness the highest card will be at the same level as the lowest. The system will then look like a long thin board miraculously

balanced near one end, the secret being that the board, like a child's toy, is denser at one end than the other.

Also solved by S. H. Eisman, A. R. Hyde, P. G. Kirmser, L. V. Mead, Leo Moser, George Mott, D. B. Mumford, C. S. Ogilvy, C. R. Perisho, Azriel Rosenfeld, and the proposer. Late solution by M. S. Klamkin.

Moser pointed out that this problem is essentially the same as problem 52 of the April 1954 *Pi Mu Epsilon Journal.*

Several solvers noted that, by the method of stacking of solution I, the top card of a deck of 52 cards will project approximately 2.268 card lengths beyond the edge of the table.

Polygonal Path Covering a Square Lattice

E 1123 [1954, 423]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N\times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

Solution by the Proposer. Assume that it can be done for $N=K$ in such a fashion that we end up in position E as shown in Figure 1. Then, as shown in

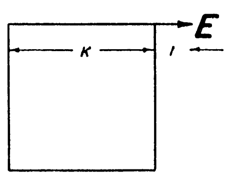

FIG. 1

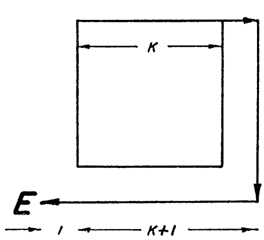

FIG. 2

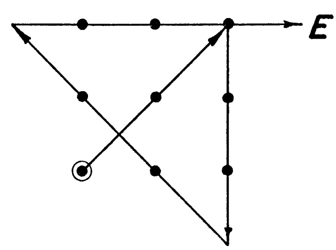

FIG. 3

Figure 2, by drawing two more lines it can be done in the same fashion for $N=K+1$. Figure 3 shows that it can be done in this fashion for $N=3$. Thus, by induction, it can be done for all $N\geqq 3$.

Also solved by G. C. Bush, Michael Goldberg, Dudley Herschbach, and D. C. B. Marsh.

No one successfully answered the question at the end of the problem.

Iterated Exponents

E 1124 [1954, 423]. *Proposed by L. J. Lander and J. L. Selfridge, U. C. L. A.*

Let $(a;0)=1$ and $(a;n+1)=a^{(a;n)}$. Find all solutions of $(a;m)=(b;n)$ in integers a, b, m, n all greater than 1.

Solution by the Proposers. Clearly $a=c^r$ and $b=c^s$. We can choose c so that $(r, s)=1$. The exponents of c in the given equation are $r(a;m-1)=s(b;n-1)$.

We may assume $r \leqq s$, and it follows that $s = rc^d$ so that $r=1$, $c=a$, and $b=a^{a^d}$. The exponents of a in the equation $(a;\ m-1) = a^d(b;\ n-1)$ are $(a;\ m-2) = d + a^d(b;\ n-2)$. Now assume that k is the largest integer such that $(a;\ k) \mid d$. Clearly a^d and hence $(a;\ m-2)$ are divisible by $(a;\ k+1)$; *i.e.*, there is no such integer k. This can occur if $a=1$ or $d=0$. Since $a>1$, we have $a=b$ and $m=n$.

Stirling Numbers

E 1125 [1954, 423]. *Proposed by Walter James, University of Minnesota*

Determine the coefficients B_j^p for the following sum:

$$\sum_{s=1}^{n} s^p = \frac{1}{(n-1)!} \sum_{j=1}^{p} \frac{B_j^p (n+j)!}{j+1} .$$

Solution by A. R. Hyde, West Hartford, Conn. If S_n represents the sum of n terms, then

$$(S_n - S_{n-1})/n = n^{p-1} = B_1^p + (n+1)B_2^p + (n+1)(n+2)B_3^p + \cdots + (n+1) \cdots (n+p-1)B_p^p.$$

By setting n equal successively to $-1, -2, \cdots, -(p-1)$, the constants may be determined so that this is an identity. Then

$$B_j^p = \frac{1}{(j-1)!} \sum_{k=1}^{j} (-1)^{p+k}\ {}_{j-1}C_{k-1} k^{p-1} = \sum_{k=1}^{j} (-1)^{p+k} \frac{k^{p-1}}{(j-k)!(k-1)!} .$$

Values of B_j^p for $j, p \leqq 9$ are as follows:

$p \diagdown$ $j=1$	1	2	3	4	5	6	7	8	9
1	1	·	·	·	·	·	·	·	·
2	-1	1	·	·	·	·	·	·	·
3	1	-3	1	·	·	·	·	·	·
4	-1	7	-6	1	·	·	·	·	·
5	1	-15	25	-10	1	·	·	·	·
6	-1	31	-90	65	-15	1	·	·	·
7	1	-63	301	-350	140	-21	1	·	·
8	-1	127	-966	1701	-1050	266	-28	1	·
9	1	-255	3025	-7770	6951	-2646	462	-36	1

Also solved by H. W. Gould, A. S. Grant, D. C. B. Marsh, Chih-yi Wang, and the proposer. Late solution by M. S. Klamkin.

Wang showed that if p is odd $B_j^p = (-1)^{j-1} S_p^j$ and if p is even $B_j^p = (-1)^j S_p^j$, where S_p^j is a Stirling number of the second kind. The calculation of these

Stirling numbers is indicated in Charles Jordan, *Calculus of Finite Differences*, and are there tabulated for $j, p \leqq 12$.

Hyde's table is easily calculated and extended if we note (as did the Proposer) that $B_j^p = (-1)^{p+j} A_j^p$, where $A_j^p = jA_j^{p-1} + A_{j-1}^{p-1}$ and $A_1^1 = 1$. Thus, after putting 1's in the first column and along the diagonal, the jth element of any row will numerically equal j(number immediately above)+(number immediately above to the left). For example, in Hyde's table, $90 = 3(25) + 15$, $1050 = 5(140) + 350$, *etc.* The signs are then taken care of by making the terms of the even diagonals negative.

In addition to solving the problem, Gould compared several known alternative methods for expanding $\sum_{s=1}^n s^p$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4623. *Proposed by L. L. Pennisi, University of Illinois, Navy Pier, Chicago*

Prove that

$$\lim_{x\to\infty} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{3\cdot 5 \cdots (2n+1)}}{\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2\cdot 4 \cdots 2n}} = \sqrt{\frac{\pi}{2}}.$$

4624. *Proposed by H. S. Shapiro, New York University*

Let J_p denote the field of integers modulo p, where p is a prime, and $J_p[x]$ the ring of polynomials with coefficients in J_p. Then the number of monic irreducible polynomials of degree n in $J_p[x]$ is equal to

$$\frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}$$

where μ is the Möbius function.

4625. *Proposed by K.-F. Moppert, Basle, Switzerland*

For $a>0$, $b>-1$, prove

$$\frac{1}{a}-\binom{b}{1}\frac{1}{1+a}+\binom{b}{2}\frac{1}{2+a}-+\cdots=\frac{\Gamma(a)\Gamma(1+b)}{\Gamma(1+a+b)}.$$

4626. *Proposed by W. A. Michael and D. A. Page, University of Illinois, Urbana*

Let A be a subset of the plane. Join each pair of distinct points of A by a line. Let A_1 consist of the points of A together with the points of intersection of distinct lines. Starting with A_1 the same construction yields a set A_2. In this way an increasing sequence of sets $A_1, A_2, \cdots$, is obtained. Let $\Omega=\cup A_n$. If A is denumerable then so is Ω, and hence Ω is not the entire plane; however, under what conditions will Ω be dense in the plane?

4627. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

In Chrystal, *Textbook of Algebra*, vol. 2, p. 225, there is the following theorem on the representation of an irrational number:

The number represented by the series

$$\sum_{n=1}^{\infty}\frac{p_n}{r_1r_2\cdots r_n}$$

is irrational provided that

(1) r_n and p_n are integers such that $0<p_n<r_n$,

(2) $r_{n+1}>r_n>1$,

(3) the sequence $\{r_1r_2\cdots r_n\}$ includes all powers of the primes.

(A) Construct a counter-example, *i.e.*, a number of the stated form which is rational even though conditions (1), (2) and (3) are satisfied.

(B) Complete the list of conditions (1), (2) and (3) so that the theorem is indeed valid.

SOLUTIONS

Two Triangles Inscribed in a Conic

4562 [1953, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, let P be a point having normal coördinates (x, y, z) and consider the points A', B', C' with normal coördinates $(-x/2, y, z)$, $(x, -y/2, z)$, $(x, y, -z/2)$. (1) The points A, B, C, A', B', C' lie on one conic S, and there is a conic with respect to which the triangles ABC and $A'B'C'$ are self-polar. (2) If AP, BP, CP cut BC, CA, AB in A_1, B_1, C_1 and if $A'P, B'P, C'P$ cut $B'C', C'A', A'B'$ in A_1', B_1', C_1', the triangles $ABC, A'B'C'$ are circumscribed about a conic Σ, the points of tangency being A_1, B_1, C_1 and A_1', B_1', C_1'. (3) The conics S and Σ

have double contact along the common polar of P with respect to these conics.

Solution by Roscoe Woods, State University of Iowa. Let (X, Y, Z) denote current coördinates. The equation of any conic, K, circumscribing the reference triangle is of the form $l/X+m/Y+n/Z=0$, where l, m, n are constants. Also the equation of any conic, K', inscribed in the reference triangle is of the form $\sqrt{LX}+\sqrt{MY}+\sqrt{NZ}=0$, where L, M, N are constants and where the ambiguous signs before the radicals are omitted for convenience. Consider now the conic K'': $(L/l)X^2+(M/m)Y^2+(N/n)Z^2=0$ for which the reference triangle is self-polar. It is easily verified that the conics K and K' are the polar reciprocals of one another with respect to the conic K''. (See C. Smith, *Conic Sections*, 1914, p. 384.)

A short calculation shows that the conic S of the problem has the equation $x/X+y/Y+z/Z=0$ and the conic for which the triangles ABC and $A'B'C'$ are self-polar has the equation $X^2/x^2+Y^2/y^2+Z^2/z^2=0$. It is also readily seen that the conic Σ referred to in the problem has the equation $\sqrt{X/x}+\sqrt{Y/y}+\sqrt{Z/z}=0$.

Part (3) is answered through use of the following theorem: *If two conics have double contact, there is a linear combination of their equations which consists of the square of the equation of their chord of contact.* Now

$$\Sigma + 4S = (X/x + Y/y + Z/z)^2 = 0,$$

where $X/x+Y/y+Z/z=0$ is the common polar of $P(x, y, z)$ with respect to the conics Σ and S. It should be pointed out that this common polar is the trilinear polar of P with respect to the triangle ABC, being the axis of perspective of the triangles ABC and $A_1B_1C_1$.

Also solved by the Proposer.

Editorial Note: The results of this problem may be obtained easily from known theorems as given in Coxeter, *The Real Projective Plane*, p. 80:

6:71. *If two triangles are self-polar for a given polarity, their six vertices lie on a conic and their six sides touch another conic.*

6:72. *Any two triangles inscribed in a conic are self-polar for some polarity.*

A Sequence Almost Contained in a Set of Sequences

4563 [1953, 716]. *Proposed by Albert Edrei, Syracuse University*

Let

$$\alpha_1, \alpha_2, \alpha_3 \cdots \tag{1}$$

be an infinite, strictly increasing sequence of positive integers, and let $B=\{b\}$ be a set of such sequences. A sequence of the form (1) is said to be *almost contained* in B if, to every b, there corresponds an integer $k=k(b)$ such that α_k, α_{k+1}, α_{k+2}, $\cdots$ is a subsequence of b.

The following theorem is easy to prove: *Let B be denumerable and such that*

the intersection of any finite number of b's is an infinite sequence. Then there exists an infinite sequence almost contained in B.

The problem is to prove that this theorem is false if the word denumerable is omitted.

Solution by Paul Erdös, University of Notre Dame. If one takes B to be the class of all sequences of integers of density 1, one obtains a solution. Clearly the intersection of any finite number of sequences of density 1 again has density 1 and is thus infinite. On the other hand clearly no infinite sequence is almost contained in B since if $a_1 < a_2 < \cdots$ is an infinite sequence, it contains an infinite subsequence $b_1 < b_2 < \cdots$ of density 0, and the complement of the b's has density 1 and does not almost contain the a's.

Also solved by B. J. Ball, G. E. Bredon, Hewitt Kenyon, O. Mourmaki, J. D. Reid, and the Proposer.

Another Summation

4564 [1953, 716]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Prove

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \frac{\pi^4}{72}, \quad \text{where} \quad S_n = \sum_{r=1}^{n} \frac{1}{r} \cdot$$

I. *Solution by J. V. Whittaker, University of California, Los Angeles.* We have, after rearrangement of the terms of the series,

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{m} \frac{1}{nm^3} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(m+n)^3} \cdot$$

We first notice that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2(m+n)^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2} - \sum_{m=1}^{\infty} \sum_{n=1}^{m} \frac{1}{m^2n^2} = \frac{\pi^4}{120} \cdot$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n(m+n)^3} = \sum_{n=1}^{\infty} \left\{ \frac{1}{m^3n} - \frac{1}{m^3(m+n)} - \frac{1}{m^2(m+n)^2} - \frac{1}{m(m+n)^3} \right\}$$

$$= \sum_{n=1}^{m} \frac{1}{m^3n} - \sum_{n=1}^{\infty} \left\{ \frac{1}{m^2(m+n)^2} + \frac{1}{m(m+n)^3} \right\} \cdot$$

Finally, summing over m from 1 to ∞, we find that

$$\left(S - \frac{\pi^4}{90}\right) = S - \frac{\pi^4}{120} - \left(S - \frac{\pi^4}{90}\right),$$

or $S = \pi^4/72$.

II. *Solution by the Proposer.* This result is a special case of a theorem due to G. T. Williams (this MONTHLY, 1953, p. 25) which may be put in the form

$$2\sum_{n=1}^{\infty}\frac{S_n}{n^p}=(p+2)\zeta(p+1)-\sum_{j=1}^{p-2}\zeta(j+1)\zeta(p-j).$$

For $p = 3$, we have

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}=\frac{1}{2}\left\{5\zeta(4)-\zeta^2(2)\right\}=\frac{\pi^4}{72}.$$

It may be obtained as easily from a result due to D. H. Browne in connection with Problem no. 4431 [1952, 472].

Also solved by W. E. Briggs and S. Chowla and P. C. Rosenbloom, Leonard Carlitz, A. E. Livingston, O. E. Stanaitis, Ernst Trost, and Chih-yi Wang.

Unit Element of a Valuation Ring

4565 [1953, 716]. *Proposed by Donald Bratton, University of Chicago*

A ring A is called a valuation ring when it is commutative and, for each couple of elements (x, y) of A, either x divides y or y divides x. Show that each valuation ring has a unit element.

Solution by D. W. Sasser, Student, Yale University. For any non-zero element x, there is a y such that $xy=x$. Every non-zero element z of A is either a multiple or a divisor of x. If $cx=z$, then $zy=cxy=cx=z\neq 0$. If $x=dz$, then $dzy=xy=x\neq 0$, whence $zy\neq 0$. Thus y is not a divisor of zero.

Now there is an a such that $ya=y$. Then for each z in A, $y(az-z)=0$, whence $az=z$. Thus a is a unit element.

Also solved by Rafael Chacon and Simon Hellerstein, I. S. Cohen and Oscar Goldman, C. H. Denbow, W. E. Deskins, M. P. Epstein, D. C. B. Marsh, R. S. Pierce, J. D. Reid, Alex Rosenberg, D. W. Sasser, W. R. Scott and Elbert Walker, David Shale, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Differential equations with applications. By Herman Betz, P. B. Burcham, and G. M. Ewing. New York, Harper, 1954. 10+310 pp. $4.50.

To quote from the preface "this is a text for an introductory course in differential equations, emphasizing particularly applications to the physical and

biological sciences and to engineering." Indeed this book has more than the usual complement of examples from the sciences, and in writing a text for the student oriented toward the applications, the authors have succeeded admirably. From the mathematical standpoint stress is placed on a variety of methods which work. For example, the formalities of the Laplace transform method are included. Questions of existence, uniqueness, and convergence receive little attention. On the other hand, the geometry of solutions is treated in detail in some cases; an example being a discussion of a limit cycle of a nonlinear equation of the van der Pol type.

An introductory chapter, which shows how differential equations arise, is followed by one on equations of the first order and degree, including Ricatti's equation. The third chapter is devoted to a large number of applications of first order equations. For example, the law of mass action is discussed in detail, complete with illustrative chemical formulas. The mutual relationship between sharks and soles in the Adriatic, which Volterra translated into a pair of equations of the first order, is given a geometric treatment. Methods of solving some implicit equations are presented in Chapter 4. Linear equations are next studied, with the "operator method" used for equations with constant coefficients. (The inverse operator introduced on page 94 does not seem to be uniquely defined.) Mechanical and electrical applications of linear equations with constant coefficients are treated in some detail. Formal properties of the Laplace transform are given and illustrated by electric circuit and beam problems. Special types of higher order equations are discussed in Chapter 8. Series solutions are introduced in Chapter 9, along with an optional section on regular singular points of second order linear equations. Chapter 10, which is devoted to graphical and numerical methods, includes the example of a limit cycle, and Runge's method. The last two chapters are on partial differential equations, Fourier series being used for heat flow and vibration problems. An appendix on hyperbolic functions is given.

Topics omitted which are of interest include the geometry of solutions of a pair of first order linear equations with constant coefficients, and boundary value problems involving Laplace's equation in two dimensions. However, the book contains more than enough for a semester's course.

There is a large number of exercises, and references for supplementary reading are given at the end of many of the chapters. It is unfortunate that references were not given for the last chapter.

The printing is excellent; no typographical errors were noticed. There might be some confusion between the notation $(\partial E/\partial S)_v$ on page 245 and $(\partial \theta/\partial x)_x$ on page 259.

E. A. Coddington
University of California at Los Angeles.

Elementary Theory of Numbers. By Harriet Griffin. New York, McGraw-Hill Book Company, 1954. ix+203 pages. $5.00.

This book stresses the elementary topics in the arithmetic theory of numbers and is written primarily for the undergraduate or immature graduate student. The author states in the preface that the presentation has been classroom tested and has proved successful. Most of the theorems are proved in detail. There are, however, some proofs which must be furnished by the careful reader. Most of the theorems are illustrated by examples, a feature that will appeal especially to the undergraduate. The problems are numerous and the instructor will find among them challenges for students of all abilities.

The chapter headings are as follows: 1. The Fundamental Laws; 2. The Linear Diophantine Equation; 3. Properties of Integers; 4. Properties of Congruences; 5. The Solution of Congruences; 6. The Theorems of Fermat and Wilson and the Möbius Function; 7. On Belonging to an Exponent; 8. Indices; 9. Quadratic Residues; 10. Some Famous Problems; 11. Polynomials; 12. Partitions. One unusual aspect of this ordering is that the Euclidean algorithm appears in Chapter 3, after the chapter on linear diophantine equations. The method of solution given in Chapter 2 is essentially equivalent to the method which uses the Euclidean algorithm. Chapters 5 to 9 are perhaps more exhaustive in the treatment of the topics involved than most elementary texts on number theory. The Waring problem, the equation $x^2+y^2=z^2$, and Fermat's last theorem are among the problems discussed in the chapter on famous problems. The chapter on polynomials begins with sets of postulates for integral domains and fields and continues with the development of some of the elementary properties of the domain of polynomials over a field. The additive theory of numbers is introduced very briefly in the final chapter.

Throughout the book brief historical highlights concerning the topic being discussed are cited. Another feature is the frequent footnote reference to periodical literature, some very recent, and most of which is within the limitations of the average student. It is the opinion of this reviewer that the book presents in an interesting manner adequate background material for students wishing to go on to advanced work.

No serious typographical errors were noted but a consistent omission of punctuation marks at the ends of mathematical expressions was noted.

V. J. VARINEAU
University of Wyoming

An Introduction to the Calculus of Finite Differences. By C. H. Richardson. New York, D. Van Nostrand Co., Inc., 1954. 6+142 pages. $5.00.

This book is precisely what its title indicates. It is an introduction to finite differences. Its contents are based on those of a semester course given by the author as an elective for juniors and seniors. It was designed primarily for students in actuarial and statistical theory at this level but it should also be of value to other undergraduates interested in applied mathematics.

The material of the book is divided into six chapters and two short appendices. The first chapter is introductory in nature and treats the basic definitions, difference formulas, and symbolic operators. The initial definitions are given for a general $\Delta x = h$ but after page 2 the discussion is limited to $\Delta x = 1$.

Chapter 2 discusses finite integration, summation of series, Stirling's numbers, and the differences of zero. Chapter 4, consisting of 24 pages, is devoted to interpolation and approximate integration. A brief introduction to the interpolation formulas of Newton, Gauss, Stirling, Bessel, and Lagrange is followed by some simple quadrature formulas and the Euler-Maclaurin sum formula.

A short Chapter 5 is devoted to Beta and Gamma functions while Chapter 6 gives an introduction to the treatment of difference equations. Brief appendices appear on mathematical induction and hyperbolic functions. (The only prerequisite assumed is an elementary course in infinitesimal calculus.)

The book is concisely yet clearly written in a style which is appropriate to the undergraduate. Occasional references are given for the benefit of the student who may wish to study some topic in detail. The illustrations are adequate and the exercises more than ample for the purpose. The author and the publisher should be commended for producing this book which provides a concise elementary text in finite differences.

P. S. Dwyer
University of Michigan

NEW BOOKS RECEIVED

Tables of Integral Transforms, Vol. II. By the Bateman Project Staff. Editor, A. Erdelyi. New York, McGraw-Hill Book Company, 1954. xvi+451 pages. $8.00.

Der Mathematische Unterricht fur die sechzehn-bis einundzwanzig Jährige Jugend in der Bundesrepublik Deutschland. By Heinrich Behnke. Göttingen, Vandenhoeck and Ruprecht, 1954. 332 pages. 20 DM.

Schaum's Outline of Theory and Problems of Plane and Spherical Trigonometry. By Frank Ayres, Jr. New York, Schaum Publishing Company, 1954. 207 pages. $1.85.

Understanding Numbers: Their History and Use (A Telecourse Syllabus). By P. S. Jones. Ann Arbor, Braun-Brumfield; available at Ulrich's Bookstore, 547–549 E. University Avenue, Ann Arbor, Michigan. 1954. 50 pages. $1.00.

An Experimental Investigation of the Mechanics of Plastic Deformation of Metals. By E. G. Thomsen, C. T. Yang, and J. B. Vierbower. Berkeley 4, California, University of California Press, 1954. 55 pages. $.75.

Graphical Method of Statistical Inference. By M. Masuyama. Tokyo, Maruzen Company, Ltd., 1954. $2.50.

Table of Sine and Cosine Integrals for Arguments from 10 to 100. By U. S.

Department of Commerce, National Bureau of Standards (Applied Mathematics Series—32). Washington, D. C., Government Printing Office, 1954. 186 pages. $2.25.

Table of the Gamma Function for Complex Arguments. By U. S. Department of Commerce, National Bureau of Standards. (Applied Mathematics Series—34). Washington, D. C. Government Printing Office, 1954. 105 pages. $2.00.

Mathematics in Western Culture. By Morris Kline. New York, Oxford University Press, 1954. xii+484 pages. $6.00.

Technical Mathematics. By H. S. Rice and R. M. Knight. New York, McGraw-Hill Book Company, 1954. xiv+748 pages. $6.50.

Elements of Algebra. By Howard Levi. New York, Chelsea Publishing Company, 1954. 160 pages. $3.25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER INSTITUTES FOR HIGH SCHOOL AND COLLEGE TEACHERS

The National Science Foundation will sponsor three institutes for teachers of mathematics in the summer of 1955. Preliminary announcements of these appear below. In each case housing will be provided in university dormitories and a limited number of stipends will be available for participants.

University of Wisconsin. This institute for both high school and college teachers will be held during the period June 27 to July 23 under the direction of Professor C. C. MacDuffee, to whom inquiries should be directed. It will stress relations between the college mathematics teacher and the high school or preparatory school teacher. Lectures will be given for each of these groups separately and jointly. The problem of the education of teachers of mathematics will also receive attention.

Oklahoma A. and M. College. This institute for college teachers will be held during the period June 13 to July 22 under the direction of Professor L. Wayne Johnson, to whom inquiries should be directed. The principal lecturers will be Professor Arthur Rosenthal of Purdue University, whose topic will be "An Introduction to the Theory of Measure and Integration"; and Professor H. S. M. Coxeter of the University of Toronto, whose topic will be "Contributions of Geometry to the Main Stream of Mathematics." Part-time lecturers will be Professor A. W. Tucker of Princeton University, who will speak on "An Introduction to the Theory of Games"; Professor Saunders MacLane of the University

of Chicago, whose topic is "The Undergraduate Program in Mathematics in the University of Chicago"; and Professor G. B. Price of the University of Kansas, who will discuss "A Universal Course in Mathematics." The sessions of the Institute will be in air-conditioned buildings.

Stanford University. This institute for college teachers will be held during the period June 27 to August 19 under the direction of Professor Harold M. Bacon, P. O. Box 1144, Stanford, California, to whom inquiries should be directed. One of the two chief lecturers will be Professor George Pólya of Stanford University, who will speak on "Calculus—Its History, Application, and Teaching." The other main speaker will be Professor I. J. Schoenberg of the University of Pennsylvania, who will discuss a topic in the field of applied mathematics. There will be shorter series of lectures on "Numerical Analysis and Computing Machines" by Professor D. H. Lehmer and "The Problem of the Undergraduate Curriculum in Mathematics" by Professor C. B. Allendoerfer.

GRADUATE INSTITUTE FOR MATHEMATICS AND MECHANICS

Indiana University has established the Graduate Institute for Mathematics and Mechanics. It includes the members of the former Graduate Institute for Applied Mathematics and is an enlarged and strengthened body whose efforts are devoted to research and advanced graduate teaching in pure and applied mathematics. Students are required to have had at least one year's previous graduate study. The members of the regular staff are: Professors T. Y. Thomas (Director), V. Hlavatý, E. Hopf, C. Truesdell, J. W. T. Youngs; Associate Professors D. Gilbarg, W. Gustin. With the assistance of an international board of thirty specialists outside Indiana University, the Institute publishes the *Journal of Rational Mechanics and Analysis*, an international periodical for the fields of its title; it appears in bi-monthly issues making annual volumes of about 800 pages.

HONORARY MATHEMATICS FRATERNITY

Chapters of Pi Mu Epsilon, national honorary mathematics fraternity, may be chartered only in colleges and universities requiring at least eight semester hours beyond calculus for a mathematics major and having an average of at least five majors per year. The institution must have on its staff at least one person who has a Ph.D. degree *in mathematics* and must have had an active mathematics club for at least one year. Faculty members of institutions meeting these requirements may write to Professor R. V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma, for further information.

SYMPOSIUM ON MATHEMATICS

The Second Symposium on "Some Mathematical Problems being Studied in Latin America," organized by the Unesco Science Cooperation Office for Latin America, was held at Mendoza, Argentina, during July 21 to 25, 1954, under the sponsorship of the University of Cuyo.

The following papers were read at the meetings:

1. *Some questions on affine differential geometry of surfaces*, by L. A. Santaló (Argentina).
2. *Some questions concerning topological vector spaces*, by L. Nachbin (Brazil).
3. *Multivalued analytic functions of Fourier transforms*, by A. Calderón (Argentina), in collaboration with R. Arens.
4. *On some divergent integrals of quantum electrodynamics*, by A. G. Domínguez (Argentina).
5. *Hilbert transforms of distributions*, by J. Horvath (Colombia).
6. *The problem of moments and Hermitian operators*, by M. Cotlar (Argentina).
7. *Absolute form of the transformation of the equations of dynamics in a curved space of n dimensions*, by G. García (Peru).
8. *Hydrodynamics: Recent advances in the theory of free boundaries*, by E. H. Zarantonello (Argentina).
9. *On some analytic applications of the expansions in series of Legendre polynomials*, by M. O. González (Cuba).
10. *Arithmetic of filters and topological spaces*, by A. Monteiro (Argentina).
11. *The theory of topological tensor products*, by A. Grothendieck (France).
12. *Algebraic operations in topology and some applications to geometric problems*, by J. Adem (Mexico).
13. *On some generalizations of the theory of functions of a complex variable*, by E. Lammel (Argentina).
14. *The functional equations of the theory of magnitudes*, by P. Pi Calleja (Argentina).
15. *A new definition of random function and its ergodic theorem*, by G. Dedebant (Argentina).
16. *Reductive subgroups of algebraic Lie groups*, by G. Mostow (U.S.A.).
17. *Problems about the definition of logic truth in semantic and syntactic systems*, by G. Klimovsky (Argentina).
18. *Singular integrals*, by A. Calderón (Argentina).

PERSONAL ITEMS

Columbia University announces the following: Assistant Professor I. M. Singer of the University of California at Los Angeles has been appointed Visiting Assistant Professor; Assistant Professor Harish-Chandra has been promoted to an associate professorship and was invited to attend a special colloquium in celebration of the centennial birth of Henri Poincaré at the University of Paris, France, October 17–31, 1954.

Harvard University reports: Visiting Assistant Professor J. T. Tate, Jr. of Columbia University has been appointed to an assistant professorship; Dr.

B. M. Dwork, formerly a student at Columbia University, has been appointed Benjamin Peirce Instructor.

At Illinois Institute of Technology: Associate Professor Haim Reingold has been named Chairman of the Department of Mathematics; Dr. R. J. Silverman of the Armour Research Foundation has been appointed to an assistant professorship; Mr. H. L. Pearson, Dr. Pasquale Porcelli, and Mr. R. E. Seall have been appointed to instructorships.

Iowa State College announces: Dr. E. H. Feller of the University of Wisconsin has been appointed to an instructorship; Dr. W. D. Lindstrom, Dr. D. E. Sanderson, Dr. R. D. Stalley, and Dr. F. M. Wright have been promoted to assistant professorships.

Kent State University reports that Assistant Professors B. B. Dressler and R. Y. Iwanchuk have been promoted to associate professorships.

Laval University announces the following: Professor Alexandre LaRue was the delegate of the University at the International Congress of Mathematicians; Dr. Arthur Dubé has been appointed to a professorship.

At Louisiana State University: Assistant Professor Heron Collins of the University of South Carolina has been appointed to an assistant professorship; Assistant Professor E. G. Kundert has been promoted to an associate professorship.

Marquette University reports: Miss Miriam Connellan, Rev. L. J. Heider, Mr. J. E. Kelley, formerly an analyst with the Department of Defense, Miss Joan Salatino, Mr. Richard Schwaller, previously a graduate assistant at the University, and Mr. Earl Swokowski, formerly a graduate assistant at the University of Wisconsin, have been appointed to instructorships.

Montana State College announces: Mr. J. L. Simpson and Dr. Hans Sagan, formerly an assistant at the University of Vienna, have been appointed to instructorships; Dr. J. E. Whitesitt has been promoted to an assistant professorship; Assistant Professor A. L. Hess has been promoted to an associate professorship; Professor Frieda M. Bull has retired with the title Professor Emeritus; Mr. D. F. DeLap has retired.

At Northwestern University: Assistant Professor M. A. Rosenlicht has received a Fulbright award for research at the University of Rome; Assistant Professor George Springer has received a Fulbright award to lecture at the University of Munster, Germany, October 1954 to June 1955; Dr. W. A. Michael, Jr., previously an assistant at the University of Illinois, and Dr. W. T. Kyner, formerly an assistant at the University of California, have been appointed to instructorships; Visiting Professor J. A. E. Dieudonné has been promoted to a professorship; Assistant Professor Daniel Zelinsky has been promoted to the position of Associate Professor; Dr. J. C. E. Dekker, Dr. M. P. Gaffney, and Dr. Alex Rosenberg have been promoted to assistant professorships.

Ohio State University reports the following: Dr. J. E. Adney, Jr., formerly an assistant at the University, Dr. J. M. Shapiro, and Mr. Clifford Spector have

been appointed to instructorships; Dr. A. D. Ziebur has been promoted to an assistant professorship.

Oklahoma Agricultural and Mechanical College announces: Dr. George Marsaglia has been appointed to an assistant professorship; Assistant Professor F. A. Graybill has been promoted to an associate professorship.

At Purdue University: Dr. L. J. Cote, formerly a graduate student at Columbia University, and Mr. S. F. Reiter, previously a research associate at Stanford University, have been appointed to assistant professorships; Dr. A. H. Copeland, Jr. of Massachusetts Institute of Technology, Mr. W. R. Fuller, formerly a mathematician with Naval Ordnance, Indianapolis, Dr. G. L. Krabbe, recently an assistant at the University of California, Dr. C. J. Neugebauer, previously an assistant at Ohio State University, Dr. C. M. Petty, formerly a research instructor at Duke University, Assistant Professor G. C. Preston of Macalester College, Mr. R. W. Randall, Jr., previously a student at Rice Institute, and Mr. D. M. Mesner, recently an assistant at Michigan State College, have been appointed to instructorships; Dr. Morris Skibinsky, research assistant at Duke University, has been appointed Research Associate in the Statistical Laboratory; Assistant Professor C. R. Putnam has been promoted to an associate professorship; Assistant Professor E. A. Trabant has been promoted to the position of Associate Professor of Mathematics and Engineering Science; Associate Professor G. H. Graves has retired with the title Professor Emeritus.

State University of Iowa reports the following: Professor M. F. Smiley, who is on leave of absence during the academic year 1954–55, spent the Fall quarter at the University of Chicago and is now at the University of Washington; Dr. H. A. Dye of the Institute for Advanced Study and Dr. H. M. Johnson of the Yerkes Observatory, University of Chicago, have been appointed to assistant professorships; Associate Professors Roscoe Woods and H. V. Price have been promoted to professorships; Professor C. C. Wylie has retired but will continue on a part-time basis.

University of Arizona announces: Assistant Professor H. D. Sprinkle of Alabama Polytechnic Institute has been appointed to an assistant professorship; Mr. D. B. Witmeyer, formerly a mathematics teacher at Duncan Arizona High School, has accepted an instructorship; Mr. D. N. Leeson, formerly a student at the University of Bridgeport, and Miss Virginia C. Clover, previously a student at Augustana College, have been appointed teaching fellows.

The University of California, Berkeley, announces: The following new appointments have been made for the year 1954–55: Visiting Associate Professor H. D. Huskey, Acting Assistant Professors S. G. Bourne and Bernard Sherman, Instructors E. A. Bishop, J. B. Butler, Jr., and H. A. Osborn; the following appointments have been made for Spring 1955: Visiting Professors A. S. Besicovitch, Charles Loewner, and N. E. Steenrod; Assistant Professor L. A. Henkin has been promoted to an associate professorship and is on leave during 1954–55

on a Fulbright fellowship; Associate Professor E. L. Lehmann has been promoted to a professorship; Acting Assistant Professor M. H. Protter has been promoted to an associate professorship; Professor Sophia L. McDonald has retired with the title of Professor Emeritus; Associate Professor R. H. Sciobereti has retired with the title of Associate Professor Emeritus; Assistant Professor L. H. Swinford has retired; Assistant Professor S. P. Diliberto is on leave of absence during 1954–55 and is at the Institute for Advanced Study; Assistant Professor Elizabeth L. Scott is on leave of absence in Europe.

University of California at Los Angeles reports: Associate Professor L. R. Sario of Massachusetts Institute of Technology has been appointed to an associate professorship; Associate Professor J. W. Green has been promoted to a professorship; Assistant Professor Alfred Horn has been promoted to an associate professorship; the following are on sabbatical leave for the year 1954–55: Professor M. R. Hestenes who is in Oslo, Norway, on a Guggenheim and Fulbright fellowship; Dr. J. D. Swift, now at the Institute for Advanced Study; Professor A. E. Taylor, who is in Geneva, Switzerland.

University of Chicago announces the following: Professor Saunders MacLane addressed the Henri Poincaré Festival in Paris, France, in October 1954; Dr. R. K. Lashof has been appointed to an instructorship; Dr. Armand Borel, formerly a member of the Institute for Advanced Study, is a visiting professor; Dr. W. H. Cockcroft, lecturer at King's College, Aberdeen, Scotland, has been appointed Visiting Lecturer.

University of Houston reports the following: Assistant Professor C. P. Benner has been appointed Chairman of the Department of Mathematics; Dr. D. H. Wright has accepted an assistant professorship.

At the University of Illinois: Professeur titulaire Roger Godement of the Universite de Nancy, France, has a position as George A. Miller Visiting Professor; Associate Professor N. S. Hawley, Jr., of Louisiana State University has been appointed to an assistant professorship; Assistant Professor M. E. Munroe has been promoted to an associate professorship.

University of Maryland announces: Dr. L. F. McAuley, part-time instructor at the University of North Carolina and Mr. W. G. Rosen, formerly a graduate assistant at the University of Illinois, have been appointed to instructorships; Assistant Professors R. A. Good, G. S. S. Ludford, and D. M. Young, Jr., have been promoted to associate professorships.

University of Massachusetts reports: Mrs. Alice Epstein and Mr. R. E. Schwartz of Ripon College have been appointed to instructorships; Mr. A. W. Wallace, formerly a student at Northeastern University, has been appointed to a part-time instructorship; Mr. R. W. Deland has been appointed a teaching fellow; Assistant Professor I. H. Rose has been promoted to an associate professorship.

The University of Oklahoma announces the following: Associate Professor Casper Goffman of Wayne University has accepted a professorship; Professor

J. O. Hassler has retired with the title of Professor Emeritus of Mathematics and Astronomy. A two day conference for mathematics teachers will be held at the University in June, 1955.

University of Rochester reports the following: Mr. W. A. Small, previously a graduate student at the University, has been appointed to an instructorship; Mr. G. C. Branche, Mr. D. M. Burton, Mr. P. L. Kingston, and Mr. T. R. Knapp have been appointed to graduate instructorships; Dr. Robert MacDowell has been promoted to an assistant professorship; Professor Wladimir Seidel is on leave and is at Notre Dame University as Visiting Professor.

At the University of Saskatchewan: Dr. Richard Blum of Acadia University and Mr. W. B. Antliff, formerly a teaching fellow at Queen's University, Ontario, have been appointed to instructorships.

University of Tennessee announces: Dr. A. A. Blank, a research associate at the University of Illinois, has been appointed to an assistant professorship; Mr. G. L. Curme, Mr. P. H. Doyle of Western Michigan College, and Mr. G. M. Speed have been appointed to instructorships.

University of Toronto reports: Professor L. J. Mordell of the University of Cambridge has been appointed Visiting Professor; Dr. Hanno Rund, seminar lecturer of Bonn University, has been appointed to an assistant professorship; Mr. R. O. A. Robinson and Mr. Ralph Wormleighton have been appointed Lecturers; Associate Professor G. deB. Robinson has been promoted to a professorship.

New York University, Washington Square College, announces the following: Assistant Professor Louis Baron has been promoted to an associate professorship; Research Assistant Professor S. C. Lowell has been promoted to a research associate professorship.

West Virginia University reports the following: Dr. E. E. Posey of the University of Tennessee has been appointed to an assistant professorship; Miss Maryelsie Hawkins, formerly a graduate assistant at the University of Illinois, has been appointed to an instructorship.

At Yale University: Dr. J. P. Jans of the University of Michigan has been appointed to an instructorship; Dr. J. T. Schwartz has been promoted to an assistant professorship.

Miss Ruth J. Abrams, formerly a technical aide for Bell Telephone Laboratories, Murray Hill, New Jersey, is a graduate student at Columbia University.

Dr. Miriam C. Ayer, member of the staff of Sandia Corporation, Albuquerque, New Mexico, has been appointed to an assistant professorship at the University of Missouri.

Associate Professor J. M. Barbour, Department of Music, Michigan State College, has been promoted to a professorship.

Mr. D. C. Barton of the University of Rochester has accepted a position as mathematician with the Eastman Kodak Company, Rochester, New York.

Dr. Gertrude Blanch, recently with the Consolidated Engineering Company,

Pasadena, California, has a position as Senior Mathematician at Wright Air Development Center, Dayton, Ohio.

Assistant Professor T. A. Botts of the University of Virginia has been promoted to an associate professorship.

Dr. Barron Brainerd has been appointed to an instructorship at the University of British Columbia.

Mr. T. F. Brophy, formerly a graduate student at New York State College for Teachers, is teaching at Stillwater Central School, New York.

Mr. L. L. Campbell of the University of Toronto has accepted a position as Scientific Service Officer with the Defense Research Board, Ottawa, Ontario, Canada.

Assistant Professor Harvey Cohn of Wayne University has been promoted to an associate professorship.

Dr. Margaret F. Conroy of Purdue University has accepted a position as Assistant Professor at Washington University.

Professor H. H. Conwell, Emeritus Dean of Beloit College, is Visiting Professor at Rockford College.

Mr. J. R. Cox, formerly a student at Lebanon Valley College, is now a graduate assistant at Pennsylvania State University.

Mr. H. B. Curtis, Jr. of Texas Agricultural and Mechanical College has been promoted to an assistant professorship.

Mr. J. M. Elkin, previously chief statistician for the Railroad Retirement Board, Chicago, Illinois, is employed as Chief Statistician by Martin E. Segal & Company, New York City.

Mr. J. G. Elliott of Ohio University is employed by Bendix Aircraft Company, Detroit, Michigan.

Assistant Professor D. H. Erkiletian, Jr. of Missouri School of Mines has been promoted to an associate professorship.

Professor G. M. Ewing of the University of Missouri is on leave of absence for the academic year 1954–55 and is a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Professor C. B. Gass, head of the Mathematics Department of Nebraska Wesleyan University, has been appointed to an associate professorship at DePauw University.

Mr. I. B. Goldberg, formerly a student at New York University, is a graduate student at Harvard University.

Dr. Harry Gonshor of the University of Southern California has accepted an assistant professorship at the University of Miami.

Mr. Basil Gordon, formerly a junior instructor at Johns Hopkins University, has been appointed a graduate assistant at the California Institute of Technology.

Mr. A. S. Gregory, formerly a student at the University of Illinois, is now a fellow at the University of Chicago.

Mr. Arshag Hajian, recently a student at the University of Chicago, is now a graduate student at Yale University.

Dr. J. K. Hale of Purdue University has accepted a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. G. R. Ingram, formerly of Montana State College, is employed by the Bureau of Public Roads, Helena, Montana.

Mr. A. A. Karwath of St. Ambrose College has been appointed to an instructorship at the University of Iowa.

Mr. R. R. Kemp, recently a student at McMaster University, is now a teaching assistant at Massachusetts Institute of Technology.

Mr. W. H. Leser has been appointed to an instructorship at Franklin and Marshall College.

Dr. R. W. Long, formerly a lecturer at the University of Pittsburgh, is employed now as a mathematician by Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

Mr. W. M. Lowney, previously a graduate student at Montana State College, is now Acting Instructor at the State College of Washington.

Mr. J. F. Manogue, formerly a student at the Catholic University of America, is a graduate student at Columbia University.

Assistant Professor Elna B. McBride of Memphis State College has been promoted to an associate professorship.

Dr. R. W. McKelvey of Purdue University is now a post-doctoral fellow at the University of Maryland.

Mr. C. E. Moulton, formerly a graduate student at the University of Buffalo, has been appointed to a professorship at Shurtleff College.

Professor F. S. Nowlan of the University of Illinois has been appointed Visiting Professor at the College of William and Mary.

Assistant Professor Anne F. O'Neill of Wheaton College, Norton, Massachusetts, has been promoted to the position of Associate Professor and Acting Head of the Department of Mathematics during 1954–55.

Assistant Professor D. B. Owen of Purdue University has a position as Statistician at the Sandia Corporation, Albuquerque, New Mexico.

Visiting Associate Professor T. K. Pan of the University of Oklahoma has been appointed to an associate professorship.

Mr. E. C. Paxhia, formerly a student at the University of Rochester, is now a graduate assistant at Washington University.

Mr. E. I. Pina, recently a part-time instructor at the University of Massachusetts, is now a junior engineer with Boeing Airplane Company, Seattle, Washington.

Professor Emeritus W. W. Rankin of Duke University has been appointed to an instructorship at the Phillips Exeter Academy, New Hampshire, for the academic year 1954–55.

Mr. P. D. Ritger, previously a research assistant in mathematics at New

York University, has been appointed to an instructorship at Stevens Institute of Technology.

Dr. Samuel Schecter of Lehigh University has a position as research associate at New York University, Institute of Mathematical Sciences.

Professor J. P. Scholz of Western College has been appointed Head of the Department of Mathematics.

Dr. Esther Seiden, formerly with the Committee on Statistics, University of Chicago, is now Assistant Professor at Howard University.

Mr. R. J. Semple, formerly a part-time instructor at Princeton University, has been appointed Lecturer at Carleton College, Ottawa, Canada.

Sister M. Walter Reginald, an instructor at the College of Saint Teresa, has been appointed to an instructorship at Dominican College, Wisconsin.

Dr. N. B. Smith, formerly a graduate student at Iowa State College, has been appointed to an instructorship at San Diego State College.

Mr. R. C. Stewart of Trinity College, Connecticut, has been promoted to an assistant professorship.

Mr. C. R. Strain, previously a mathematician for Engineering Research Associates, Arlington, Virginia, is now a programmer and digital computer for Remington-Rand, New York City.

Mr. C. H. Taft of West Virginia University has accepted a position as mathematician with the National Security Agency, Washington, D. C.

Assistant Professor D. L. Thomsen, Jr. of the Pennsylvania State University has accepted a position as Applied Science Representative with International Business Machines Corporation, Philadelphia, Pennsylvania.

Mr. D. E. Thoro, previously a mathematician for R. C. A. Service Company, Cocoa, Florida, has been appointed to an instructorship at the University of Florida.

Assistant Professor G. L. Walker of Purdue University is now Head of the Mathematical Section of the American Optical Company, Southbridge, Massachusetts.

Mr. W. G. Weideman, formerly in military service, is now a graduate student at the University of Michigan.

Miss Irene S. Welna, recently a student at Saint Joseph College, is teaching at Bloomfield Junior High School, Connecticut.

Assistant Professor C. S. Wolfe of Shepherd College is now a teaching assistant at the University of Maryland.

Mr. P. W. Zehna of Colorado State College of Education has been appointed to an instructorship at the University of Kansas.

Dr. A. Zirakzadeh of the University of Colorado has been appointed to an instructorship at the State College of Washington.

Assistant Professor J. V. Limpert of St. Lawrence University died on September 2, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 61 persons have been elected to membership by the Board of Governors on applications duly certified.

A/2C ERNEST ANDERSON, Keesler Air Force Base, Mississippi.

P. G. ARCHER, B. A.(Buffalo) Mathematician, Bell Aircraft Corporation, Niagara Falls, N. Y.

R. H. AYERS, Student, Lebanon Valley College, Annville, Pa.

G. E. BARDWELL, M.S.(Colorado) Research Asso., University of Denver.

JOAN M. BEAUCHAMP, B.A.(Keuka) Teacher, Middlesex Valley Central School, Rushville, N. Y.

A. H. BLESSING, B.A.(Buffalo) Instr., University of Buffalo.

W. J. BLUNDON, M.A.(Columbia U.) Professor, Memorial University of Newfoundland, St. John's, Newfoundland, Canada.

L. L. BRASSAW, JR., B.S.(U. S. Naval Academy) Teaching Fellow, University of Buffalo.

REV. A. E. CAHILL, O.S.B., A.B.(St. Benedict's) Instr. in Physics and Mathematics, Belmont Abbey College, Belmont, N. C.

LAMBERTO CESARI, Ph.D.(Pisa) Professor, Purdue University.

C. Y. CHAO, M.S.(State U. of Iowa) Instr., Coe College.

ELLA L. CLEMENT, A.B.(State U. of Iowa) Teacher, Douglas High School, Oklahoma City, Okla.

MIRIAM E. CONNELLAN, M.A.(Catholic U.) Instr., Marquette University.

D. C. DAVIS, B.A.(Oklahoma) Grad. Asst., University of Oklahoma.

P. A. DAVIS, Student, University of British Columbia.

GERTRUDE V. DECKER, M.A.(Hofstra) Teacher, East Rockaway High School, N. Y.

E. T. DENMARK, JR., M.S.(Florida S. U.) Instr., Chipola Junior College, Marianna, Fla.

MARY G. DIMAGGIO, A.B.(Montclair S. T. C.) Teacher, Closter High School, N. J.

J. E. DIXON, Student, William Jewell College.

E. E. DOUGHERTY, Student, Sacramento State College.

WILLARD DRAISIN, Student, Brooklyn College.

MRS. JOY B. EASTON, M.S.(W. Va. U.) Secretary, Mathematics Department, West Virginia U.

E. E. FLOYD, Ph.D.(Virginia) Asso. Professor, University of Virginia.

C. L. GAPE, B.A.(Buffalo) Instr., University of Buffalo.

LEONARD GILLMAN, Ph.D.(Columbia U.) Asst. Professor, Purdue University.

WILLIAM GRANET, M.A.(Columbia U.) Asst. to Director, Office of Statistical & Research Services, Boston University.

S. L. GREITZER, M.A.(Columbia U.) Teacher, Yeshiva University and Bronx High School of Science.

L. W. GUNTER, M.S.(Wisconsin) Instr., Western Michigan College of Education.

A. B. HARPER, JR., S.B.(M.I.T.) Analyst, Arthur D. Little Company, Nashua, N. H.

ROBERT HAYES, M.A.(Columbia U.) Chairman, Mathematics Department, Hempstead High School, N. Y.

R. L. HELMBOLD, M.S.(Carnegie I. T.) Teaching Asst., Carnegie Institute of Technology.

LESTA HOEL, M.A.(Oregon) Supervisor of Mathematics, Portland, Oregon.

A. A. J. HOFFMAN, Student, University of Texas.

HARRIETT R. JUNIOR, M.S.(Howard) Instr., Hampton Institute.

J. G. KEMENY, Ph.D.(Princeton) Professor, Dartmouth College.

JOACHIM LAMBEK, Ph.D.(McGill) Asso. Professor, McGill University.

W. H. LESER, A.M.(Pennsylvania) Instr., Franklin & Marshall College.

EDNA M. LOVE, M.A. (Pittsburgh) Teacher, Derry Township High School, Derry, Pa.

S. F. MACK, Ph.D. (California) Asst. Professor, Pennsylvania State University.

R. C. MEACHAM, Ph.D. (Brown) Asso. Professor, University of Florida.

R. A. MOORE, Ph.D. (Washington U.) Instr., Yale University.

J. P. NOLAN, Norman, Oklahoma.

R. J. OEDY, A.B. (Indiana) Grad. Student, Iowa State College.

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THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the University of Manitoba in Winnipeg, Manitoba, on October 16, 1954. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors J. W. Lawson, R. P. Winter and Sister M. Leontius, Chairman of the Section, presided at the respective sessions.

Forty-seven persons attended the meeting including the following thirty members of the Association:

F. J. Arena, J. M. Calloway, C. S. Carlson, N. J. Divinsky, R. J. Dowling, K. L. Hankerson, Hildegarde H. Howden, Diane M. Johnson, G. K. Kalisch, J. W. Lawson, W. S. Loud, Walter Lyche, K. O. May, W. H. McBride, W. H. McEwen, W. R. McEwen, N. S. Mendelsohn, E. O. Nelson, J. C. Peterson, P. A. Rognlie, P. C. Rosenbloom, L. W. Sheridan, Sister M. Thomas á Kempis, Sister Mary Leontius, F. C. Smith, R. C. Staley, O. E. Stanaitis, K. W. Wegner, R. P. Winter, F. L. Wolf.

By invitation of the Executive Committee, Professor G. K. Kalisch of the University of Minnesota delivered an address at the morning session entitled "Canonical Forms of Finite and Infinite Matrices." The abstract of this address follows:

This address is devoted principally to a discussion of certain canonical forms of certain finite and infinite matrices arising from linear transformations of bilinear or quadratic forms in finite or infinite complex or real Hilbert spaces. The following five equivalence relations and certain associated canonical forms—sometimes confined to special classes of matrices—were discussed: (1) equivalence ($M \sim M' = PMQ$, P and Q non-singular); (2) unitary (orthogonal) equivalence ($M \sim M' = UMV$, U and V unitary (orthogonal)); (3) similarity ($M \sim M' = P^{-1}MP$, P non-singular); (4) conjunctivity (congruence) ($M \sim M' = P^*MP$, P non-singular with complex (real) entries); (5) unitary (orthogonal) similarity ($M \sim M' = U^*MU$, U unitary (orthogonal)). A number of unsolved problems connected with the preceding equivalence relations were also briefly discussed. In the infinite case, some attention was paid to certain sequence spaces other than l_2.

The following short papers were presented:

1. *A lemma on positive definite matrices*, by Professor E. O. Nelson, University of North Dakota.

The purpose of this paper is to show that a matrix

$$B = \begin{Vmatrix} a_{11} \cdots a_{1k} & b_1 \\ \cdot \cdot \cdot \cdot \cdot \cdot & \cdot \\ a_{k1} \cdots a_{kk} & b_k \\ b_1 \cdots b_k & b_{k+1} \end{Vmatrix}$$

formed by bordering a positive definite k by k matrix

$$A = \begin{Vmatrix} a_{11} \cdots a_{1k} \\ \cdot \cdot \cdot \cdot \\ a_{k1} \cdots a_{kk} \end{Vmatrix}$$

with any constants $b_1, \cdots, b_{k+1}$ is also positive definite if the determinant of the matrix B is greater than zero.

2. *The use of statistics in plastic film evaluation*, by Dr. L. W. Sheridan, General Mills, Minneapolis, Minnesota.

This paper presents some statistical methods considered in arriving at numerical values from laboratory tests for use in writing some of the specifications for polyethylene film used in fabricating stratospheric balloons.

3. *A note on an integral test for convergence of series of arbitrary terms*, by Professor O. E. Stanaitis, St. Olaf College.

Let $f(x)$ denote a function of a real variable on the interval $1 \leqq x < \infty$. If $f'(x)$ exists and is integrable over the same interval, and if $\int_1^\infty f'(x)dx$ is absolutely convergent, then the series $\sum_{x=1}^\infty f(x)$ and the integral $\int_1^\infty f(x)dx$ converge or diverge together. It has been shown that the theorem holds if a derivative $f^{(n)}(x)(n>1)$ exists and the integral $\int_1^\infty f^{(n)}(x)dx$ is absolutely convergent.

4. *Nilpotent matrices*, by Professor B. Noonan, University of Manitoba, introduced by Professor J. W. Lawson.

The main theorem demonstrates that properly triangular square matrices with elements in a field and sets of such matrices are nilpotent. The proofs are made elementary by considering only the principal diagonals of the matrices, namely, the first secondary diagonal in each matrix containing a non-zero element. It is further shown that in any set of properly triangular n by n matrices of nilpotency $n-r$, all matrices having the rth secondary diagonal as principal diagonal have at

least one zero element in this principal diagonal, and it is the same element in all such matrices. The converse theorem is also given.

5. *On a construction of the regular pentagon*, by Professor F. J. Arena, North Dakota Agricultural College.

In this paper the author gives a construction of the regular pentagon which is not well-known. The construction is as follows:

In any circle with center at O, let AOB and COD be two perpendicular diameters. Bisect OB at E. With EC as radius and E as center, strike an arc intersecting OA at F. Then in the circle $CADB$ draw the chords CG, GH, HI and IJ all equal to CF and $CGHIJC$ is the required pentagon.

The author then proves that the polygon $CGHIJC$ is a regular pentagon.

6. *Some simple but significant calculus problems in economics*, by Professor K. O. May, Carleton College.

If y is the total costs of a firm and x its total output, it is quite realistic to assume that $y=mx+b$ where b is the overhead and m is the marginal cost. If the best use of existing resources is made, m and b can be changed only by modifying technology, methods or organization. Typically, $m=f(b)$, where $f'(b)<0$ and $f''(b)>0$. It is easy to determine the best cost function for a given output, *i.e.*, that which minimizes cost for that output. The real problem, however, is to choose a cost function best suited to some anticipated time series of outputs in the future. It is a simple calculus problem to show that the cost over a period of time is minimized by choosing the cost function that yields minimum cost for the anticipated average output. This yields also the maximum profit. The problem may be modified by assuming that cost is some non-linear function. By defining total costs and average outputs in terms of integrals, one may acquaint the student with economic interpretations of mathematical manipulations usually associated with centroids and moments. Indeed for a quadratic cost function, it is easily shown that the best cost function is determined by the mean and the variance of the outputs.

7. *Matrices poorly behaved with respect to the computation of their inverses*, by Professor N. S. Mendelsohn, University of Manitoba.

If A is a square matrix of real numbers with non-vanishing determinant, it has a unique left inverse which is also a right inverse. For some matrices A there are matrices X^* which are almost left inverses but which are very poor as right inverses. In fact we can establish the following theorem:

Let ϵ and K be arbitrary positive numbers. There exist, for each $n\geqq 2$, n by n matrices X and A such that every element of XA differs from the corresponding element of the identity matrix by less than ϵ while every element of AX is larger than K.

A connection between the numbers ϵ, K and the separation of the characteristic values of A is obtained.

8. *Sequences illustrating uniform and dominated convergence*, by Professor W. S. Loud, University of Minnesota.

If a sequence of functions $\{f_n(x)\}$ has a pointwise limit $f(x)$ on $a\leqq x\leqq b$, sufficient conditions for $\int_a^b f_n(x)dx$ to approach $\int_a^b f(x)dx$ (assuming all integrals exist) are that the convergence be uniform, or that the convergence be uniform on every closed subinterval with the sequence being dominated by an integrable function on the entire interval. Taking α, β, γ and δ as positive, the sequence $f_n(x)=n^\alpha x^\beta e^{-n^\gamma x^\delta}$ on $0\leqq x\leqq 1$ converges pointwise to zero. It is uniformly convergent if $\alpha\delta<\beta\gamma$, and satisfies the second condition if $\alpha\delta<(\beta+1)\gamma$. It can be verified that the integrals converge to zero if $\alpha\delta<(\beta+1)\gamma$. A sequence having similar properties on the same interval is $f_n(x)=n^\alpha x^\beta/(1+n^\gamma x^\delta)$ where it is assumed in addition that $\alpha<\gamma$.

9. *A new definition of the radical*, by Professor N. J. Divinsky, University of Manitoba.

This note is a report on the recent Russian definition of the Brown and McCoy radical. It considers papers by Andrunakievich in 1948 and 1952 and by Kurochkin in 1949. Use is made of co-united multiplication: $a \circ b = a + b - ab$; and the strange concept of a co-united right ideal I which is a subset of a ring A such that $a \circ x$ is in I for every a in I and x in A and such that for every a, b and c in I, $a-b+c$ is in I. The set I is not a right ideal in the ordinary sense; in fact, it is not closed with respect to addition. All the usual theorems on the radical are proved using this new point of view.

F. C. SMITH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUSIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 23, 1955.

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, April 22–23, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April 22–23, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

EMPLOYMENT OPPORTUNITIES

The University of Alberta, Edmonton, Alberta. (1) Assistant or Associate Professor of statistics; appointment to be effective September 1, 1955. (2) Sessional Lecturer in Mathematics for eight months from September 1, 1955; prospect of permanent appointment. Applications including transcript of academic record, curriculum *vitae*, publications, names of two references, and recent photograph or snapshot to be sent to Walter H. Johns, Dean, Faculty of Arts and Science, University of Alberta.

University of British Columbia, Vancouver 8, Canada. Instructor, Ph.D., Mathematics Department.

By action of the Board of Governors this Employment Opportunities section will be discontinued. Employers are referred to the Employment Registrar now maintained at all principal meetings of the mathematical organizations.

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ON THE NUMBER OF VANISHING TERMS IN AN INTEGRAL CUBIC RECURRENCE

MORGAN WARD, California Institute of Technology

Introduction. Let

$$(T): T_0, T_1, T_2, \cdots, T_n, \cdots$$

be an integral cubic recurrence; that is, the initial values T_0, T_1, T_2 of (T) are integers and

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, \cdots).$$

Here P, Q and R are fixed integers and $R \neq 0$. The polynomial

$$f(z) = z^3 - Pz^2 + Qz - R$$

and the recurrence (T) are said to be associated. If

$$g(z) = T_0 z^2 + (T_1 - PT_0)z + (T_2 - PT_1 + QT_0),$$

then for $|z|$ sufficiently large,

$$g(z)/f(z) = \sum_1^\infty T_{n-1} z^{-n}$$

is a generating function for (T).

The two most interesting cases are when $g(z) = df(z)/dz$ and $g(z) = 1$; we denote the corresponding recurrences by (S) and (H).

If $f(z)$ has distinct roots u, v, w, then S_n is simply the Newtonian sum of the nth powers of the roots, while H_{n+2} as a symmetric function is the homogeneous product sum of the roots of degree n [4]:

$$S_n = u^n + v^n + w^n, \qquad H_{n+2} = \sum u^{r_1} v^{r_2} w^{r_3}.$$

Here the second sum is extended over all non-negative integers r_i such that $r_1 + r_2 + r_3 = n$. (H) is of particular importance in the algebra of recurring sequences [1], [2]; the corresponding recurrences of order two are the well known Lucas functions [3].

An integer $k \geqq 0$ is called a "zero" of the recurrence (T) if $T_k = 0$. For example, 0 and 1 are zeros of (H). We determine here the maximum number of possible zeros of (T) when its associated polynomial has integral roots subject to a restriction to be described presently.

Prior work on this problem has been confined to the cases when $f(z)$ has complex roots and $R = \pm 1$ [6], [7] or when $(T) = (S)$ and $P = 0$ [8]. There is an important general result of Kurt Mahler's [5] in this connection. Let us call both (T) and its polynomial $f(z)$ "degenerate" or "nondegenerate" according as the ratio of any pair of different roots of $f(z)$ is, or is not, a root of unity.

Mahler showed that if (T) is non-degenerate, $|T_n|$ tends to infinity with n. Hence

The total number of zeros of any non-degenerate recurrence (T) *is finite.*

We prove here the following more precise result.

THEOREM 1. *If the associated polynomial of an integral cubic recurrence is both non-degenerate and has integral roots which are co-prime in pairs, then at most three terms of the recurrence can vanish.*

The exact determination of the number of zeros of a given recurrence (T) under these hypotheses may be very difficult. For example, it is easy to see that (S) can have at most one zero; but the assertion that if $S_1 \neq 0$, (S) has no zeros is essentially Fermat's last theorem. On the other hand, it follows from the results of this paper that the sequence (T) with initial values $T_0=0$, $T_1=1$ and $T_2=0$ has no other zeros.

The plan of the paper is sufficiently indicated by the section headings. In the conclusion we mention some unsolved problems concerning the zeros of (H) suggested by the investigation.

2. Preliminary lemmas. We denote the roots of $f(z)$ by u, v and w. Then u, v and w are integers co-prime in pairs with distinct absolute values, since $f(z)$ is non-degenerate.

The general term T_n of (T) is of the form

$$T_n = Uu^n + Vv^n + Ww^n \tag{2.1}$$

where U, V, W are rational and different from zero, since (T) is of order three. Consequently, k and l are zeros of (T) if and only if

$$\begin{aligned} Uu^k + Vv^k + Ww^k &= 0 \\ Uu^l + Vv^l + Ww^l &= 0. \end{aligned} \tag{2.2}$$

On solving (2.2) for the ratios $U:V:W$, we obtain

LEMMA 2.1. *If k and l are distinct integers and $l>k\geqq 0$, a necessary and sufficient condition that k and l be zeros of (T) is that U, V and W of formula* (2.1) *satisfy the conditions*

$$\frac{Uu^k}{w^{l-k}-v^{l-k}} = \frac{Vv^k}{u^{l-k}-w^{l-k}} = \frac{Ww^k}{v^{l-k}-u^{l-k}}.$$

We note for future use two simple corollaries of this result.

LEMMA 2.2. *If $T_0=0$, then a necessary and sufficient condition that $T_n=0$ for $n>0$ is that*

$$\frac{U}{w^n-v^n} = \frac{V}{u^n-w^n} = \frac{W}{v^n-u^n}. \tag{2.3}$$

LEMMA 2.3. *If $T_0=0$ and if $l>k>0$, then a necessary condition that $T_l=T_k=0$ is that*

$$\frac{w^k - v^k}{w^l - v^l} = \frac{u^k - w^k}{u^l - w^l} = \frac{v^k - u^k}{v^l - u^l}. \tag{2.4}$$

LEMMA 2.4. *Let t be a real variable and n any real number greater than one. Then the function $(t^n-1)/(t+1)$ increases steadily if $t>0$ and $(t^n+1)/(t+1)$ increases steadily if $t>1$.*

For the derivatives of the functions are positive under the stated conditions.

LEMMA 2.5. *Let r and s be co-prime integers and k and l positive integers. Then*

$$(r^k - s^k, r^l - s^l) = r^d - s^d \qquad \text{where } d = (k, l). \tag{2.5}$$

Here (x, y) denotes the greatest common divisor of the integers x and y.

For denote the left side of (2.5) by m. Since d divides k and l, r^d-s^d divides r^k-s^k and r^l-s^l. Consequently, r^d-s^d divides m. It suffices then to show that m divides r^d-s^d.

Since m divides r^k-s^k, it is prime to both r and s. Hence there exists a positive integer t with the property that m divides r^t-s^t but m does not divide r^n-s^n if $0<n<t$. Then *m divides r^n-s^n if and only if t divides n.* For let $n=2qt\pm c$, where $0\leqq c<t$ and let $a=qt$, $b=qt\pm c$. Then if $b\geqq a$, $r^n-s^n=(r^a-s^a)(r^b-s^b)+(rs)^a(r^c-s^c)$, and if $a\geqq b$, $r^n-s^n=(r^a-s^a)(r^b-s^b)-(rs)^b(r^c-s^c)$. In either case, since m divides r^n-s^n and r^a-s^a and is prime to r and s, the minimal property of t is contradicted unless $c=0$ or $c=t$.

Now m divides both r^k-s^k and r^l-s^l. Hence t divides both k and l. Therefore, t divides d, and m divides r^d-s^d, completing the proof.

3. Recurrences with three zeros. Let (T) be a recurrence with $T_0=0$ and at least two other zeros, k and l. We may assume that

$$0 < k < l, \qquad T_n \neq 0, \qquad 0 < n < k. \tag{3.1}$$

Then by Lemma 2.3, the equalities (2.4) must hold. Let $d=(k, l)$. Since u, v and w are co-prime in pairs, if we divide the numerator and denominator of each of the fractions in (2.4) by the corresponding integers w^d-v^d, u^d-w^d and v^d-u^d, we obtain by Lemma 2.5 three equal fractions in their lowest terms. Hence corresponding numerators and denominators must be equal up to sign; that is

$$\begin{aligned} \frac{w^k - v^k}{w^d - v^d} &= \pm \frac{u^k - w^k}{u^d - w^d} = \pm \frac{v^k - u^k}{v^d - u^d}, \\ \frac{w^l - v^l}{w^d - v^d} &= \pm \frac{u^l - w^l}{u^d - w^d} = \pm \frac{v^l - u^l}{v^d - u^d}. \end{aligned} \tag{3.2}$$

Consider now the equalities

$$\frac{w^n - v^n}{w - v} = \pm \frac{u^n - w^n}{u - w} = \pm \frac{v^n - u^n}{v - u}, \qquad n \geqq 1. \tag{3.3}$$

Observe that each of the equalities (3.2) may be put into this form by letting $u' = u^d$, $v' = v^d$, $w' = w^d$, taking n equal to k/d or l/d and then dropping the primes.

Let a, b and c be the absolute values of u, v and w respectively. We may evidently assume that

$$w = c > b > a > 0; \qquad v = \pm b, \; u = \pm a. \tag{3.4}$$

For changing the signs of all the roots of $f(z)$ merely multiplies T_n by $(-1)^n$. There are four cases according to the choices of sign of u and v.

Case	*Roots of $f(z)$* u	v	w	*Equalities* (3.3)
1	a	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - a^n}{b - a}$
2	$-a$	b	c	$\dfrac{c^n - b^n}{c - b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - (-a)^n}{b + a}$
3	a	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - a^n}{c - a} = \dfrac{b^n - (-a)^n}{b + a}$
4	$-a$	$-b$	c	$\dfrac{c^n - (-b)^n}{c + b} = \dfrac{c^n - (-a)^n}{c + a} = \dfrac{b^n - a^n}{b - a}$

Both case 1 *and case* 2 *are impossible if* $n > 1$. In case 1, this statement is evident, since $c > b > a > 0$. In case 2, if $n > 1$, we have

$$\frac{c^n \pm a^n}{c + a} = \frac{b^n \pm a^n}{b + a}.$$

Now let $c = ax$ and $b = ay$. Then

$$\frac{x^n \pm 1}{x + 1} = \frac{y^n \pm 1}{y + 1} \qquad \text{with } x > y > 1 \text{ and } n > 1$$

contradicting Lemma 2.4.

Both cases 3 *and* 4 *are impossible if n is even. For assume n* is even. Then $n > 1$, and in case 3 we have

$$\frac{b^n - a^n}{b + a} = \frac{c^n - a^n}{c - a}.$$

But $(b^n - a^n)/(b+a) < (b^n - a^n)/(b-a) < (c^n - a^n)/(c-a)$ since $c > b > a > 0$.

In case 4, we have

$$\frac{c^n - b^n}{c + b} = \frac{c^n - a^n}{c + a}.$$

Hence if $b = cx$ and $a = cy$, then

$$\frac{1 - x^n}{1 + x} = \frac{1 - y^n}{1 + y} \qquad \text{with } 1 > x > y > 0.$$

But by Lemma 4, $(1 - t^n)/(1+t)$ steadily decreases if $t > 0$.

We now apply these results to the equalities (3.2) in accordance with the remark following (3.3). First, *d must be odd*. For if d is even, u^d, v^d, w^d are positive, and case 1 applies; that is, $n = 1$ so that $k = d$ and $l = d$ contrary to (3.1). Since n must be odd, k and l *must both be odd*. Hence $w^k - v^k$ and $w^d - v^d$ are of like sign. Therefore, the first equality (3.2) may be written as

$$\frac{w^k - v^k}{w^d - v^d} = \frac{u^k - w^k}{u^d - w^d} = \frac{v^k - u^k}{v^d - u^d}.$$

But since $T_0 = T_k = 0$, this equality and lemma 2.2 imply that

$$\frac{U}{w^d - v^d} = \frac{V}{u^d - w^d} = \frac{W}{v^d - u^d}.$$

Hence $T_d = 0$ by Lemma 2.2. But $0 < d \leqq k$. Hence $d = k$ by condition (3.1). We have thus proved:

THEOREM 2. *Let (T) be an integral cubic recurrence whose associated polynomial is non-degenerate and has integral roots which are co-prime in pairs, and assume that the first two zeros of (T) are 0 and k. Then if k is even, (T) has no other zeros. If k is odd, any other zero of (T) must be an odd multiple of k.*

It follows immediately that the recurrence with initial values 0, 1, 0 has no other zeros. On the other hand, the recurrence (H) of the introduction has $H_3 = P = 0$ if $u + v + w = 0$. *Hence there exist recurrences with three zeros.*

4. Proof of Theorem 1. We will now use Theorem 2 to give a proof by contradiction of Theorem 1. Let (T) and $f(z)$ satisfy the hypotheses of the theorem, and suppose (T) has more than three zeros. Let the first four zeros of (T) be k_1, k_2, k_3, k_4, so that $0 \leqq k_1 < k_2 < k_3 < k_4$.

The recurrence (T') defined by $T'_n = T_{n+k_1}$ is associated with $f(z)$ and its first three zeros are 0, $k_2 - k_1$, and $k_3 - k_1$. Both $k_2 - k_1$ and $k_3 - k_1$ are odd by Theorem

2. Hence their difference k_3-k_2 is even. The recurrence (T'') defined by $T_n'' = T_{n+k_2}$ is associated with $f(z)$ and its first three zeros are 0, k_3-k_2, and k_4-k_2. But k_3-k_2 is even, contradicting Theorem 2.

5. Conclusion. It follows from Theorem 2 that if (T) has three zeros, say k_1, k_2 and k_3, then $d=k_2-k_1$ is odd and the zeros lie in a recurrence (T^*) defined by $T_n^*=T_{k_1+nd}$ whose associated polynomial $f^*(z)$ has for its roots the dth powers of the roots of $f(z)$. Since the initial values of (T^*) are 0, 0, and T_2^*, $T_n^*=T_2^*H_n^*$ where (H^*) is the recurrence associated with $f^*(z)$ discussed in the introduction. We are thus led to consider the zeros of the recurrence (H).

We have seen that H_3 can vanish. The next possibility is that $H_5=0$, giving the diophantine equation

$$(u+v+w)(u^2+v^2+w^2)+uvw=0. \tag{5.1}$$

Trivial solutions of (5.1) evidently violate our hypotheses on $f(z)$. Whether or not (5.1) has non-trivial solutions appears to be unknown. The more general question of whether H_{2n+1} can vanish when $n>1$ appears to be of a difficulty comparable to the Fermat problem for the recurrence (S).

The simplest case when $(T)\neq(H)$ would have three zeros would be when $d=3$ and $k_3=k_1+5d$ making $H_5^*=0$. The related diophantine equation is obtained from (5.1) by replacing u, v, w by their cubes. It would imply then that the diophantine system

$$u^3+v^3+w^3=4^\epsilon s^3, \qquad u^6+v^6+w^6=2^\epsilon t^3, \qquad \epsilon=0 \text{ or } 1,$$

has a non-trivial solution, which appears unlikely.

It is tempting to conjecture in view of these remarks that under our hypotheses on $f(z)$, (H) is the *only* recurrence which can have more than two zeros. It follows of course from Lemma 2.2 that there exist recurrences (T) with arbitrarily assigned zeros k and l. But it may be shown by simple examples that neither Theorem 1 or Theorem 2 is true if we change our hypotheses on the roots of $f(z)$.

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ON THE MINIMALITY OF THE VARIATIONAL PRINCIPLES OF CLASSICAL PARTICLE MECHANICS

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1. Introduction. We shall be concerned here mainly with the variational principles which are associated with the names of Hamilton, Jacobi, Maupertuis and Hilbert. Each principle involves a functional in the form of an integral. The requirement of the vanishing of the first variation of the functional provides the differential equations characterizing the motion. For this reason these principles are called variational or stationary principles. If, in addition, the actual motion minimizes the functional, then the principle is called a minimum principle. We shall show here that some of the classical variational principles are minimum principles under certain conditions and only stationary principles under other conditions; others are always minimum principles, while still others are always only stationary principles. Some facts of this type are well known,* but it is hoped that the treatment presented here not only provides more complete results, but also is more elementary in method, simpler in development and yields results in more convenient form.

2. Notation having a fixed significance throughout the remainder of this paper. We shall consider a system $\mathfrak{M}$ of N particles. Let the configuration of $\mathfrak{M}$ be described by $n=3N$ Cartesian coordinates $[z_1, z_2, \cdots, z_n]$ in the usual manner; *i.e.*, $z_{3r-2}, z_{3r-1}, z_{3r}$ are the coordinates of the r-th particle $(r=1, 2, \cdots, N)$. The configuration of $\mathfrak{M}$ is then represented by a point $z=\{z_1, z_2, \cdots, z_n\}$ in n-dimensional Euclidean space E_n. The following summation convention will be used: an index appearing more than once in the same term will be understood to be summed over its range unless otherwise specified. The range of i and j will be $[1, 2, \cdots, n]$; the range of σ will be $[1, 2, \cdots, k]$, where k will be introduced below. Unless otherwise stated it will be understood that any statement in which any of these indices appear unsummed, applies when such indices are assigned each value in their range.

The mass of the r-th particle $(r=1, 2, \cdots N)$ is denoted by $m_{3r-2}=m_{3r-1}=m_{3r}$, and $m=\min_i m_i$ while $M=\max_i m_i$.

We shall suppose that $\mathfrak{M}$ is subjected to certain forces, the details of which will be considered subsequently; under the influence of these forces the configuration of $\mathfrak{M}$ during the time interval D: $t_0 \leqq t \leqq t_1$ is described by the E_n-valued function $x(t)=\{x_1(t), x_2(t), \cdots, x_n(t)\}$. Let $x(t_0)=x^0$ and $x(t_1)=x^1$. We shall suppose that the origin of the coordinate system has been chosen at x^0 and that $t_1-t_0=\delta>0$. The function x will be called the *actual motion* or the *solution*. If Δ is any time interval $(\tau_0 \leqq t \leqq \tau_1)$ then any E_n-valued function $\xi(t)=\{\xi_1(t), \xi_2(t), \cdots, \xi_n(t)\}$ having a continuous derivative on Δ and such that $\xi(\tau_0)=x^0$ and $\xi(\tau_1)=x^1$ will be called a Δ-*motion of* $\mathfrak{M}$. A region Σ of $E_n \times D$

* *Cf. e.g.*, E. T. Whittaker, *Analytical Dynamics*, Dover 1944, pp. 250–253; also W. F. Osgood, *Mechanics*, Macmillan 1937, pp. 381–388.

will be called a *tube about the solution* if (a) for each t in D, $(x(t), t)$ is in Σ, and (b) for each (z, t) in Σ and each ϵ in $0 \leqq \epsilon \leqq 1$, $(x(t)+\epsilon[z-x(t)], t)$ is in Σ. If ξ is a D-motion and if, for each t in D, $(\xi(t), t) \in \Sigma$ then we say that ξ *lies in* Σ.

The following notational abbreviations will also be used. If g is a function of the ν variables, $y_1, y_2, \cdots, y_\nu$, and $y = \{y_1, y_2, \cdots, y_\nu\}$ then we shall write $\partial g(y)/\partial y_\mu = g_{,\mu}(y)$, $\partial^2 g(y)/\partial y_\mu \partial y_\rho = g_{,\mu\rho}(y)$ *etc.*; if $u(t) = \{u_1(t), u_2(t), \cdots, u_n(t)\}$ is an E_n-valued function on D then $[\int_{t_0}^{t_1} u_i u_i dt]^{1/2}$ is written as $\|u\|$.

The forces that we shall consider stem from three sources: (i) from a potential function $\Phi(z, v, t)$ dependent on the position, velocity and time [the force associated with this is $-(\partial\Phi/\partial z_i) + (d/dt)/(\partial\Phi/\partial v_i)$]; (ii) from constraints $\phi^\sigma(z, v, t) = 0$. If it should happen that no constraints are present we take $k = 0$ so that σ ranges over the null set and the terms involving ϕ^σ may be considered absent. We assume that Φ and ϕ^σ are continuous and have two continuous and bounded derivatives with respect to their first $2n$ real arguments for all (z, t) in some tube S, about the solution, and for all values of v; (iii) forces which are due neither to the generalized potential nor to the constraints. These will be denoted by $\{F_i(t; \xi)\}$. For each t, $F_i(t; \xi)$ is a functional defined over a subset of D-motions which includes x. We assume that there exist k functions $\{\lambda_\sigma(t)\}$ which together with the n components of the solution $\{x_i(t)\}$ satisfy the n Lagrange equations

$$(1) \qquad \frac{d}{dt}\left[m_i \dot{x}_i - \Phi_{,n+i}(Q) + \lambda_\sigma \phi^\sigma_{,n+i}(Q)\right] + \Phi_{,i}(Q) - \lambda_\sigma \phi^\sigma_{,i}(Q) = F_i(t; x),$$

(no sum on i) and the k constraint equations $\phi^\sigma(Q) = 0$, where $Q = (x(t), \dot{x}(t), t)$.

The following notation will be used to specify the force situation under consideration. $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{V}_4$ signifies that $\Phi(z, v, t)$ is respectively $V(z, v, t)$, $V(v, t)$, $V(z, t)$, $V(z)$; $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ signifies that $\phi^\sigma(z, v, t)$ is respectively $f^\sigma(z, v, t)$, $f^\sigma(z, t)$, workless; $\mathcal{J}_1$, $\mathcal{J}_2$ signifies that $F_i(t; \xi)$ are present or absent, respectively. Thus, for example, the notation $\mathcal{V}_4\mathcal{C}_3\mathcal{J}_2$ designates the conservative case.

3. Lemmas. We give now some simple inequalities which will be used subsequently.

LEMMA 1. *Let* $u(t) = \{u_1(t), u_2(t), \cdots, u_n(t)\}$ *be continuous and have a sectionally continuous derivative on* D. (a) *If* $u(t_0) = u(t_1) = 0$, *then* $\|\dot{u}\| \geqq \pi\|u\|/\delta$; (b) *if only* $u(t_0) = 0$, *then* $\|\dot{u}\| \geqq \pi\|u\|/2\delta$.

Proof: (a) Let

$$a_{i\nu} = \frac{2}{\sqrt{\delta}} \int_{t_0}^{t_1} \dot{u}_i \cos \frac{\nu\pi}{\delta}(t - t_0) dt \qquad (\nu = 1, 2, \cdots),$$

then

$$2\|\dot{u}\|^2 = 2\int_{t_0}^{t_1} \dot{u}_i \dot{u}_i dt = a_{i\nu}a_{i\nu} \geqq \frac{a_{i\nu}a_{i\nu}}{\nu^2} = \frac{2\pi^2}{\delta^2}\int_{t_0}^{t_1} u_i u_i dt = \frac{2\pi^2}{\delta^2}\|u\|^2.$$

Remark: The inequality cannot be improved, since if $u_i(t) = \sin [\pi(t-t_0)/\delta]$, then $\delta\|\dot{u}\| = \pi\|u\|$. (b) By extending $u_i(t)$ by reflection in the line $t = t_1$, the conditions of Lemma 1 are satisfied on the interval $t_0 \leqq t \leqq t_1 + \delta$ of length 2δ. Then

$$2\|\dot{u}\|^2 = \int_{t_0}^{t_1+\delta} \dot{u}_i\dot{u}_i dt \geqq \frac{\pi^2}{(2\delta)^2}\int_{t_0}^{t_1+\delta} u_iu_i dt = \frac{2\pi^2}{4\delta^2}\int_{t_0}^{t_1} u_iu_i dt = \frac{2\pi^2}{4\delta^2}\|u\|^2.$$

Again the inequality cannot be improved, for if $u_i(t) = \sin [\pi(t-t_0)/2\delta]$ then $2\delta\|\dot{u}\| = \pi\|u\|$.

LEMMA 2. *Let $\omega(\epsilon)$ have two continuous derivatives on $0 \leqq \epsilon \leqq 1$ and $\omega'(0) = 0$. If* $\min_{0\leqq\epsilon\leqq 1} \omega''(\epsilon) \geqq 2b$ *then $\omega(0) \leqq \omega(1) - b$. If* $\max_{0\leqq\epsilon\leqq 1} \omega''(\epsilon) \leqq -2c$ *then $\omega(0) \geqq \omega(1) + c$.*

Proof: $\omega(1) - \omega(0) = \int_0^1\int_1^\epsilon \omega''(s)dsd\epsilon$.

4. Hamilton's principle. We assume forces of type $\mathcal{V}_1\mathcal{C}_1\mathcal{F}_1$. For each D-motion ζ define the functional

$$I(\zeta) = \int_{t_0}^{t_1}\left[\frac{1}{2} m_i\dot{\zeta}_i^2 - V(\zeta, \dot{\zeta}, t) + \lambda_\sigma f^\sigma(\zeta, \dot{\zeta}, t) + F_i(t; x)\zeta_i\right] dt.$$

Now let X be an arbitrary, but fixed D-motion. If we can show that $I(x) \leqq I(X)$ we have a minimum principle. [If there are no forces F_i present and if we require that X satisfy the constraints, then $I(X)$ and $I(x)$ become $\int_{t_0}^{t_1}(T-V)dt$ evaluated along the respective motions (T is the kinetic energy); *i.e.*, if one shows that, for arbitrary $\zeta \neq x$, $I(x) < I(\zeta)$ then $\int_{t_0}^{t_1}(T-V)dt$ is less when computed along the actual motion than when computed along any other path satisfying the constraints.] Let $\alpha = X - x$, and for $0 \leqq \epsilon \leqq 1$ define $\omega(\epsilon) = I(x + \epsilon\alpha)$. Now $\omega(0) = I(x)$, $\omega(1) = I(X)$, while from equation (1) and the fact that $\alpha(t_0) = \alpha(t_1) = 0$ it follows that $\omega'(0) = 0$. Thus Lemma 2 will allow us to establish the results we seek if we can find bounds for $\omega''(\epsilon)$. Now

$$\omega''(\epsilon) = \int_{t_0}^{t_1}\{m_i\dot{\alpha}_i^2 + [\lambda_\sigma f^\sigma_{,ij}(R) - V_{,ij}(R)]\alpha_i\alpha_j + 2[\lambda_\sigma f^\sigma_{,(n+i)j}(R) - V_{,(n+i)j}(R)]\dot{\alpha}_i\alpha_j + [\lambda_\sigma f^\sigma_{,(n+i)(n+j)}(R) - V_{,(n+i)(n+j)}(R)]\dot{\alpha}_i\dot{\alpha}_j\} dt, \tag{2}$$

where $R = (x(t) + \epsilon\alpha(t), \dot{x}(t) + \epsilon\dot{\alpha}(t), t)$, and bounds on this can be established under various conditions. We shall select some of the simple ones.

THEOREM 1a. *Let the forces be of the type $\mathcal{V}_3\mathcal{C}_2\mathcal{F}_1$ and let $c_{ij}(z, t)$ be the symmetric matrix $[V_{,ij}(z, t) - \lambda_\sigma f^\sigma_{,ij}(z, t)]$. Let $\mu(z, t)$ be the largest characteristic value of $[c_{ij}(z, t)]$ and let $\mu = \sup_{(z,t)\in S} \mu(z, t)$. If $\mu \leqq 0$ or, if $\mu > 0$ and $\delta < \pi(m/\mu)^{1/2}$ then, for any D-motion ζ lying in S, $I(x) \leqq I(\zeta)$. Indeed*

$$I(\zeta) \geqq I(x) + \tfrac{1}{2}([m\pi^2/\delta^2] - \mu)\cdot\|\zeta - x\|^2.$$

Proof: $[V_{,ij}(z, t) - \lambda_\sigma f^\sigma_{,ij}(z, t)]\alpha_i\alpha_j \leqq \mu(z, t)\alpha_i\alpha_i$. By Lemma 1 $\omega''(\epsilon) \geqq \frac{1}{2}([m\pi^2/\delta^2] - \mu)\|\alpha\|^2$ so that Lemma 2 is applicable.

Remark: Let $C = \sup_{\chi \in S} \sqrt{c_{ij}(\chi)c_{ij}(\chi)}$. Then $\mu \leqq C$ and we may also conclude that $I(\zeta) \geqq I(x) + \frac{1}{2}(m\pi^2\delta^{-2} - C) \cdot \|\zeta - x\|^2$.

This result is sometimes phrased "Hamilton's Integral is a minimum if the duration is sufficiently short." On the other hand no fixed time interval is sufficiently short for all fields, even in the conservative case. This can be seen in the following way. Assume the forces are $\mathcal{V}_3\mathcal{C}_3\mathcal{J}_2$ with $V = \kappa\Psi(z, t)$. We shall say that the field is strengthened when κ is increased. Assume that $\Psi_{,rr}(z, t) \geqq \rho > 0$ and continuous in some slab of $E_n \times D: -\infty < z_i < \infty$, $t_0 \leqq \tau_0 < t < \tau_1 \leqq t_1$ for some r in $[1, 2, \cdots, n]$. We then let $\eta_r(t)$ be a function which is continuously differentiable and zero outside of $\tau_0 < t < \tau_1$ and positive inside. Let $\eta_i(t) \equiv 0$ when $i \neq r$. Let $a = \int_{t_0}^{t_1} m_r \dot{\eta}_r^2 dt$ and $b = \int_{t_0}^{t_1} \eta_r^2 dt$. Then $b > 0$ and $\omega''(\epsilon) \leqq a - \kappa\rho b$. Thus if $\kappa > a/\rho b$ then $\omega''(\epsilon) < 0$. Hence we have proved

THEOREM 1b. *Consider a field with potential $\Psi(z, t)$ such that $\Psi_{,rr}(z, t) \geqq \rho > 0$ and continuous in some slab of $E_n \times D$ for some r and such that for each κ_0 there is a $\kappa > \kappa_0$ such that there is a solution $\xi(t)$ to the equations $m_i\ddot{\xi}_i + \kappa\Psi_{,i}(\xi, t) = 0$ (no sum on i), $\xi(t_0) = x^0$, $\xi(t_1) = x^1$. Then it is always possible to strengthen the field and find a D-motion $x + \eta$ such that $I(x) > I(x + \epsilon\eta)$ for every $\epsilon > 0$.*

It is impossible to state this as a maximum principle since an examination of equation (2) for the present case and consideration of the fact that it is possible to find arbitrarily small functions with arbitrarily large derivatives shows that there are always many η's such that $I(x + \epsilon\eta)$ has a proper minimum when $\epsilon = 0$.

As an example we may consider the motion of a mass on a spring of spring constant κ. Let x be the displacement from equilibrium, $x(t_0) = 0$, $x(t_1) = x^1 \neq 0$. Here $V = \frac{1}{2}\kappa x^2$. Then, by Theorem 1a, the actual motion minimizes Hamilton's Integral if $\delta^2 < m\pi^2/\kappa$. On the other hand if, following Theorem 1b, we choose, *e.g.*, $\eta(t) = \sin \pi(t - t_0)/\delta$, then $\omega''(\epsilon) = m\pi^2/2\delta - \kappa\delta/2$, so that the actual motion does not minimize Hamilton's Integral if $\kappa > m\pi^2/\delta^2$. (We assume in each case that $\delta\sqrt{\kappa/m}$ is not an integral multiple of π so that the desired motion is possible.) Since the period is $2\pi\sqrt{m/\kappa}$, our result shows that the actual motion minimizes Hamilton's Integral if the time interval is less than half the period while it does not minimize the integral if the time interval exceeds half the period.

THEOREM 1c. *Let the forces be of type $\mathcal{V}_2\mathcal{C}_3\mathcal{J}_1$. For each t in D and each v in E_n let $\gamma(v, t)$, $\Gamma(v, t)$ denote the smallest and largest characteristic values of the symmetric matrix $[V_{,ij}(v, t)]$. Let $\gamma = \inf_{v,t} \gamma(v, t)$ and $\Gamma = \sup_{v,t} \Gamma(v, t)$. Let ζ be any D-motion. If $\Gamma < m$ then we have the minimum principle*

$$I(\zeta) \geqq I(x) + \frac{\pi^2(m - \Gamma)}{2\delta^2} \|\zeta - x\|^2;$$

but if $\gamma > M$ then we have the maximum principle

$$I(x) \geqq I(\zeta) + \frac{\pi^2(\gamma - M)}{2\delta^2} \|\zeta - x\|^2.$$

It is not difficult to construct fields satisfying either of these conditions.

5. Generalized coordinates. We now suppose that the system is described by generalized coordinates $q=(q_1, q_2, \cdots, q_n)$ with constraints $\mathfrak{f}^\sigma(q, \dot{q}, t)=0$. Here the transformation $Hz=q$ is a homeomorphism H from E_n to an n-dimensional Riemannian space $\mathfrak{E}$. Each tube Σ about the solution in $E_n \times D$ is mapped into a region $H\Sigma$ in $\mathfrak{E} \times D$ and the curves $\xi(t)$ are mapped into curves $H\xi(t)$. Since the values of the kinetic energy, potential energy and forces are the same whether computed in terms of the q's or the x's (t is the same in either case), the integrals we have been considering will have the same value whether computed along $\xi(t)$ or $H\xi(t)=q(t)$. Thus the remarks made above concerning the minimality of the integral when taken along the solution are also applicable in case the computations are made in generalized coordinates.

This remark applies also to the results in the remainder of the paper (to apply it to section (7) below replace the n here by $2n$).

6. Least action principle. Here we consider only forces of type $\mathcal{V}_4\mathcal{C}_3\mathcal{F}_2$.* With this assumption equation (1) shows that the energy $\frac{1}{2}m_i\dot{x}_i^2 + V(x)$ is a constant, say h, for all t in D. For each interval Δ: $\tau_0 \leqq t \leqq \tau_1$ and each Δ-motion ξ let $J(\xi) = \int_{\tau_0}^{\tau_1}[\frac{1}{2}m_i\dot{\xi}_i^2 - V(\xi) + h]dt$. Now let $\mathfrak{D}$ be an arbitrary but fixed interval $\mathfrak{t}_0 \leqq t \leqq \mathfrak{t}_1$ with $\mathfrak{t}_1 - \mathfrak{t}_0 = \mathfrak{d} > 0$ and $\mathfrak{d} \neq \delta$ and let $\mathfrak{X}$ be an arbitrary but fixed $\mathfrak{D}$-motion. We then inquire how $J(\mathfrak{X})$ compares with $J(x)$. (Of course if we demand of $\mathfrak{X}$ that for each t in $\mathfrak{D}$

$$\frac{1}{2}m_i\dot{\mathfrak{X}}_i^2 + V(\mathfrak{X}) = h, \tag{3}$$

then we are comparing $\int 2Tdt$ along the motions described by $\mathfrak{X}$ and x.)

In the integral for $J(\mathfrak{X})$ change the variable of integration from t to u by the formula $t = \mathfrak{t}_0 + \mathfrak{d}(u - t_0)/\delta$ and let $X(u) = \mathfrak{X}(t)$. Then $X(u)$ is a D-motion. Let $\alpha(t)$ be $X(t) - x(t)$ for each t in D and let

$$\omega(\epsilon) = \int_{t_0}^{t_1} \left\{\frac{m_i(\dot{x}_i + \epsilon\dot{\alpha}_i)^2}{2(1+\epsilon\theta)} + [h - V(x+\epsilon\alpha)][1+\epsilon\theta]\right\} dt,$$

where $\theta = \mathfrak{d}/\delta - 1 > -1$ and $0 \leqq \epsilon \leqq 1$. Now $\omega(0) = J(x)$, $\omega(1) = J(\mathfrak{X})$ and $\omega'(0) = 0$. So again the desired results can be established by the use of Lemma 2. Now

$$\omega''(\epsilon) = \int_{t_0}^{t_1} \left\{\frac{m_i(\theta\dot{x}_i - \dot{\alpha}_i)^2}{(1+\epsilon\theta)^3} - 2\theta V_{,i}(x+\epsilon\alpha)\alpha_i - V_{,ij}(x+\epsilon\alpha)\alpha_i\alpha_j(1+\epsilon\theta)\right\} dt. \tag{4}$$

Again there are many force situations for which $\omega''(\epsilon)$ can be shown to be

* The principle can be extended to the non-conservative case. *Cf.* Whittaker, *op. cit.*, p. 248.

bounded. We consider here only one simple one. Assume that $x(t)$ is not identically zero. Let $[\sup_{(z,t)\in S} \Sigma_i V_{,i}^2(z)]^{1/2} = \Lambda$ and $\sup_{(z;t)\in S} \Omega(z) = \Omega$, where $\Omega(z)$ is the maximum characteristic value of the matrix $[V_{,ij}(z)]$. Then with the help of Lemma 1 we have

$$\omega''(\epsilon) \geqq \frac{\pi^2 m}{4}(1 + \epsilon\theta)^{-3}\delta^{-2}\|\theta x - \alpha\|^2 - 2|\theta|\Lambda\sqrt{\delta}\|\alpha\| - (1 + \epsilon\theta)\Omega\|\alpha\|^2;$$

hence

$$\omega''(\epsilon) \geqq \left[\frac{\pi^2 m}{4}(1 + \epsilon\theta)^{-3}\delta^{-2} - (1 + \epsilon\theta)\Omega\right]\|\alpha\|^2$$
$$- 2|\theta| \left[\Lambda\delta^{1/2} + \frac{\pi^2 m}{4}\|x\|(1 + \epsilon\theta)^{-3}\delta^{-2}\right]\|\alpha\| + \frac{\pi^2 m}{4}\theta^2(1 + \epsilon\theta)^{-3}\delta^{-2}\|x\|^2,$$

which is of the form $A\|\alpha\|^2 - 2B\|\alpha\| + C$ with $C > 0$. Thus if $\|\alpha\|$ is sufficiently small, say less than $\mathfrak{a}$, then $\omega''(\epsilon) > 0$. To put this in more physical form observe that $\|\alpha\| \leqq (n\delta)^{1/2} \sup_{i,t} |\alpha_i(t)|$. Hence if $\sup_{i,t} |\alpha_i(t)| < (\mathfrak{a})/(n\delta)^{1/2}$ we have $\omega''(\epsilon) > 0$. To bring these results back to the interval $\mathfrak{D}$ let $\mathfrak{x}(t) = x(t_0 + (\delta/\mathfrak{d})(t - t_0))$. This proves

THEOREM 2. *Let the forces be of the type $\mathcal{V}_4\mathcal{C}_3\mathcal{F}_2$ and the solution x not identically zero. Let $\mathfrak{S}$ denote the projection of S on E_n and $\mathfrak{D}$ be any interval $\mathfrak{t}_0 \leqq t \leqq \mathfrak{t}_1$ with $\mathfrak{t}_1 - \mathfrak{t}_0 = \mathfrak{d} \neq \delta$. Let $\mathfrak{x}(t) = x(t_0 + \delta(t - \mathfrak{t}_0)/\mathfrak{d})$, $t \in \mathfrak{D}$. Then there is a $\mathfrak{b} > 0$ such that if $\mathfrak{X}(t)$ is any $\mathfrak{D}$-motion whose values lie in $\mathfrak{S}$ and satisfy the condition $\sup_{i;t\in\mathfrak{D}} |\mathfrak{X}_i(t) - \mathfrak{x}_i(t)| < \mathfrak{b}$ then $J(x) < J(\mathfrak{X})$.*

One might remark that, in the above discussion, if $A > 0$ and $C > B^2/A$ then $\omega''(\epsilon) > 0$ for all values of $\|\alpha\|$, so that the last inequality in the conditions of the above theorem as well as the conditions $\mathfrak{d} \neq \delta$ and $x \not\equiv 0$ could then be dropped.

If we require of $\mathfrak{X}(t)$ that it be a $\mathfrak{D}$-motion satisfying (3) then: (a) $J(\mathfrak{X}) = \int_{t_0}^{t_1} 2T dt$; (b) $J(\mathfrak{X}) = 2\int_{x^0}^{x^1} \sqrt{h - V(\mathfrak{X})} ds$, [where $(ds)^2 = \frac{1}{2}m_i(d\mathfrak{X}_i)^2$]; (c) if also $V = 0$ then $J(\mathfrak{X}) = 2h(\mathfrak{t}_1 - \mathfrak{t}_0)$; and (d) if also $V = 0$ then $J(\mathfrak{X}) = 2\sqrt{h}\int_{x^0}^{x^1} ds$. Thus statements about the minimality of $J(x)$ when compared to $J(\mathfrak{X})$ for any $\mathfrak{X}$ satisfying (3) are also statements about the minimality of (a) Maupertuis' Least Action Principle, (b) Jacobi's Least Energy Principle, (c) The principle of least time and (d) the geodesic principle. Indeed in the last two cases equation (4) shows immediately that $\omega''(\epsilon) \geqq 0$ so that these are truly *minimum* principles.

7. Hamilton's Modified Principle. We introduce now points $(\xi, \psi) = \{\xi_1, \xi_2, \cdots, \xi_n, \psi_1, \psi_1, \cdots, \psi_n\}$ in $E_n \times E_n$. The motion of $\mathfrak{M}$ will now be described by an $E_n \times E_n$-valued function of t: $(x(t), p(t)) = \{x_1(t), x_2(t), \cdots, x_n(t), p_1(t), p_2(t), \cdots, p_n(t)\}$, $t \in D$. This we shall call the *actual phase path*. For any interval Δ: $\tau_0 \leqq t \leqq \tau_1$ we shall define a Δ-*phase motion* to be any $E_n \times E_n$-valued function $(\xi(t), \psi(t))$ which has a continuous derivative on Δ and satisfies the conditions $\xi(t_0) = x^0$, $\xi(t_1) = x^1$. Let the forces be of the type $\mathcal{V}_3\mathcal{C}_3\mathcal{F}_2$. The actual

phase path is then, according to Lagrange's equations, a D-phase path satisfying the $2n$ equations

$$
\begin{aligned}
&p_i = m_i\dot{x}_i \text{ (no sum)},\\
&\dot{p}_i = -V_{,i}(x, t).
\end{aligned}
\tag{5}
$$

For each D-phase path (ζ, π) define

$$K(\zeta, \pi) = \int_{t_0}^{t_1}\left[\pi_i\dot{\zeta}_i - \frac{\pi_i^2}{2m_i} - V(\zeta, t)\right]dt.$$

Let (X, P) be a D-phase motion. Let $\alpha = X - x$, $\beta = P - p$ and

$$
\begin{aligned}
\omega(\epsilon) &= K(x + \epsilon\alpha, p + \epsilon\beta)\\
&= \int_{t_0}^{t_1}\left[(p_i + \epsilon\beta_i)(\dot{x}_i + \epsilon\dot{\alpha}_i) - \frac{(p_i + \epsilon\beta_i)^2}{2m_i} - V(x + \epsilon\alpha, t)\right]dt.
\end{aligned}
$$

Then $\alpha(t_0) = \alpha(t_1) = 0$, $\omega(0) = K(x, p)$, $\omega(1) = K(X, P)$ and $\omega'(0) = 0$. Now

$$\omega''(\epsilon) = -\int_{t_0}^{t_1}[2\alpha_i\dot{\beta}_i + \beta_i^2/m_i + V_{,ij}(x + \epsilon\alpha, t)\alpha_i\alpha_j]dt.$$

This shows that we can always find a path (X, P) arbitrarily close to (x, p) such that $\omega''(\epsilon) > 0$ and another such that $\omega''(\epsilon) < 0$. This follows from the fact that $|\beta_i|$ can be arbitrarily small while $|\dot{\beta}_i|$ is arbitrarily large. At the same time α_i could be arbitrarily small in absolute value but with suitable variations in sign so that the first term in the integrand will determine the sign of the whole integrand and be either positive or negative at our choice. Hence

THEOREM 3. *Hamilton's Modified Principle cannot be stated as a minimum (or a maximum) principle for a conservative field regardless of the assumptions regarding the field or the length of the time interval.*

We shall consider finally Hilbert's Variational Principle. Let the forces be of type $\mathcal{V}_4\mathcal{C}_3\mathcal{J}_2$. The actual phase path is a D-phase motion satisfying the equations (5), and now also

$$\frac{p_i^2}{2m_i} + V = h.$$

For each interval Δ: $\tau_0 \leqq t \leqq \tau_1$ and any Δ-phase path (ξ, ψ) define

$$G(\xi, \psi) = \int_{\tau_0}^{\tau_1}\left[\dot{\xi}_i\psi_i - \frac{\psi_i^2}{2m_i} - V(\xi) + h\right]dt.$$

Now let $\mathfrak{D}$ be an arbitrary but fixed interval $\mathfrak{t}_0 \leqq t \leqq \mathfrak{t}_1$ where $\mathfrak{t}_1 - \mathfrak{t}_0 = \mathfrak{d} > 0$ and

let $(\mathfrak{X}, \mathfrak{P})$ be an arbitrary but fixed $\mathfrak{D}$ phase-motion. In the integral defining $G(\mathfrak{X}, \mathfrak{P})$ let $t=t_0+\mathfrak{d}(u-t_0)/\delta$ and let $X(u)=\mathfrak{X}(t)$, $P(u)=\mathfrak{P}(t)$. Let $\alpha(t)=X(t)-x(t)$, $\beta(t)=P(t)-p(t)$ and

$$\omega(\epsilon) = \int_{t_0}^{t_1} (p_i + \epsilon\beta_i)(\dot{x}_i + \epsilon\dot{\alpha}_i) - [1 + \epsilon\theta]\left[\frac{(p_i + \epsilon\beta_i)^2}{2m_i} + V(x + \epsilon\alpha) - h\right] dt,$$

where $\theta = \mathfrak{d}/\delta - 1$.

Then $\omega(0) = G(x, p)$, $\omega(1) = G(\mathfrak{X}, \mathfrak{P})$, $\omega'(0) = 0$. Now

$$\omega''(\epsilon) = -\int_{t_0}^{t_1} \left\{2\alpha_i\dot{\beta}_i + [1 + \epsilon\theta]\left[\frac{\beta_i^2}{m_i} + V_{,ij}(x + \epsilon\alpha)\alpha_i\alpha_j\right]\right.$$

$$\left. + 2\theta\left[\frac{(p_i + \epsilon\beta_i)\beta_i}{m_i} + V_{,i}(x + \epsilon\alpha)\alpha_i\right]\right\} dt.$$

The same sort of considerations apply here as to the previous case and so the principle can not be stated as a minimum (or a maximum) principle.

The writer wishes to thank Professors F. Koehler, A. Milgram, and C. Langenhop for helpful discussions concerning the material presented here.

ON THE REDUCTION OF A MATRIX TO DIAGONAL FORM

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Given an $n \times n$ matrix $A = (a_{ij})$ with elements in a field F, one wishes to know when it is possible to find a matrix P so that the matrix $D = PAP^{-1}$ is in diagonal form. Of course even when this is possible it will be necessary, in general, to take the elements of P and therefore of D in some extension field of F, specifically in the algebraic closure of F. Nevertheless it is desirable to have a rational criterion for the diagonalizability of A, *i.e.*, a criterion which can be verified entirely within the given field F irrespective of where the elements of P and D lie. In this paper we give such a criterion.

We shall restrict attention to the case of fields of characteristic zero, and we denote by I the identity $n \times n$ matrix.

By the degree of a matrix A we shall mean the degree of its minimal polynomial. Thus if A has degree s then $I, A, \cdots, A^{s-1}$ are linearly independent while $I, A, \cdots, A^{s-1}, A^s$ are linearly dependent over F.

THEOREM. *Let A be an $n \times n$ matrix of degree s with elements in a field F of characteristic zero. Let S denote the trace and Δ the determinant*

$$\Delta = \det\,[S(A^{i+j})], \qquad (0 \leqq i, j \leqq s - 1).$$

Then A can be reduced to diagonal form (in some extension field of F) if and only if $\Delta \neq 0$.

Proof. If A can be reduced to diagonal form then

$$PAP^{-1} = B = \begin{pmatrix} \lambda_1 I_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r I_r \end{pmatrix}$$

where $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A and I_k is the n_k-rowed identity matrix ($n_1 + \cdots + n_r = n$). Evidently $r = s$ and for $u = 1, 2, 3, \cdots, S(A^u) = S(B^u)$, hence

$$S(A^{i+j}) = n_1\lambda_1^{i+j} + \cdots + n_r\lambda_r^{i+j}, \qquad (0 \leqq i, j \leqq r-1).$$

This implies, by the product rule for determinants

$$\Delta = \det\,[S(A^{i+j})] = \begin{vmatrix} n_1 & n_2 & \cdots & n_r \\ n_1\lambda_1 & n_2\lambda_2 & \cdots & n_r\lambda_r \\ \cdot & \cdot & \cdot & \cdot \\ n_1\lambda_1^{r-1} & n_2\lambda_2^{r-1} & \cdots & n_r\lambda_r^{r-1} \end{vmatrix} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{r-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{r-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda_r & \cdots & \lambda_r^{r-1} \end{vmatrix}$$

$$= n_1 \cdots n_r V^2,$$

where V is the (non-vanishing) Vandermonde determinant of $\lambda_1, \cdots, \lambda_r$.

Conversely, suppose A cannot be put in diagonal form. We see from the Jordan canonical form that if $\lambda_1, \cdots, \lambda_r$ are the distinct characteristic roots of A and since s is the degree of A, $s \geqq r$. As before let λ_k have multiplicity n_k so for some k, $n_k > 1$ (or else A is diagonalizable). It is then easily seen (again from the Jordan canonical form) that $(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_r I) \neq 0$, so that $s > r$. We again have

$$S(A^{i+j}) = n_1\lambda_1^{i+j} + \cdots + n_r\lambda_r^{i+j}, \qquad (0 \leqq i, j \leqq s-1),$$

but since Δ is the determinant of an $s \times s$ matrix we have

$$\Delta = \det\,[S(A^{i+j})] = \begin{vmatrix} n_1 & \cdots & n_r & 0 & \cdots & 0 \\ n_1\lambda_1 & \cdots & n_r\lambda_r & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n_1\lambda_1^{s-1} & \cdots & n_r\lambda_r^{s-1} & 0 & \cdots & 0 \\ & & & & & \end{vmatrix} \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{s-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda_r & \cdots & \lambda_r^{s-1} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0.$$

Example. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Clearly the degree of A is 2. We have

$$A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \quad \text{so} \quad \Delta = \begin{vmatrix} S(I) & S(A) \\ S(A) & S(A^2) \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 5 & 29 \end{vmatrix} = 33 \neq 0;$$

hence A is reducible to diagonal form. We can choose

$$P = \begin{pmatrix} -3+\sqrt{33} & 4 \\ -3-\sqrt{33} & 4 \end{pmatrix} \quad \text{and then} \quad D = \begin{pmatrix} \frac{1}{2}(5+\sqrt{33}) & 0 \\ 0 & \frac{1}{2}(5-\sqrt{33}) \end{pmatrix}.$$

The above theorem follows also from certain well-known results in the theory of linear algebras. To consider the problem from this viewpoint, we fix the matrix A and the field F as well as an algebraic closure $\overline{F}$ of F.

We say that A is reducible over F if, by a multiplication of the type QAQ^{-1}, where Q is an $n \times n$ matrix with elements in F, A can be transformed to a matrix of the form

$$\begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix}$$

where, for some m, $1 \leqq m \leqq n-1$, M_1 is an $m \times m$ matrix, M_3 is an $(n-m) \times (n-m)$ matrix, and M_2 is an $m \times (n-m)$ matrix. Otherwise we say that A is irreducible. We say that A is completely reducible if, by a multiplication of the above type, A can be transformed to a matrix of the type

$$\begin{pmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & N_t \end{pmatrix}$$

where each N_i is an irreducible $n_i \times n_i$ matrix ($n_1 + \cdots + n_t = n$). In particular, (for $t=1$) an irreducible matrix is completely reducible.

It can be shown that if A is irreducible over F then its minimal polynomial is irreducible over F; more generally, A is completely reducible over F if and only if its minimal polynomial is the product of distinct factors each irreducible over F, or equivalently, if and only if its minimal polynomial is the product of distinct linear factors over $\overline{F}$. On the other hand, over $\overline{F}$ the latter condition on the minimal polynomial is necessary and sufficient for the diagonalizability of A. Since the minimal polynomial of A is an invariant, depending only on the field containing the elements of A, it follows that A is diagonalizable over $\overline{F}$ if and only if A is completely reducible over F.

A set of $n\times n$ matrices with elements in F is called an algebra over F if it is a vector space over F (necessarily finite-dimensional) and is closed under matrix multiplication. The algebra over F generated by A is the smallest algebra of $n\times n$ matrices over F which contains A and I. This algebra consists of linear combinations over F of powers of A (including $I=A^0$) and a basis for the vector space underlying the algebra is $I, A, \cdots, A^{r-1}$ when A is of degree r.

A set $\mathfrak{M}$ of $n\times n$ matrices over F may be regarded as a set of linear endomorphisms of an n-dimensional vector space V over F with a fixed basis. A subspace W of V is said to be invariant under $\mathfrak{M}$ if each endomorphism in $\mathfrak{M}$ maps W into itself. An invariant subspace is said to be simple under $\mathfrak{M}$ if it contains no other invariant subspace except (0).

An algebra $\mathfrak{A}$ over F of $n\times n$ matrices containing I is said to be semi-simple if V is the direct sum of simple subspaces invariant under $\mathfrak{A}$. If $\mathfrak{A}$ is generated by A then it is known that $\mathfrak{A}$ is semi-simple if and only if A is completely reducible over F.

A theorem in the theory of algebras states that an algebra $\mathfrak{A}$ of matrices over a field F of characteristic zero is semi-simple if and only if the function $S(XY)$ (where $X, Y\in\mathfrak{A}$ and S denotes the trace function) is non-degenerate, *i.e.*, if $X\in\mathfrak{A}$ is such that $S(XY)=0$ for every $Y\in\mathfrak{A}$, then $X=0$.

Combining these results we see that if F is of characteristic zero and $\mathfrak{A}$ is the algebra over F generated by A, then the following are equivalent:

(1) A is diagonalizable over $\overline{F}$;
(2) the minimal polynomial of A is the product of distinct factors each irreducible over F;
(3) A is completely reducible over F;
(4) $\mathfrak{A}$ is semi-simple;
(5) $S(XY)=0$ for fixed $X\in\mathfrak{A}$ and all $Y\in\mathfrak{A}$ implies $X=0$.

Now let $x_0, \cdots, x_{r-1}$ be indeterminates and let $X=\sum_{i=0}^{r-1}x_iA^i$. Then

$$S(XA^j) = S\left(\sum_{i=0}^{r-1} x_iA^{i+j}\right) = \sum_{i=0}^{r-1} S(A^{i+j})x_i, \quad (0 \leqq j \leqq r-1).$$

It is clear that the function $S(XY)$ will be degenerate on $\mathfrak{A}$ if and only if there exist elements $\alpha_0, \cdots, \alpha_{r-1}\in F$, not all zero, such that

$$\sum_{i=0}^{r-1} S(A^{i+j})\alpha_i = 0, \qquad (0 \leqq j \leqq r-1).$$

Thus the non-degeneracy of $S(XY)$ is equivalent to the requirement that there exists only the trivial solution for a set of r homogeneous linear equations over F in r unknowns. This requirement is satisfied if and only if the determinant of coefficients, $\det [S(A^{i+j})]$, is non-vanishing.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—IV (AN INEQUALITY DUE TO BERGSTRÖM)

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1. Introduction. In a recent note,* Bergström proved the following interesting inequality:

Let A and B be positive definite matrices and let A_{ii}, B_{ii} denote the sub-matrices obtained by deleting the i-th row and column. Then

$$\frac{|A+B|}{|A_{ii}+B_{ii}|} \geqq \frac{|A|}{|A_{ii}|} + \frac{|B|}{|B_{ii}|}, \tag{1}$$

where $|\ |$ represents the determinant.

Bergström's proof is essentially a verification. We present two proofs below, the second of which lays bare the origin of the result.

2. First proof. The first proof is an immediate consequence of the result:

LEMMA 1: *If A is positive definite, then*

$$\phi(A) = \frac{|A|}{|A_{ii}|} = \min_x \sum_{i,j=1}^{N} a_{ij}x_ix_j, \tag{1}$$

where x is constrained by

$$x_i = 1. \tag{2}$$

From this it is clear that $\phi(A+B) \geqq \phi(A)+\phi(B)$.

We shall not present the proof, which is easily obtained by the use of a Lagrange multiplier, since Lemma 1 is a special case of the more general result established in the next section.

3. Second proof. We begin by establishing

LEMMA 2: *If A is positive definite, then*

$$(x, Ax)(y, A^{-1}y) \geqq (x, y)^2, \tag{1}$$

for all x and y.

Here (x, y) denotes the inner product of x and y and (x, Ax) the quadratic form $\sum_{ij}a_{ij}x_ix_j$. A^{-1} is the inverse of A.

Proof of Lemma 2: Reduce A to diagonal form by an orthogonal matrix T,

* H. Bergström, "A triangle inequality for matrices," *Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949*, Oslo, 1952, pp. 264–267.

i.e., $T'AT=L$, $T'=T^{-1}$. Let $x=Tu$, $y=Tv$. Then (1) becomes

$$\left(\sum_{i=1}^{N} \lambda_i u_i^2\right)\left(\sum_{i=1}^{N} v_i^2/\lambda_i\right) \geqq \sum_{i=1}^{N} u_i v_i, \tag{2}$$

which is the Cauchy-Schwarz inequality.

Since the inequality becomes an equality for suitable choice of x, we have

$$\min_x \frac{(x, Ax)}{(x, y)^2} = \frac{1}{(y, A^{-1}y)} = \psi(A). \tag{3}$$

From this it is immediate that

$$\psi(A+B) \geqq \psi(A) + \psi(B). \tag{4}$$

The case $y_i=1$, $y_j=0$, $j\neq i$ yields Bergström's result.

A REMARK ON MATRIC POLYNOMIALS AND SIMILARITY OF MATRICES

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Let F be a field and let F_n be the set of all n-rowed square matrices with elements in F. It is customary to call $A(x)=A_m x^m + \cdots + A_1 x + A_0$ a *matric polynomial* if A_i is in F_n $(i=0, \cdots, m)$. If C is also in F_n, we define, as usual, $A_R(C)=A_m C^m + \cdots + A_1 C + A_0$. Our purpose is to note a rather important consequence of the following innocent lemma.

LEMMA. *If $B(x)$ is also a matric polynomial, then*

$$(A(x)B(x))_R(C) = A_m B_R(C)C^m + \cdots + A_1 B_R(C) + A_0 B_R(C).$$

Proof. This is clear, since $A(x)B(x)=A_m B(x)x^m + \cdots + A_1 B(x)x + A_0 B(x)$.

COROLLARY 1. *If $B_R(C)=0$, then $(A(x)B(x))_R(C)=0$.*

Our next Corollary is quite useful in certain parts of matric theory.

COROLLARY 2. *If M and N are in F_n and $P(x)(Ix-M)Q(x)=Ix-N$ for matric polynomials $P(x)$ and $Q(x)$ which have matric polynomial inverses, then $P=Q_R(N)$ is non-singular and $N=P^{-1}MP$.*

Proof. We have $(Ix-M)Q(x)=P^{-1}(x)(Ix-N)$ and it is a consequence of Corollary 1 that $((Ix-M)Q(x))_R(N)=0$. Apply the Lemma to obtain $Q_R(N)N - MQ_R(N)=0$, or $PN=MP$. Now $Q^{-1}(x)Q(x)=I$, and let us write $Q^{-1}(x) = U_t x^t + \cdots + U_1 x + U_0$. Apply the Lemma again, this time to $I=(Q^{-1}(x)Q(x))_R(N)$ to obtain

$$I = U_t Q_R(N)N^t + \cdots + U_1 Q_R(N)N + U_0 Q_R(N) = U_t PN^t + \cdots + U_1 PN + U_0 P.$$

Now use $PN=MP$ repeatedly to find that

$$I = U_t M^t P + \cdots + U_1 MP + U_0 P = (Q_R^{-1}(M))P.$$

This proves that P is non-singular and $N=P^{-1}MP$ follows from $PN=MP$.

Our proof of Corollary 2 may be put to use in the actual *computation* of the matrix P which yields canonical forms for A in F_n under similarity. Thus we may reduce $Ix-A$ to Smith canonical form $S(x)$ by elementary operations. We carefully record the *column* operations used, being as economical as we can in our use of such operations. Let J be the Jordan canonical matrix* and let K be the classical canonical matrix* of A. It is easy to list elementary operations which replace $S(x)$ by $Ix-J$. We then have $P(x)(Ix-A)Q(x)=Ix-J$ and hence also $J=P^{-1}AP$ with $P=Q_R(J)$, $P^{-1}=Q^{-1}(A)$. To obtain $Q^{-1}(x)$ we apply to I the elementary column operations which led to $Q(x)$ *in reverse order*. One may carry out an analogous procedure to obtain P_1 (and P_1^{-1}) such that $P_1^{-1}AP_1=K$.

Our proof of Corollary 2 and our remarks concerning computation are valid if F is a division ring. It is not difficult to adapt our proof to the more general case of σ-similarity of matrices ($N=P^{-1}MP^{\sigma}$, σ an automorphism of F) which is appropriate to a discussion of the matrices which represent a *semi-linear* transformation T with automorphism σ. For simplicity we have chosen to write our proof for the case σ the identity automorphism, T a linear transformation.

Although we believe that our *proof* of Corollary 2 is new, we should like to note in conclusion that all of the results we have stated for F a *field* may be found in many of the large number of recent texts on matric theory. For the case of a division ring, the reader should consult the book *The Theory of Rings* by N. Jacobson.

* We follow N. Jacobson, *Lectures in Abstract Algebra II, Linear Algebra*, pp. 93–94, in our use of these terms.

PARTICULAR INTEGRALS OF LINEAR DIFFERENTIAL EQUATIONS

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In this note a convenient method is developed for obtaining a particular solution of the linear differential equation with constant coefficients

$$(D^n + a_1D^{n-1} + \cdots + a_n)Y = f(x). \tag{1}$$

The method is simpler than the Lagrange method of "variation of parameters," especially when the auxiliary equation has repeated roots. Also, the procedure is more easily extended to the case of simultaneous equations than is Lagrange's method.

We start by assuming a solution of the form

$$Y = \int_0^x G(x-\xi)f(\xi)d\xi, \tag{2}$$

where the function G is to be determined.† Then

† The reader who is familiar with operational methods will recognize a well known theorem on convolution integrals. This type of solution is also suggested by the formula resulting from Lagrange's method when the auxiliary equation has distinct roots.

$$Y' = G(0)f(x) + \int_0^x G'(x - \xi)f(\xi)d\xi. \tag{3}$$

If now it is specified that $G(0)$ be zero, then

$$Y' = \int_0^x G'(x - \xi)f(\xi)d\xi. \tag{4}$$

Similarly,

$$Y'' = G'(0)f(x) + \int_0^x G''(x - \xi)f(\xi)d\xi, \tag{5}$$

and if we take $G'(0) = 0$ this reduces to

$$Y'' = \int_0^x G''(x - \xi)f(\xi)d\xi. \tag{6}$$

Proceeding in this manner, we find that

$$Y^{(n)} = G^{(n-1)}(0)f(x) + \int_0^x G^{(n)}(x - \xi)f(\xi)d\xi, \tag{7}$$

where the condition $G^{(k)}(0) = 0$ has been applied to G and its first $(n-2)$ derivatives. Substitution into (1) yields

$$G^{(n-1)}(0)f(x) + \int_0^x [G^{(n)}(x-\xi) + a_1G^{(n-1)}(x-\xi) + \cdots + a_nG(x-\xi)]f(\xi)d\xi = f(x), \tag{8}$$

and it is seen that (5) is a solution of (7) provided $G(s)$ is a solution of the homogeneous equation, and

$$G^{(n-1)}(0) = 1.$$

Thus G is completely determined by the homogeneous equation and the initial conditions $G(0) = G^{(k)}(0) = 0$, $k = 1, \cdots, (n-2)$, $G^{(n-1)}(0) = 1$.

As an example we consider the equation

$$(D^4 - 2a^2D^2 + a^4)Y = f(x). \tag{9}$$

The solution to the homogeneous equation which satisfies the conditions $Y(0) = Y'(0) = Y''(0) = 0$, $Y'''(0) = 1$, is

$$\frac{1}{2a^3}[(ax)\cosh(ax) - \sinh(ax)]. \tag{10}$$

Hence a particular solution of (17) is

$$\frac{1}{2a^3}\int_0^x [a(x - \xi)\cosh a(x - \xi) - \sinh a(x - \xi)]f(\xi)d\xi. \tag{11}$$

The present method is also advantageous in the case of simultaneous equations for which the method of variation of parameters is not directly applicable. For example, consider the pair of equations:

$$
\begin{aligned}
&[a_0D^n + a_1D^{n-1} + \cdots + a_n]Y + [b_0D^n + b_1D^{n-1} + \cdots + b_n]Z = f(x),\\
&[c_0D^n + c_1D^{n-1} + \cdots + c_n]Y + [d_0D^n + d_1D^{n-1} + \cdots + d_n]Z = g(x),
\end{aligned}
\tag{12}
$$

where the a's, b's, c's and d's are constants. A particular solution of this system can be obtained by taking

$$
\begin{aligned}
Y &= \int_0^x G_1(x-\xi)f(\xi)d\xi + \int_0^x G_2(x-\xi)g(\xi)d\xi,\\
Z &= \int_0^x H_1(x-\xi)f(\xi)d\xi + \int_0^x H_2(x-\xi)g(\xi)d\xi.
\end{aligned}
\tag{13}
$$

Substituting into the system, we find that G_1 and H_1, G_2 and H_2 are solutions of the homogeneous equations subject to the conditions

$$
\begin{aligned}
&\begin{cases} G_1(0) = G_1^{(k)}(0) = 0 \\ H_1(0) = H_1^{(k)}(0) = 0, \ [k = 0, 1, \cdots, (n-2)] \end{cases}
&&\begin{cases} a_0G_1^{(n-1)}(0) + b_0H_1^{(n-1)}(0) = 1 \\ c_0G_1^{(n-1)}(0) + d_0H_1^{(n-1)}(0) = 0 \end{cases}\\
&\begin{cases} G_2(0) = G_2^{(k)}(0) = 0 \\ H_2(0) = H_2^{(k)}(0) = 0, \ [k = 0, 1, \cdots, (n-2)] \end{cases}
&&\begin{cases} a_0G_2^{(n-1)}(0) + b_0H_2^{(n-1)}(0) = 0 \\ c_0G_2^{(n-1)}(0) + d_0H_2^{(n-1)}(0) = 1. \end{cases}
\end{aligned}
\tag{14}
$$

It should be noted that the determinant $[a_0d_0 - b_0c_0]$ can be assumed to be non-vanishing, for otherwise it would be possible to reduce the order of the system by one through suitable linear combinations of the equations.

As an example, consider the system

$$
\begin{aligned}
[D^2 - a^2]Y - 2aDZ &= f(x),\\
2aDY + [D^2 - a^2]Z &= g(x).
\end{aligned}
\tag{15}
$$

The solution of the homogeneous equations which satisfies the conditions $Y(0) = Z(0) = 0$, $Y'(0) = 1$, $Z'(0) = 0$ is

$$
Y = x\cos(ax), \qquad Z = -x\sin(ax), \tag{16}
$$

and the solution satisfying the conditions $Y(0) = Z(0) = 0$, $Y'(0) = 0$, $Z'(0) = 1$ is

$$
Y = x\sin(ax), \qquad Z = x\cos(ax). \tag{17}
$$

Therefore a solution to the non-homogeneous equations is

$$
\begin{aligned}
Y &= \int_0^x (x-\xi)\cos a(x-\xi)f(\xi)d\xi + \int_0^x (x-\xi)\sin a(x-\xi)g(\xi)d\xi\\
Z &= -\int_0^x (x-\xi)\sin a(x-\xi)f(\xi)d\xi + \int_0^x (x-\xi)\cos a(x-\xi)g(\xi)d\xi.
\end{aligned}
\tag{18}
$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

EQUATIONS REDUCIBLE TO LINEAR FORM

E. A. NORDHAUS, Michigan State College

In standard texts on ordinary differential equations the general solution of Bernoulli's equation $dy/dx = yA(x) + y^nB(x)$, where $n \neq 0, 1$, is obtained by reducing it to linear form using the transformation $z = y^{1-n}$. This artifice is usually unmotivated and although the observations made in this note may possibly be embedded somewhere in the extensive literature, it is apparently not widely known that there is a large class of first order differential equations which may be reduced to linear form by a transformation of the dependent variable.

It will be shown (under suitable differentiability assumptions for the functions involved) that an equation $dy/dx = f(x, y)$ can be reduced to linear form if and only if the right member has the form $f(x, y) = A(x)M(y) + B(x)N(y)$, where $MN' - NM' = cN$, c a non-zero constant. An effective transformation is $z = M/N$. Throughout this note accents are used to indicate derivatives taken with respect to the argument of the function involved. In the case of the Bernoulli equation we may choose $M(y) = y$, $N(y) = y^n$ and verify that $MN' - NM' = (n-1)N$, $z = M/N = y^{1-n}$. Additional examples may be readily constructed by selecting $N(y)$ and solving the resulting linear equation $dM/dy = MN'/N - c$. A particular solution is $M(y) = N\int dy/N$.

The proof is immediate.* We note that an equation (1) $y' = f(x, y)$ can be put into the linear form (2), $z' = a(x)z + b(x)$, by a transformation (3), $z = q(y)$, if and only if (2) can be put into form (1) by (3). This implies that $q'(y)y' = a(x)q(y) + b(x)$, so $f(x, y)$ must have the form $A(x)M(y) + B(x)N(y)$ with $M = kq/q'$ and $N = h/q'$. Then $q = (h/k) \cdot (M/N)$, and since

$$q' = \frac{h}{k} \cdot \frac{NM' - MN'}{N^2} = \frac{h}{N},$$

it follows that $NM' - MN' = kN$.

The method used above may be readily extended to the reduction of equations of higher order to linear form, and similar necessary and sufficient conditions are obtainable. However, as might be expected, the form of the function f grows increasingly specialized, so that these extensions seem to be of little interest.

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* The author wishes to thank the referee for suggesting the following simplified proof.

MEAN VALUE THEOREMS FOR FUNCTIONS WITH LEFT AND RIGHT DERIVATIVES

W. R. UTZ, University of Missouri

This note is concerned with the following generalizations of Rolle's theorem and two well-known mean value theorems. Throughout the note $f'_-(x)$ and $f'_+(x)$ denote left and right derivatives, respectively.

THEOREM 1. *If $f(x)$ is a continuous function for $a \leqq x \leqq b$, $f(a)=f(b)=0$ and $f'_-(x)$, $f'_+(x)$ exist in $a<x<b$, then there exist numbers ξ, p, q with $a<\xi<b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$pf'_+(\xi) + qf'_-(\xi) = 0.$$

THEOREM 2. *If $f(x)$ is a continuous function for $a \leqq x \leqq b$, $f'_+(x)$ and $f'_-(x)$ exist for $a<x<b$, then there exist numbers ξ, p, q with $a<\xi<b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$\frac{f(b) - f(a)}{b - a} = pf'_+(\xi) + qf'_-(\xi).$$

THEOREM 3. *If $f(x)$ and $g(x)$ are continuous for $a \leqq x \leqq b$ and $f'_+(x)$, $f'_-(x)$, $g'_+(x)$, $g'_-(x)$ exist on $a<x<b$, then there exist numbers ξ, p, q with $a<\xi<b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$[pf'_+(\xi) + qf'_-(\xi)](g(b) - g(a)) = [pg'_+(\xi) + qg'_-(\xi)](f(b) - f(a)).$$

Theorems 1 and 2 are given by Jovan Karamata [*Sur la formule des accroissements finis*, Srpska Akad. Nauka. Zbornik Radova 7, Matematički Inst., vol. 1, 1951, pp. 119–124 (Serbo-Croatian. French summary.)] and Theorem 3 is given by V. Vučković [*Quelques extensions des théorèmes de moyenne*, Srpska Akad. Nauka. Zbornik Radova, vol. 18, Matematički Inst. 2, 1952, pp. 159–166 (Serbo-Croatian. French summary.)]. However, in both papers the authors maintain that a stronger conclusion is secured, *viz.*, "$p \geqq 0$, $q \geqq 0$" is, in each theorem, replaced by "$p>0$, $q>0$." To see that one can not secure this conclusion in any one of the three theorems let $a=-1$, $b=1$,

$$f(x) = \begin{cases} x^3 - 1, & x \geqq 0; \\ -x - 1, & x < 0. \end{cases}$$

In Theorems 1 and 2, q must be 0 and if we take $g(x)=x$, then q must be 0 in Theorem 3.

The author has found the theorems suitable exercises for advanced calculus classes. The proofs of Theorems 2 and 3 follow from Theorem 1 by arguments parallel to those used to secure the corresponding theorems for differentiable functions from Rolle's theorem.

We shall sketch a proof of Theorem 1. If $f(x) \equiv 0$ one may take, as usual,

$\xi=(b-a)/2$, $p=q=1/2$. If $f(x)\neq 0$ then $f(x)$, being continuous, assumes a positive maximum or negative minimum (or both) in $a<x<b$. Let $x=\xi$, $a<\xi<b$, be such that $f(\xi)>0$ is a maximum or $f(\xi)<0$ is a minimum. If $f'_-(\xi)=f'_+(\xi)=f'(\xi)$, one continues as in Rolle's theorem with $p=q=1/2$. Suppose $f'_-(\xi)\neq f'_+(\xi)$. If $f'_-(\xi)=0$, let $p=0$, $q=1$; if $f'_+(\xi)=0$, let $p=1$, $q=0$. If $f'_-(\xi)f'_+(\xi)\neq 0$, then this product is negative as can be seen from the definitions of $f'_-(x)$ and $f'_+(x)$. In this final case the equations $p+q=1$ and $pf'_+(\xi)+qf'_-(\xi)=0$ are satisfied by

$$p=\frac{f'_-(\xi)}{f'_-(\xi)-f'_+(\xi)}\geqq 0,\qquad q=\frac{f'_+(\xi)}{f'_+(\xi)-f'_-(\xi)}\geqq 0.$$

ON EQUIVALENCE RELATIONS

W. T. GUY, JR., The University of Texas

Probably many teachers who teach both elementary and advanced courses are frequently struck with the idea that some of the concepts and troublesome spots in elementary courses could be cleared up much more quickly, as well as more easily, with the "machinery" of the advanced courses. Despite the fact that usually this more powerful structure can not be brought to bear on the elementary work, it is felt that we should actively look for cases in which it can be used with profit for our students. The purpose of this note is to set forth another plea that the concept of equivalence relation be introduced early in the students' training—even as early as in the algebra course of our present day high schools.

College and university teachers of my acquaintance frequently complain that the students do not know or appreciate the difference between an identity, a definition, and a conditional equation. I have accused college algebra students of using the equal sign as the "pause that refreshes" when they come to a place where they hesitate or have to stop to think. After deciding what to do they start writing just to the right of this "pause" sign, although they did not intend to do so originally. Many of us have received papers from students containing symbols with equal signs sprinkled around like salt and pepper, similar to

$$y^2+4y=-4=(y+2)^2=0=y+2=0=y=-2,\text{ or}$$
$$\sum 1/n^2=\sum 1/n^p=p>1=\text{converges by the }p\text{-series test.}$$

Needless to say, such scribbling shows a complete lack of understanding of or appreciation for an equal sign.

Recently a graduate student who was supposed to be well along in his work said that the usual hypotheses for an equivalence relation were redundant. He referred to the

DEFINITION. *An equivalence relation* (*symbol* $\sim$) *is said to be defined over a*

non-empty set S of elements if and only if given any two elements a and b of S either $a \sim b$ (read a equivalent to b) or $(a \sim b)'$ (read a not equivalent to b), but not both, such that

(i) *$a \sim a$ for every a of S,*

(ii) *$a \sim b$ implies $b \sim a$ for every a and b of S, and*

(iii) *$a \sim b$ and $b \sim c$ imply $a \sim c$ for every a, b, and c of S.*

Freshmen students quickly see that our usual use of an equal sign is a special instance of an equivalence relation defined over the set of objects under discussion. They also realize that this definition does not define an equal sign.

The student cited above was disturbed by the hypothesis $a \sim a$ and expressed surprise when sets with relations were exhibited such that any two of the three hypotheses were fulfilled whereas the remaining one was not. Such examples can be constructed without end and three have been selected to be displayed here. The hypotheses are satisfied vacuously in one of then. This was done deliberately since it has been my experience that such examples will stimulate quite a heated class discussion, after which the students will be able to construct more self-satisfying examples. It is felt that after such discussions and examples the students involved will have a much better appreciation for the equal symbol as it is is usually used.

Example 1. Let the elements of the set S_1 be denoted and arranged in a vertical column as indicated. By $x \sim y$, for any x and y of S_1, we mean that x and y are distinct and that x and y are in the same horizontal row. By $(x \sim y)'$ we mean that $x \sim y$ is false.

$$\begin{array}{ll} S_1: & a \\ & b \\ & c \end{array}$$

Here hypothesis (i) is not fulfilled whereas (ii) and (iii) are.

Example 2. Let the elements of the set S_2 be denoted and arranged as indicated. By $x \sim y$, for any x and y of S_2, we mean that x and y denote the same element or that y is to the "right of" x. By $(x \sim y)'$ we mean that $x \sim y$ is false.

$$\begin{array}{ll} S_2: & a,\ b \\ & c \\ & d \end{array}$$

Here hypothesis (ii) is not fulfilled whereas the others are.

Example 3. Let the elements of S_3 be denoted and arranged as indicated. By $x \sim y$, for any x and y of S_3, we mean that x and y denote the same element or that y is "next to" x. By $(x \sim y)'$ we mean that $x \sim y$ is false.

$$S_3: \quad a,\ b,\ c,\ d$$

Here hypothesis (iii) is not fulfilled whereas the others are.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1156. *Proposed by W. A. Hockings, Calumet, Michigan*

The conversation of problem E 1126 continues:

Jones: "Both Brown and I also have ranches in Todd County. They are oblong in shape and a whole number of miles in length and width."

Brown: "Yes, and your ranch is nine times as large as mine."

Smith: "I know the length of Jones' ranch, but I can't figure the area."

Green: "I know the area of Jones' ranch, but I can't determine the length without more information. Is the width less than half the length?"

Smith: "Before you answer, Jones, let me see if I can now determine the area." He deliberates for a minute. "No, I still do not know the area."

Jones answers the question.

Green: "I now know the length."

Smith: "Although I did not hear the answer to Green's question, I now know the area."

What are the dimensions of Jones' ranch in miles?

E 1157. *Proposed by T. F. Mulcrone, St. Charles College, Grand Coteau, La.*

Determine the smallest collection of positive integers such that for any integer k, $0<k\leqq c$, where c is a fixed integer, there is a subcollection whose sum is k.

E 1158. *Proposed by J. E. Hanson, Johns Hopkins University*

Find the roots of

$$\sum_{i=0}^{n}\binom{n+i+1}{2i+1}x^i = 0,\ n \geqq 1.$$

E 1159. *Proposed by Albert Newhouse, University of Houston*

In finding the greatest common divisor (a, b) of a and b (either integers or polynomials) by the Euclidean algorithm there appear, in the successive steps, quotients q_i and remainders r_i. Show that $(a, b)=as+bt$, where

$$s = (-1)^{k-1}\begin{vmatrix} q_2 & 1 & 0 \cdots 0 \\ -1 & q_3 & 1 \cdots 0 \\ 0 & -1 & q_4 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}, \quad t = (-1)^{k}\begin{vmatrix} q_1 & 1 & 0 \cdots 0 \\ -1 & q_2 & 1 \cdots 0 \\ 0 & -1 & q_3 \cdots 0 \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}.$$

E 1160. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

SOLUTIONS

Sum of Cubes as Difference of Two Squares

E 1108 [1954, 194 and 643]. *Proposed by René Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\sum_{i=0}^{k}(n+ai)^3$ as a difference of two integer squares.

II. *Comment by W. B. Carver, Cornell University.* This problem, and the proposer's solution [1954, 644], might lead one to suppose that when n, k, and $a+1$ are all odd the sum of the cubes *cannot* be expressed as the difference of two integer squares. This is not the case, as, for instance, when $n=5$, $k=3$, $a+1=5$, the sum of the four cubes is $7964=1992^2-1990^2$. It might be of some interest to ask what are necessary and sufficient conditions on the positive integers n, k, and a such that the sum of cubes

$$S = \sum_{i=0}^{k} (n + ai)^3$$

cannot be expressed as a difference of two squares. The answer is that we must have

$$n \text{ odd}, \quad \left.\begin{matrix} k \\ a+1 \end{matrix}\right\} \equiv 1 \bmod 4.$$

Any integer S, except when $S \equiv 2 \bmod 4$, may be *very simply* expressed as the difference of two squares thus

$$S = [(S+1)/2]^2 - [(S-1)/2]^2 \quad \text{when} \quad S \text{ is odd},$$
$$S = [S/4+1]^2 - [S/4-1]^2 \quad \text{when} \quad S \equiv 0 \bmod 4.$$

In the remaining case, $S \equiv 2 \bmod 4$, S cannot be the difference of two squares, and the sum of the cubes has this form if and only if n, k, and a satisfy the conditions given above.

The Length of Smith's Ranch

E 1126 [1954, 470]. *Proposed by J. V. Pennington, Houston, Texas*

Smith: "Down in Todd County, which is a 23-mile square, I have a ranch—oblong in shape—measuring a whole number of miles each way."

Jones: "Hold on a minute. I happen to know the area of your ranch; let me see if I can figure its length." He figures furiously. "I need more information. Is the width more than half the length?"

Smith answers the question.

Jones: "I now know the length."

Brown: "I too know the area and, although I did not hear your answer to Jones' question, I too can tell you the length."

Green: "I did not know the area, but I can now tell you the length."

How many miles long is the ranch?

Solution by Frank Kocher, Pennsylvania State University. Assume the sides of the ranch are parallel to the sides of the county. Now, the area of the ranch has more than one factorization, and in at least one factorization $2W<L$ and in at least one factorization $2W>L$. Had there been only one factorization of each type Brown would have needed the answer to Jones' question. Hence, the area must have at least three factorizations with either only one of them of the type $2W<L$ or else only one of them of the type $2W>L$. Consideration of the multiplication table up to 22×23 shows a number of cases of integers with three factorizations, none with more. If the width were more than half the length no unique length would be determined, but if the width is less than half the length, the length is 20 miles. The area is either 180 or 120 sq. mi.

It is interesting to note that if Jones, Brown, Green, and we do not assume the sides of the ranch parallel to the sides of the county, then the area also is uniquely determined, and is 180 sq. mi., for a 5×24 oblong may fit into a 23×23 square.

Also solved by P. M. Anselone, J. L. Botsford, Julian Braun, P. L. Chessin, W. A. Hockings, A. R. Hyde, M. S. Klamkin, Sidney Kravitz, Octave Levenspiel, D. C. B. Marsh, C. S. Ogilvy, Walter Penney, L. A. Ringenberg, Arnold Walfisz, and the proposer.

Number of Squares on an $m\times n$ Checker Board

E 1127 [1954, 470]. *Proposed by Edmund DeWan, Harpur College*

Enumerate the total number of squares appearing on an $m\times n$ checker board, $m\geqq n$. Solve the analogous problem for rectangular parallelopipeds.

Solution by Julian Braun, Aberdeen Proving Ground, Maryland. The number of squares measuring i units on a side (the side of a smallest square being taken as the unit) is $(m+1-i)(n+1-i)$ and $i_{\max}=n$, whence the total number of squares, T_2, is given by

$$T_2 = \sum_{i=1}^{n} (m + 1 - i)(n + 1 - i) = \sum_{i=0}^{n} (m - i)(n - i)$$
$$= n(n + 1)(3m + 1 - n)/6.$$

Similarly, the number of cubes, T_3, in a rectangular parallelopiped subdivided into an $m \times n \times p$ "checkerblock," $m \geqq p$, $n \geqq p$, is

$$T_3 = \sum_{i=1}^{p} (m + 1 - i)(n + 1 - i)(p + 1 - i) = \sum_{i=0}^{p} (m - i)(n - i)(p - i)$$
$$= p(p + 1)[6mn - (p - 1)(2m + 2n - p)]/12.$$

It may also be asked how many rectangles and rectangular parallelopipeds appear in the respective manifolds. The former is given by

$$\sum_{i=0}^{m} (m - i) \sum_{j=0}^{n} (n - j) = mn(m + 1)(n + 1)/4$$

and the latter by

$$\sum_{i=0}^{m} (m - i) \sum_{j=0}^{n} (n - j) \sum_{k=0}^{p} (p - k) = mnp(m + 1)(n + 1)(p + 1)/8.$$

The extension to an N dimensional manifold is evident.

Also solved by P. M. Anselone, A. P. Boblétt, J. E. Freund, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, Leo Moser, T. F. Mulcrone, Walter Penney, L. A. Ringenberg, Azriel Rosenfeld, Milton Scharf, R. E. Shafer, C. W. Trigg, Alan Wayne, and the proposer.

Geometrical Solution of an Old Calculus Problem

E 1128 [1954, 470]. *Proposed by C. S. Ogilvy, Hamilton College*

Show that the point P at eye level from which a picture high on a vertical wall subtends the maximum angle can be found as follows. With eye-level line as directrix and bottom of picture as focus, draw a parabola; from the intersection of this parabola with a line through the center of the picture parallel to the directrix, a line perpendicular to the directrix meets it at P.

Solution by Leon Bankoff, Los Angeles, Calif. It is clear that a circle (O) can be drawn through the top T and the bottom B of the picture, and tangent to the directrix at P. It follows that O is equidistant from P, T, and B, and is the point of concurrence of

a) the perpendicular bisector of TB,

b) the perpendicular to the eye-level line at P,

c) the parabola with focus at B or T and with the eye-level line as directrix,

d) the ellipse with foci B and T and with semi-major axis equal to OP.

Also solved by Norman Anning, J. E. Darraugh, A. L. Epstein, Herta

Freitag, B. K. Gold, A. R. Hyde, J. M. Kingston, J. D. E. Konhauser, D. C. B. Marsh, T. F. Mulcrone, M. J. Pascual, Walter Penney, L. A. Ringenberg, Azriel Rosenfeld, S. H. Sesskin, Sister M. Stephanie, D. R. Sudborough, J. A. Tierney, C. W. Trigg, Chih-yi Wang, Roscoe Woods, David Zeitlin, and the proposer.

Editorial Note. The above treatment applies equally well to the case of a slanted picture.

A Power-free Arithmetic Progression

E 1129 [1954, 470]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find an integral arithmetic progression with an arbitrarily large number of terms such that no term is a perfect rth power for $r=2, 3, \cdots, N$. Is this still possible if $N=\infty$?

Solution by Azriel Rosenfeld, Columbia University. The progression 2, 6, $10, \cdots, 4k+2, \cdots$ can contain no perfect powers whatsoever. For, a power of an odd integer is odd, and a power of an even integer must be divisible by 4.

Also solved by W. E. Briggs, A. R. Hyde, Leo Moser, J. B. Muskat, and R. E. Shafer.

An obvious solution, pointed out by Hyde, is any arithmetic progression with common difference $d=0$ and with the (invariant) term chosen so as not to be an integral power of an integer, for example the progression $3, 3, 3, \cdots,$ Hyde, Moser, and Shafer offered deeper solutions to the problem.

A Criterion for a Triangle to be Isosceles

E 1130 [1954, 470]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the perpendicular bisector of the median BB' of triangle ABC and the tangent at B to the circumcircle of triangle ABC cut the line AC in points M and N respectively. Show that triangle ABC is isosceles with vertex at A if and only if $AM/AN=3/4$.

I. *Solution by Chih-yi Wang, University of Minnesota.* Let the coordinates of A, B, C be $(-a, 0)$, $(0, b)$, $(c, 0)$ respectively. Then the coordinates of M and N can be easily found as, respectively,

$$\left(\frac{b^2}{a-c}-\frac{a-c}{4}, 0\right) \quad \text{and} \quad \left(\frac{b^2+ac}{a-c}, 0\right).$$

Let $AM/AN=k$. Then we have, after simplification,

$$(c+a)-\sqrt{a^2+b^2}=(\sqrt{4-4k}-1)\sqrt{a^2+b^2}. \tag{1}$$

The left side of (1) is equal to 0 if and only if $k=3/4$.

II. *Solution by Roscoe Woods, State University of Iowa.* It is obvious that the points M and N are not finite unless $a\neq c$. Let P denote the midpoint of the

median BB' and let ϕ denote the acute angle between the median BB' and the side AC. From the right triangle $B'PM$, it follows that $B'M = BB'/(2 \cos \phi)$. By substituting the value of $\cos \phi$ from the triangle $B'BC$ and making use of the fact that the square of the length of the median BB' is $(2a^2+2c^2-b^2)/4$ and that $AM = b/2 + B'M$, a short reduction shows that $AM = b(4c^2-b^2)/(4c^2-4a^2)$. Since $AN(AN-b) = NB^2 = c^2 + AN^2 - 2c\,AN \cos A$, we find that, upon replacing $\cos A$ by its value $(b^2+c^2-a^2)/2bc$ and collecting terms, $AN = bc^2/(c^2-a^2)$. It now follows that $AM/AN = (4c^2-b^2)/4c^2$. The solution is completed by noting that this ratio is $3/4$ if and only if $b=c$.

Also solved by J. L. Botsford, A. R. Hyde, Josef Langr, Beckham Martin, Margaret Olmsted, L. A. Ringenberg, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4617 [1954, 719]. *Correction.*

In place of the condition $x^2+y \leqq 1$, read $x^2+y^2 \leqq 1$.

4628. *Proposed by L. Carlitz, Duke University*

Let p be an odd prime and let $P_m(x)$ denote the Legendre polynomial of degree m, where $p=2m+1$. Show that

$$P_m(3) \equiv P_m(-3) \equiv \begin{cases} 0 \pmod{p} & (m \text{ odd}) \\ 2a \pmod{p} & (m \text{ even}), \end{cases}$$

where a is the unique odd integer determined by

$$p = a^2 + b^2, \qquad a \equiv b + 1 \pmod{4}.$$

4629. *Proposed by M. Henriksen and S. Perlis, Purdue University*

A commutative ring R is called a radical ring (in the sense of Jacobson) if every element has a quasi-inverse; *i.e.*, for every a in R, there is an x in R such that $a+x+ax=0$. In particular, if every element of R is nilpotent, then R is a radical ring. Need every subring of a radical ring be a radical ring?

4630. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Having given two confocal conics C_1 and C_2, let M_1 on C_1 and M_2 on C_2 be so chosen that the tangents at these points are perpendicular. Show that the envelope of M_1M_2 is another conic having the same foci and having asymptotes passing through the intersections of C_1 and C_2.

4631. *Proposed by F. H. Northover, Memorial University of Newfoundland*

Evaluate the unending continued fraction

$$1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\cdots.$$

4632. *Proposed by H. W. Becker, Station WOW, Omaha, Nebraska*

As quoted in L. E. Dickson, *History of the Theory of Numbers*, vol. II, p. 504, "C. Gill treated the problem to find n squares the sum of any $n-1$ of which is a square. He gave his solution for $n=5$ and remarked that the smallest numbers given by his formulas are so very large as to discourage any attempt to compute them." Perhaps allowing duplications among the n values, find a computable formula for general n, or for a particular $n \geqq 5$.

SOLUTIONS

Binomial Coefficients

4566 [1953, 716]. *Proposed by C. C. Chevalley, Columbia University, and Paul Erdös, University of Notre Dame*

Show that the following relations are impossible for $n > 2$:

$$\binom{2n}{n} = 2\binom{2n}{k}, \qquad \binom{2n+1}{n} = 2\binom{2n+1}{k}.$$

Solution by Ernst Trost, Technikum Winterthur, Zürich, Switzerland. We consider first the relation

$$\text{(1)} \qquad \frac{(2n)!}{n!n!} = \binom{2n}{n} = 2\binom{2n}{k} = 2\,\frac{(2n)!}{k!(2n-k)!}, \qquad n > 1,$$

where, of course, we can suppose $0 < k < n$. From (1) follows

$$\text{(2)} \qquad \frac{2n-k}{k+1}\cdot\frac{2n-k-1}{k+2}\cdots\frac{n+2}{n-1}\cdot\frac{n+1}{n} = 2.$$

Equation (2) is impossible for $k+1=n>1$, hence $n-k>1$. Because each of the $n-k$ factors on the left is >1 we may assume that $2n-k<2(k+1)$, whence $k+1 \geqq 2(n-k) > 2$. Now we make use of the following theorem of Sylvester:* If

* For a proof see P. Erdös, A theorem of Sylvester and Schur, *Journal Lond. Math. Soc.*, vol. 9, 1934, pp. 282–288.

$r>s$, then in the set of s consecutive integers $r, r+1, r+2, \cdots, r+s-1$ there is a number having a prime divisor greater than s. Hence we infer that there is a prime $p \geqq 2(n-k)>2$ dividing one of the integers $k+1, \cdots, 2n-k-1$. If $k+1 \equiv 0 \pmod{p}$ the next integer having the prime divisor p is $k+1+p \geqq 2n-k+1$, which implies that in (2) a factor $p^e (e \geqq 1)$ occurs only once. If p^e occurs in the denominator of (2) the quotient cannot be integral, if p^e occurs in the numerator an integral quotient must be of the form $tp^e (t \geqq 1)$. Thus (1) is impossible.

Let us now consider

$$\text{(3)} \qquad \frac{(2n+1)!}{n!(n+1)!} = \binom{2n+1}{n} = 2\binom{2n+1}{k} = 2\,\frac{(2n+1)!}{k!(2n-k+1)!}$$

with $n>2$ and $0<k<n$. From (3) follows

$$\text{(4)} \qquad \frac{2n-k+1}{k+1}\cdot\frac{2n-k}{k+2}\cdots\frac{n+3}{n-1}\cdot\frac{n+2}{n} = 2.$$

Equation (4) is impossible for $k+1=n>2$, hence $n-k>1$. Further we can assume that $2n-k+1<2(k+1)$ and therefore $k+1>2(n-k)>2$. Applying again the theorem of Sylvester we now infer that in the set $k+1, \cdots, 2n-k+1$ there is a number having a prime divisor $q \geqq 2(n-k)+1>3$. A factor $q^g (g \geqq 1)$ occurs only once in the set, for we have $k+1+q \geqq 2n-k+2$. Multiplying (4) through by $n+1$ we get a contradiction in the same way as above provided that q does not divide $n+1$. If q divides $n+1$ then we have from (4)

$$2 = \prod_{i=k+1}^{n} \frac{2(n+1)-i}{i} \equiv \pm 1 \pmod{q},$$

which is impossible because $q>3$. This completes the proof.

Editorial Note. The Proposers suggest the more general problem of showing: for fixed rational a the equation

$$\binom{2n}{n} = a\binom{2n}{k}$$

has only a finite number of solutions.

Simultaneous Equations

4567 [1953, 716]. *Proposed by R. S. Underwood, Texas Technological College*

Prove that the following simultaneous equations in the four unknowns x, y, z, u have one and only one solution in terms of the parameters a, b, where $0<a<5$, $0<b<5$:

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{5-a} + \frac{u^2}{5-b} = 1$$

$$(x - z - 4)^2 + (y - u - 2)^2 = 5.$$

Solution by Chih-yi Wang, University of Minnesota. Let the left member of the first given equation be denoted by $F(a, b)$. Since the right member is independent of a and b, by setting $\partial F/\partial a = 0$ and $\partial F/\partial b = 0$ we obtain, respectively

$$x/z = \pm a/(5 - a), \qquad y/u = \pm b/(5 - b).$$

Then the four unknowns must be of the following forms:

$$(1) \qquad x = ka, \qquad z = \pm k(5 - a), \qquad y = mb, \qquad u = \pm m(5 - b)$$

where k and m are the respective constants of proportionality whose values have to be determined. It follows from the second given equation, if the solution is real valued, that $x - z$ and $y - u$ must be independent of a and b respectively; we must, therefore, choose the negative signs of (1). By substituting (1), with negative signs for z and u, into the two given simultaneous equations, we get, after simplification,

$$(2) \qquad 5k^2 + 5m^2 = 1, \qquad 5k^2 - 8k + 5m^2 - 4m = -3.$$

By solving (2) we obtain two double roots, namely, $k = 2/5$ and $m = 1/5$. Hence the unique solution is $x = 2a/5$, $y = b/5$, $z = -2(5 - a)/5$, $u = -(5 - b)/5$.

Also solved by A. R. Hyde, D. C. B. Marsh, and the Proposer.

A De Moivre Quintic

4568 [1954, 51]. *Proposed by A. W. Walker, University of Toronto*

It can be shown (*e.g.* by taking y as independent variable) that if $y^5 - 5y^3 + 5y - 5x = 0$, then $(4 - 25x^2)y'' - 25xy' + y = 0$ (*Mathematical Gazette*, vol. 29, 1945, p. 223). Any three distinct roots of the above quintic must therefore be linearly related. Investigate this algebraically, and derive the following factored form of the quintic (readily verified by expansion):

$$(y - y_1)(y - y_2)(y - Ay_1 - Ay_2)(y + y_1 + Ay_2)(y + Ay_1 + y_2) = 0,$$

where $A^2 - A - 1 = 0$.

Solution by Louis Weisner, Hunter College, New York City. Substituting $y = z + z^{-1}$, we obtain $z^{10} - 5xz^5 + 1 = 0$. If we put

$$R_1 = \sqrt[5]{\frac{5x + \sqrt{25x^2 - 4}}{2}}, \qquad R_2 = \sqrt[5]{\frac{5x - \sqrt{25x^2 - 4}}{2}},$$

the values of the radicals being chosen so that $R_1R_2 = 1$, it follows easily that the roots of the quintic are $y_1 = \epsilon R_1 + \epsilon^4 R_2$, $y_2 = \epsilon^4 R_1 + \epsilon R_2$, $y_3 = \epsilon^2 R_1 + \epsilon^3 R_2$, $y_4 = \epsilon^3 R_1 + \epsilon^2 R_2$, $y_5 = R_1 + R_2$, where ϵ is a primitive fifth root of unity. Hence

$$y_3 = -y_1 - Ay_2, \qquad y_4 = -Ay_1 - y_2, \qquad y_5 = A(y_1 + y_2),$$

where $A = -\epsilon^2 - \epsilon^3$. It is readily verified that $A^2 - A - 1 = 0$. The five roots hav-

ing now been expressed in terms of y_1 and y_2, the factored form of the quintic is evident.

Also solved by R. H. Bruck, M. S. Klamkin, D. C. B. Marsh, Chih-yi Wang, Roscoe Woods, and the Proposer.

Editorial Note. The Proposer, after having submitted an algebraic solution, finds that $y=2\cos\theta$, $x=\frac{2}{5}\cos 5\theta$ reduces the quintic to a trigonometric identity and provides an easy way of verifying the differential equation and the relations among the roots.

A Function Intermediate Between x^n and e^x

4569 [1954, 51]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Find explicitly a function $f(x)$ such that $f\{f(x)\}$ is of the order of magnitude of e^x. (In other words, find a function intermediate between x^n and e^x.)

I. *Solution by Edgar Reich, the Rand Corporation, Santa Monica, California.* Take $f(x)$ as $[e^x]$ when x is not an integer, and as $x+1/x$ when x is an integer. The brackets indicate the greatest integer function.

II. *Solution by I. N. Baker, Adelaide University, South Australia.* We can find a continuous solution and one, in fact, such that $f\{f(x)\}=e^x$, this being a particular case of the problem of the "iteration" of e^x. See U. T. Boedewadt, *Mathematische Zeitschrift*, vol. 49, p. 497. The method is due to Abel.

$f(x)$ may be written in the form

$$f(x) = g^{-1}\{g(x) + \tfrac{1}{2}\}$$

where $g(x)$ has the property $g(e^x)=g(x)+1$ and where $g^{-1}(x)$ is the function inverse to $g(x)$. Define inductively

$$\begin{aligned}
&\exp\,[n, x] = \exp\,[1, \exp\,[n-1, x]],\\
&\exp\,[0, x] = x, \qquad \exp\,[1, x] = e^x,\\
&\log\,[n, x] = \log\log\,[n-1, x]\\
&\log\,[0, x] = x, \qquad \log\,[1, x] = \log x.
\end{aligned}$$

Then a solution to the problem is yielded by

$$\begin{aligned}
g(x) &= e^x - 1, &&\text{for } -\infty < x \leqq 0\\
&= x, && 0 < x \leqq 1\\
&= \log\,[n, x] + n, && \exp\,[n, 0] < x \leqq \exp\,[n, 1] = \exp\,[n+1, 0],\\
& && n = 1, 2, 3, \cdots.
\end{aligned}$$

Finally

$$\begin{aligned}
f(x) &= \log\,(e^x + \tfrac{1}{2}), &&\text{for } -\infty < x \leqq -\log 2,\\
&= e^x - \tfrac{1}{2}, && -\log 2 < x \leqq 0,
\end{aligned}$$

$$= \exp [n, \log [n, x] + \tfrac{1}{2}], \qquad \exp [n, 0] < x \leqq \exp [n, \tfrac{1}{2}],$$

$$= \exp [n + 1, \log [n, x] - \tfrac{1}{2}], \qquad \exp [n, \tfrac{1}{2}] < x \leqq \exp [n, 1],$$

$$n = 0, 1, 2, \cdots .$$

It is a simple matter to verify directly that $f(x)$ is a continuous function and that it satisfies the requirements of the problem.

k-Chromatic Graphs

4570 [1954, 52]. *Proposed by G. A. Dirac, King's College, London, England*

(1) If $d \geqq k \geqq 2$, show that there exist regular connected k-chromatic graphs of degree d and of arbitrarily high order.

(2) If $k \geqq 4$, construct a k-chromatic graph which does not contain a complete k-graph as a subgraph, and in which the degree of every node except one is $k-1$.

(See also problem 4526 [1954, 352].)

Solution by the Proposer. (1) If $k \geqq 2$ there exist regular connected k-chromatic graphs of degree k* and of every order: Let $G_1, G_2, \cdots, G_{2n}$ be $2n$ mutually disjoint complete k-graphs,† where n is as large as we please, and let the nodes of G_m be $a_{m1}, a_{m2}, \cdots, a_{mk}$, $(m=1, 2, \cdots, 2n)$. For $1 \leqq i \leqq n$ and $1 \leqq j \leqq k$, join $a_{2i,j}$ to $a_{2i-1,j}$ or to $a_{2i+1,j}$ according as j is odd or even. (Here $a_{2n+1,j}$ is to mean a_{1j}.) The graph so obtained is k-chromatic, connected and regular of degree k.

If $d \geqq k \geqq 2$, starting with a graph Γ which is k-chromatic, connected and regular of degree $d-1$, we can construct a graph which is k-chromatic, connected and regular of degree d: Let $\Gamma_1; \Gamma_2; \cdots; \Gamma_d$, whose nodes are $a_{11}, a_{12}, \cdots, a_{1m}; a_{21}, a_{22}, \cdots, a_{2m}; \cdots; a_{d1}, a_{d2}, \cdots, a_{dm}$, respectively, be d mutually disjoint graphs, isomorphic to Γ in such a way that $a_{1i}, a_{2i}, \cdots, a_{di}$ correspond for $1 \leqq i \leqq m$. Let $b_1, b_2, \cdots, b_m$ be m new nodes, b_j being joined to $a_{1j}, a_{2j}, \cdots, a_{dj}$ for $1 \leqq j \leqq m$. The graph so obtained is k-chromatic, connected and regular of degree d.

This completes the proof of (1).

(2) Take two mutually disjoint complete $(k-1)$-graphs G_1 and G_2 and join one node a_1 of G_1 to one node b_1 of G_2. Then construct a new node c and join c to every other node except a_1 and b_1. The resulting graph is k-chromatic, does not contain a complete k-graph, and every node except c is of degree $k-1$.

(Note the theorem of I. F. Stone: *The only k-chromatic graph for $k \geqq 4$ in which the degree of every node is $k-1$ is the complete k-graph.*)

Also solved by R. L. Brooks and W. T. Tutte.

* A linear *graph* is the figure obtained by joining certain points or *nodes* by line segments. If it requires k colors to color the nodes so that no two nodes of the same color are connected by a line, the graph is *k-chromatic*. The *degree* of a node is the number of segments ending there. A *regular* graph has all its nodes of the same degree. The *order* of a graph is the total number of nodes.

† A *complete* k-graph has k nodes, each connected to every other. Hence it is regular, of degree k-1, and is k-chromatic.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Algebra for College Students, 2nd edition. By J. R. Britton and L. C. Snively. New York, Rinehart & Co., Inc., 1954. xiii+474+Tables and Index, $4.25.

College Algebra. By H. G. Apostle. New York, Henry Holt & Co., 1954. xii+394 +Tables and Index, $4.50.

College Algebra, 3rd edition. By P. K. Rees and F. W. Sparks. New York' McGraw-Hill Book Co., Inc., 1954. xiii+414+Tables and Index, $4.25.

Understanding College Algebra. By E. R. Smith, S. Selby and M. Kleiman. New York, Dryden Press, 1954. xvi+505+Appendix, Tables and Index, $3.50.

The four books under review afford some choice in treatment but are fairly standard in content. Just how standard the material is can be gained by comparing them with texts of thirty years ago. Two such texts were found to differ largely in giving much shorter treatments of the elementary material. Evidently, modern texts are fighting a losing battle in closing the gap between high school preparation and college work.

Britton and Snively has 189 pages before reaching quadratics. Their book does a thorough job of ironing out high school deficiencies. Evidently, not even simple arithmetic is to be taken for granted. Students offering a poor unit of high school algebra could use this book. The more advanced parts are not treated quite as completely as in the other books also under review, but all essential topics are adequately presented. There is a short chapter on approximate numbers. Graphs are used more freely than is usual. A small feature is several pages given to formulas from geometry. Large numbers of worked problems are carefully discussed. Both Horner's Method and Successive Approximations are treated, but the brief treatments of the cubic and quartic equations found in the other three books are omitted. Students using this book can gain a thorough drill in the mechanics of algebra as well as a feeling that algebra can be highly useful. However, it gives little to lead one to think of algebra as an ever-expanding subject.

Apostle's book devotes 142 pages to reaching quadratics. However, the book has a new approach. It devotes the first thirty-eight pages to an axiomatic treatment of the fundamental operations. The author felt that not all of the treatment should be taken at once. Enough remains to give the student a strong feeling that algebra is quite as logical as geometry. There is plenty of room for differing on how this should be done, but the author should be commended for treating college students as adults even though they may be deficient in elementary mathematics. Certainly, high school material deserves a new approach in college. Repetition alone is not apt to solve the student's problem. The author distinguishes between a ratio and a fraction similar to Euclid Book V. The other

volumes make no such distinction. Unless a more complete semantic discussion is given, one feels tempted to agree with them. Mathematical Induction is treated as usual. The proof using the Principle of Contradiction is also presented as different from Mathematical Induction rather than merely a second proof. Hobson's Real Variables Volume 1 seems to take the latter view. This book has a number of worked examples, some of which are rather briefly handled. In arrangement the pages are a little crowded but not unduly so.

The book by Rees and Sparks has 148 pages before quadratics. The logical treatment is brief, quite in contrast with the Apostle book. It has numerous worked examples set forth on large uncrowded pages. A student should find the book easy reading. The authors stress the flexibility of the book. All of these items are easy on the teacher. An abundance of satisfactory problems gives one a feeling of the usefulness of the subject. The question should be raised if these pleasant features do not also cause the student to get a feeling that algebra is only a bundle of unrelated topics complete in themselves. This criticism can always be raised about the usual college algebra text.

The book by Smith, Selby and Kleiman is more between Apostle's and Rees and Sparks' in its logical introduction. There is a distinct effort to make the rules plausible before stating them. Rules stated when counter examples are not available can hardly impress a student. It has about as much elementary material as Rees and Sparks. Features not in the other books are the Sigma notation, the idea of the sequence and some notion of limits. There is a fair chapter on series if the teacher can teach it before running out of time. The teacher will find the proof of the binomial theorem by mathematical induction incomplete. Appendix A on computing π is obsolete and should be corrected and brought up-to-date. Over fifty brief references to the history of mathematics should be welcomed by the student. The problems and general treatment are quite well done.

All four of these books are quite usable. They have pointed up all too plainly the gradual decline in high school algebra. They have tried to meet the problem. They are strong on drill and learning by doing. Their departure from tradition has been slight. In view of the present trend to add something modern to elementary mathematics, they are conservative.

J. W. HURST
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Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Co., 1954. 609 pages. $5.90.

This book is a freshman text of the so-called integrated type. Taken substantially in its entirety, it is suitable for a two-semester five-hour course, but the material is so arranged that a proper selection could be a suitable basis for a shorter course emphasizing trigonometry, or analytic geometry, or college algebra.

The twenty-five chapters of the book seem to fall more or less into four main

divisions. The first of these comprises the first seven chapters, in which the emphasis is on application of algebra to geometric ideas. Starting with some algebraic review, the student is led from the function concept and rectangular co-ordinates through equations of loci and curve tracing; from slope of a line and slope of a curve, to limits, continuity, derivative, and the calculus of polynomials insofar as it applies to relative maximum and minimum, with application to curve tracing and word problems, motion in a straight line, and related time-rate problems.

The next six chapters, eight to thirteen, involve standard topics usually given in a college algebra course, culminating in a substantial chapter on statistics. This group of chapters begins, after a short discussion of exponents and scientific notation, with logarithms and logarithmic computation, and, after a glance at the topic of variation, proceeds through progressions, permutations, combinations, the binomial theorem and probability, to the chapter on statistics. Here emphasis is placed on mean, standard deviation, and the binomial distribution, with some discussion of quality control. Later in the book there is a chapter on use of log paper and method of least squares applied to fitting a straight line to a set of data. This chapter could well be studied in conjunction with the statistics chapter. Other algebraic topics as determinants and their use, and the algebra of complex numbers, are deferred to later chapters after the trigonometry and more geometry of straight lines have been developed.

The next five chapters of the book include standard topics in trigonometry emphasizing the analytic side and leading up to the chapters on analytic geometry in polar co-ordinates and complex numbers with the polar or vector point of view prominent.

There is a return to the calculus in a chapter involving some integration of polynomials with applications to problems of motion and areas. No introduction of the concept of differential is made, but the symbol of integration is used with the usual notation, $\int f(x)dx$. This presents the student with an unnecessary conceptual mystery, and the presentation suffers somewhat, as a result.

The remainder of the book, except for the algebraic portions referred to above, is devoted to conventional topics in analytic geometry which were not touched on in the first seven chapters: the geometry of the straight line, conics, parametric equations, and a chapter on the analytic geometry of space. There is a concluding chapter which is intended to stimulate student interest in mathematics. Here the selection of topics is wide and includes short discussions of matrix multiplication, angle trisection, magic squares, perfect number, Fermat's Last Theorem, the four-color problem, cryptography, base of a number system, and computing machines.

There are many commendable things that may be said for this volume. It is well written, from the point of view of both clarity and freshness; the problems are interesting, the format is good, and the diagrams well done. The book is inviting to read, and should be a stimulating start in the student's study of collegiate mathematics. The book is also a step in the right direction in satisfying

the need for early introduction of the calculus to engineering and scientific students so that a more substantial course in this basic science can be given in the sophomore year.

A unique feature of the book is found in the extensive skeleton solutions, with numerous sketches and graphs included in the answers to the odd problems. There is also a self test provided at the end of each chapter.

Here and there one might voice a slight note of objection. On page 13, the authors speak of "finding rational factors." Previously no mention has been made of the word *rational;* true, there is a footnote on the page, in which the definition of a rational number is implied, but this definition might well be introduced before using the term. In connection with the introduction of the imaginary numbers, the term *complex* is used as if referring exclusively to nonreals, and this is also implied in the analysis given on page 16. The idea of a discontinuity in a locus is discussed before any mention is made of continuity. One might also question the value of the exposure of freshmen to the ϵ, δ definition of a limit and might wonder at the failure to use previously-developed concepts of the calculus in the chapters on conics—for example, in showing the reflector property of the parabola. However, these are mere minor notes of discord in a volume which is, in this writer's opinion, a piece of work well done and a very welcome addition to the literature of collegiate mathematics.

NATHAN SCHWID
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Emerging Practices in Mathematics Education, Twenty-Second Yearbook of the National Council of Teachers of Mathematics, Washington, D. C., National Council of Teachers of Mathematics, 1954.

A valuable contribution to the literature on the teaching of mathematics, this book of articles was compiled under the editorship of John R. Clark with the cooperation of John Kinsella, Joy E. Mahachek, Phillip Peak, and Veryl Schult, and with the assistance of three members of the yearbook Committee of the National Council, *viz.*, Frank Lankford, W. D. Snader, and F. L. Wren. The book is composed of six parts, with the following respective self-explanatory titles: Various Provisions for Differentiated Mathematics Curriculums; Laboratory Teaching in Mathematics; Teacher Education; New Emphases in Subject Matter; The Evaluation of Mathematics Learning; and (annotated) Bibliography of "What Is Going on in Your School?"—1952–1953. Fifty-six different authors contributed, by invitation, the 51 articles distributed among the first five parts. The sixth part is a condensation of a department of "The Mathematics Teacher" over the period indicated in the title.

In Part One, 15 articles are arranged under five headings. Contributors discuss their analyses and/or successful experiences with problems associated with (1) reorganization of content, (2) homogeneous grouping of pupils, (3) effective classroom procedures where homogeneous grouping is undesirable or not feasible, (4) preparation of the public to accept and even contribute to changes

in a field usually regarded as static and inflexible, and (5) appropriate guidance procedures to accompany new programs. Part Two contains ten articles describing the use and merits of commercial and home- and pupil-made visual and other material aids to instruction, and of television and radio as media of instruction at various levels and for divers purposes. Part Three comprises 12 articles on a few of the many problems arising in connection with pre-service and in-service training of teachers. The discussions provide evidences of new awakenings in the areas of the induction of neophytes and of continuing professional growth among veteran teachers which must prove heartening to many who have long awaited these developments. In Part Four, new departures in teaching geometry, new mathematics, and approximate numbers are discussed. Part Five consists of seven articles; the evaluation procedures described convince one that some teachers are no longer content to rely solely on standardized tests nor even on their own time-tested home-made examinations. The manifestly deepening interest in the pupil as a dignified human being is encouraging. The list of Selected References following Part Six is a significant part of the book—indeed, some of the items whose titles it includes may prove more important than some of those in the body of the book.

The rarity of typographical and other errors compliments the industry of the authors and the diligence of the editors. However (page 321, line 12), 1.73 is a rather rough approximation to $\sqrt{2}$, and attention must be called, *e.g.*, to the first sentence of paragraph 3 on page 314 and to paragraph 2 on page 324.

In a book of this kind the several authors are expected to be in at least partial disagreement on some points. In their juxtaposition, the articles on approximate numbers in Part Four tend to fulfill and sharply emphasize this usual expectation.

In a period of almost explosive expansion in the number and diversity of marvelously interesting and worthwhile activities in which professional mathematicians are engaged, it is surprising and somewhat disappointing that the pertinent articles here devote as little attention as they do to prospective careers in mathematics. Perhaps this deficiency has been or can be corrected in the classroom or in the guidance procedures.

The criticisms offered are obviously constructive and certainly minor when compared with the high quality and good conception and execution of the book.

WADE ELLIS
Oberlin College

Analytic Geometry. By Gordon Fuller. Cambridge, Addison-Wesley Publishing Company, Inc., 1954. 205 pages. $3.85.

Analytic Geometry. By E. S. Smith, Meyer Salkover and H. K. Justice. New York, John Wiley and Sons, Inc., 1954. 306 pages. $4.00.

Both of these texts start in the usual way, giving the first two chapters to a discussion of rectangular coordinates and plotting, developing the elementary

formulae such as the distance between two points, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and the slope $m=(y_2-y_1)/(x_2-x_1)$. After developing these formulae which assume the straight line in Chapter 2, each takes up and develops the straight line equations in Chapter 3.

Smith, Salkover and Justice give a good discussion of the conic sections in Chapter 4 and, in the following chapters, treat the circle, parabola, ellipse and hyperbola as special conics. This is done in a logical way with considerable emphasis on the eccentricity. Transformation of coordinates is left to Chapter 9.

Fuller takes up transformation of coordinates in Chapter 4 and discusses the individual conics in succeeding chapters, placing more emphasis on the simplified forms. There is also a chapter on the slope of a curve which is a good introduction to the idea of the calculus.

Both discuss curve plotting from empirical data and both have a good selection of plane curves of degree higher than two as well as some curves in polar coordinates. Each gives a good discussion of space coordinates, brief, but sufficient for the purpose for which they were written.

There are lists of problems conveniently distributed through the texts and answers are given for the odd-numbered problems.

They are valuable additions to the list of publications in this subject.

C. O. Williamson
College of Wooster

NEWS AND NOTICES

Edited by Edith R. Schneckenburger, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF AMERICAN SOCIETY FOR ENGINEERING EDUCATION

The Annual Meeting of the American Society for Engineering Education will be held June 20–24, 1955, at Pennsylvania State University, State College, Pennsylvania. This meeting is attended each year by over 2000 leading educators and educational administrators from engineering colleges throughout the country. The Society this year wishes to extend a hearty welcome to guests of foreign nations who are interested in scientific and technological education. A program dealing with international aspects of engineering education will be included.

The Annual Meeting program this year will feature at one of its General Sessions the reports of two Society Committees. One of these—the Committee on Evaluation of Engineering Education—has been making a comprehensive

study of the objectives and curricular goals of engineering education, with particular emphasis on the changes which are needed to keep engineering education abreast of the advancing frontiers of science and technology. This Committee contains 40 members, who are leading engineering educators and educational administrators in colleges throughout the United States. The final report of this Committee will be presented at the 1955 Annual Meeting.

Another Committee report deals with the teaching of the humanities, the social sciences and the business studies in the engineering curriculum. The Committee is endeavoring to seek out those educational programs which show fresh and imaginative approaches which show promise of making exceptional contributions in terms of educational achievement. Since many engineers enter supervisory and managerial positions, this area of education is deemed of great importance in broadening the outlook and giving a cultural background to the engineer.

The Engineering College Research Council (ECRC) will have a General Session dealing with Education in Research. The Engineering College Administrative Council (ECAC) will feature a General Session dealing with Methods of Interesting Secondary School Students in Engineering as a Career. The ECAC and the ECRC will hold a combined dinner with a prominent speaker.

The ASEE conference will feature over 75 conferences dealing with all areas of engineering education, physics, mathematics, international relations, research, and others bearing upon the broad subject of engineering education.

All sessions will be open to foreign guests who wish to attend. The printed program for the Annual Meeting will be available about the end of April, 1955. For further information write to the Secretary of the ASEE, Northwestern University, Evanston, Illinois. For information regarding housing accommodations, write to Professor Kenneth Holderman, Pennsylvania State University, State College, Pennsylvania.

SUMMER SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

The fifth Summer Seminar of the Canadian Mathematical Congress will be held at the University of Manitoba, Winnipeg, Manitoba, August 17 to September 9, 1955. The topic of this year's seminar is Analysis. Announcement will be made later of the titles of the Research Lectures. As in the past, instructional courses are designed for graduate students and as refresher courses for others; they assume little previous knowledge, but do assume some mathematical maturity. The object of the sub-seminar is to provide a forum for further treatment and discussion of the material of the Research Lectures.

The Research Lecturers, who will give two lectures a week, are: Heinrich Behnke, Westfälische Wilhelms Universität, Münster; W. K. Hayman, University College of the South West, Exeter, England; Jean Leray, Collège de France. The Sub-Seminar Leader is V. Hlavatý.

The following instructional courses will be given: Differential Equations by G. F. D. Duff, Queen's University; Combinatorial Analysis, N. Mendelsohn,

University of Manitoba; Theory of Transforms, D. B. Sumner, McMaster University.

It is expected that those travelling to the seminar by rail will travel at reduced fares within the Dominion of Canada. Lodging and board will be arranged at reasonable prices through the facilities of the University of Manitoba, and we hope to obtain additional accommodation for those desirous of attending with their families. Fee for the seminar is $10.

For further information please write to: Canadian Mathematical Congress, Chemistry Building, McGill University, Montreal, Canada.

SUMMER SCHOLARSHIP PROGRAM OF ELECTRODATA CORPORATION

ElectroData Corporation will continue the summer scholarship program which was inaugurated last year. The purpose of the program is to train graduate students of applied mathematics in industrial methods and techniques. Training in digital computing equipment and the organization and management of a corporation that produces this equipment will be provided for a period of ten weeks during the summer of 1955.

Students are selected upon requests of the departmental chairman. The scholarship is in the form of free grants to the universities involved.

For further information write to: Dr. Paul Brock, ElectroData Corporation, 717 North Lake Avenue, Pasadena 6, California.

RESEARCH CONFERENCE ON NUMBER THEORY

A Research Conference on Number Theory sponsored by the National Science Foundation will be held on June 22, 23 and 24, 1955, at the California Institute of Technology. The provisional arrangements include sessions on the following topics: 1. Applications of modular functions to number theory. 2. Problems of additive number theory. 3. The use of computing machines in number theory. 4. Diophantine equations. 5. Class field theory, quadratic forms, and the Riemann hypothesis in function fields.

Inquiries regarding the Conference should be addressed to Professor A. L. Whiteman, Chairman of the Organizing Committee, University of Southern California, Los Angeles 7, California.

PERSONAL ITEMS

Dr. Paul Brock, manager of ElectroData Corporation's technical services department, has been re-elected Chairman of the Southern California Section of the Society for Industrial and Applied Mathematics.

Associate Professor Edith R. Schneckenburger of the University of Buffalo represented the Association at the inauguration of Chancellor C. C. Furnas of the University of Buffalo.

Professor John von Neumann of the Institute for Advanced Study has been appointed by President Eisenhower to a five year term as a member of the

Atomic Energy Commission. The appointment must be confirmed by the Senate.

Alabama Polytechnic Institute announces the following: Mr. C. V. Aucoin, formerly a research fellow at the Institute, and Mrs. Clair Aucoin, previously a teaching fellow at the Institute, have been appointed to instructorships; Professor H. C. Wang is on leave of absence and is at the Institute for Advanced Study for the year.

Bucknell University reports: Assistant Professor W. K. Smith has been promoted to an associate professorship; Mr. Gregory Wulczyn has been appointed to an instructorship.

At Denison University: Associate Professor Marion Wetzel has been appointed Chairman of the Department of Mathematics; Instructor Andrew Sterrett, Jr., is directing the core course in general education in mathematics.

Ohio University reports the following: Mr. Howard Becksfort of Syracuse University has been appointed to an assistant professorship; Mr. F. C. DeSua of the University of Pittsburgh and Miss Nancy M. Scribano, formerly a student at Northwestern University, have been appointed to instructorships.

University of Arkansas announces that Miss Theresa Renner and Mr. L. R. Tappan have been appointed to instructorships.

At the University of British Columbia: Dr. R. R. Christian of Yale University has been appointed to an instructorship; Associate Professor Douglas Derry has been promoted to a professorship; Assistant Professors B. N. Moyls and W. H. Simons have been promoted to associate professorships; Professor Douglas Derry, who was on leave of absence for a year, has returned from France where he held a Canadian Government Overseas Fellowship; Professor S. A. Jennings is serving as Acting Head of the Department of Mathematics for the year 1954–55.

University of Kentucky reports the following: Dr. W. M. Faucett, formerly a teaching fellow at Tulane University, and Dr. J. D. Riley, previously mathematician at the Naval Ordnance Laboratory, Silver Spring, Maryland, have been appointed to assistant professorships; Mr. W. C. Swift has received a General Electric Fellowship and is working on Summability Theory under the direction of Professor V. F. Cowling.

Professor O. W. Albert of the University of Redlands has retired with the title Emeritus Professor.

Associate Professor B. H. Arnold has been granted a sabbatical leave from Oregon State College for the Spring term of 1955 and is at the University College of Wales, Aberystwyth, Wales.

Dr. Dean C. Benson of Iowa State College has been appointed to an assistant professorship at South Dakota School of Mines and Technology.

Dr. R. L. Blair of the University of California, Davis, has been promoted to an assistant professorship.

Mr. C. E. Buie, formerly a teacher at Pasadena City College, has been appointed to a professorship at Mt. San Antonio College.

Miss Mary J. Burns, previously a student at Duquesne University, has a position as a technician with Westinghouse Atomic Power Division, Bettis Field, Pittsburgh, Pennsylvania.

Assistant Professor L. J. Burton of Lake Forest College has been promoted to an associate professorship.

Mr. C. H. Cook, aerophysics engineer for Consolidated Vultee Aircraft, Fort Worth, Texas, has accepted a position as Senior Mathematician for the Glenn L. Martin Aircraft Corporation, Baltimore, Maryland.

Assistant Professor Trevor Evans of Emory University has returned after a two-year leave of absence at the Institute for Advanced Study and the University of Chicago.

Miss Catherine S. Feeley, formerly an auditor for Northern States Company, Inc., Chicago, Illinois, is now employed as a mathematician by Chicago Midway Laboratories, University of Chicago.

Mr. M. J. Forray of the Polytechnic Institute of Brooklyn is on leave of absence for the year 1954–55 and has a position as Senior Stress Analyst at Republic Aviation Corporation, Farmingdale, New York.

Miss Joyce B. Friedman of the Defense Department, Washington, D. C., has accepted a position as a mathematician with the ACF Electronics, Alexandria, Virginia.

Mrs. Ruth M. Frisch, formerly a graduate assistant at Syracuse University, is a teaching assistant at the University of Southern California.

Dr. T. M. Gallie, Jr., previously an assistant at Rice Institute, has been appointed to an instructorship at Duke University.

Mr. R. M. Gordon of the University of California has accepted a position as a mathematical analyst with the Lockheed Aircraft Corporation, Burbank, California.

Mr. E. P. Graney, formerly a research associate at the Willow Run Research Center, University of Michigan, has a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Associate Professor A. E. Halteman of the University of Idaho is on leave of absence and is at the University of Oregon.

Assistant Professor H. M. Hughes of the University of California has accepted an appointment as an analytical statistician at the School of Aviation Medicine, Randolph Field, Texas.

Associate Professor D. H. Hyers of the University of Southern California has been promoted to a professorship.

Dr. Jack Indritz of the University of Minnesota has been appointed to an assistant professorship at Washington University.

Associate Professor W. J. Jaffe of the Newark College of Engineering has been promoted to a professorship.

Mrs. Carol R. Karp, previously of New Mexico College of Agriculture and Mechanic Arts, is now a graduate student at the University of California.

Mrs. Katharine B. Keppler, recently at Milwaukee-Downer Seminary, is

Head of the Department of Mathematics of the Masters School, Dobbs Ferry, New York.

Mr. R. B. Kiltie, formerly a student at Columbia University, is now a graduate student at New York University.

Mr. E. H. Kingsley, previously a mathematician at the Aerial Measurements Laboratory, Northwestern University, Technical Institute, is now an operations analyst for the United States Air Force, Eglin Air Force Base, Florida.

Assistant Professor L. C. Knight, Jr., of Muskingum College has been promoted to an associate professorship.

Mr. E. R. Lancaster, recently a mathematician with the I.B.M. Corporation, Poughkeepsie, New York, has been appointed to an instructorship at Newark College of Engineering.

Associate Professor J. A. Larrivee of the University of Vermont has been appointed to an associate professorship at Worcester Polytechnic Institute.

Assistant Professor Octave Levenspiel of the Department of Chemical Engineering of Oregon State College has been appointed to an assistant professorship in Chemical Engineering at Bucknell University.

Mr. D. B. MacMillan of Columbia University has accepted a position as Mathematician with the Knolls Atomic Power Laboratory, Schenectady, New York.

Assistant Profesor Dale Maness of Baylor University has been appointed to an instructorship at the University of Kansas.

Mr. P. J. McCarthy, previously an assistant at the University of Notre Dame, has been appointed to an instructorship at the University.

Professor C. J. McCormick of Illinois State Normal University has been appointed Acting Head of the Department of Mathematics.

Mr. D. T. Mitchell, previously a student at Wabash College, is now a graduate student and teaching assistant at Purdue University.

Dr. O. B. Moan, a quality control staff engineer at Hughes Aircraft Company, Culver City, California, has accepted a position as a research specialist with the Lockheed Aircraft Corporation, Van Nuys, California.

Mr. S. I. Neuwirth, formerly a biometrician and assistant to the Secretary of the Committee on Research, American Medical Association, Chicago, Illinois, has accepted a position as Director of the Statistics Division of Mutual Insurance Advisory Association, New York City.

Dr. E. E. Osborne of the University of Connecticut has accepted a position with the National Bureau of Standards, Los Angeles, California.

Mr. W. E. Pace of the Virginia Polytechnic Institute has been promoted to an assistant professorship.

Mr. P. B. Patterson of the University of Florida has been promoted to an assistant professorship.

Associate Professor C. R. Perisho of Nebraska Wesleyan University has been appointed to an instructorship at State Teachers College, Mankato, Minnesota.

Mr. H. R. Rouse, formerly a teaching fellow at Vanderbilt University, has been appointed to an instructorship at Elizabethtown College.

Miss Elsie C. Rump, previously a teacher at Madison High School, Madison, Kansas, is teaching at Curtis School, Wichita City Schools, Kansas.

Dr. J. U. Russell, recently a graduate assistant at the University of Illinois, has been appointed to an assistant professorship at Southwestern at Memphis.

Mr. W. H. Sawyer, formerly a student at the University of Oklahoma, is teaching at Chemawa Junior High School, Riverside, California.

Associate Professor E. B. Shanks of Vanderbilt University has been promoted to a professorship.

Mr. J. L. Spenceley, previously a teacher at Grand Haven High School, has been appointed to an instructorship at Flint Junior College, Michigan.

Dr. J. K. Sterrett, formerly a mathematician with the Naval Research Laboratory, Washington, D. C., has a position as Chief of Aeroballistics, Directorate of Test Operations, Air Force Armament Center, Eglin Air Force Base, Florida.

Mr. L. R. Stidham, previously an assistant at New Mexico College of Agriculture and Mechanic Arts, has accepted a position as a data analysis supervisor at the Telecomputing Corporation, White Sands Proving Ground, New Mexico.

Assistant Professor R. G. Stoneham of San Diego State College has accepted a position as Mathematician at the Radiation Laboratory of the University of California.

Associate Professor G. L. Tiller of Southwestern at Memphis has been appointed to an associate professorship at the University of Georgia, Atlanta Division.

Mr. J. W. Toole has been appointed to an instructorship at St. Peter's College, Jersey City, New Jersey.

Assistant Professor R. Z. Vause, Jr., of Clemson Agricultural College has been appointed to an assistant professorship at Memphis State College.

Assistant Professor Robert Weinstock of Stanford University has been appointed to an assistant professorship at the University of Notre Dame.

Mr. E. H. Weiss, formerly an engineer with the Engineering and Research Corporation, Riverdale, Maryland, has accepted a position as Mathematician with the M. W. Kellogg Company, Jersey City, New Jersey.

Mr. E. F. Whittlesey of Bates College has been appointed to an instructorship at Trinity College, Connecticut.

Dr. H. H. Wicke of Lehigh University has a position as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor R. E. Wild of the University of Idaho is a teaching assistant at the University of California, Los Angeles.

Mr. G. P. Williams, previously a statistician for the Army Ordnance, Cincinnati Ordnance District, Cincinnati, Ohio, is now an operations research analyst for General Electric Company, Cincinnati, Ohio.

Dr. R. J. Wisner of the University of British Columbia has been appointed to an assistant professorship at Haverford College.

Mr. W. B. Woolf, formerly a graduate student at Claremont Graduate School, is at the University of Michigan on a fellowship.

Assistant Professor J. W. Young of the University of Florida has accepted a position as a mathematician in the Applied Science Division, I.B.M. Corporation, Atlanta, Georgia.

Dr. R. E. Zink of Purdue University is now in military service.

Dr. G. A. Campbell, who had retired from his position with American Telegraph and Telephone Company, New York City, died on November 10, 1954. He was a charter member of the Association.

Emeritus Professor J. A. Caparo of the University of Notre Dame died on July 12, 1954. He was a charter member of the Association.

Miss Georgie T. Davis of Virginia Polytechnic Institute died on May 25, 1954.

Reverend J. A. Hearn of St. Joseph's College, Philadelphia, Pennsylvania, died on December 11, 1953.

Mrs. Kathryn B. Rolfe of Modesto, California, died on August 3, 1954.

Sister Mary Paula of Marygrove College died on October 7, 1954. She had been a member of the Association for twenty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-eighth annual meeting of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Thursday, December 30, 1954, in conjunction with the annual meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Society for Industrial and Applied Mathematics. About seven hundred and twenty-five persons were registered, including the following four hundred and twelve members of the Association:

S. S. Abhyankar, J. E. Adney, Jr., M. I. Aissen, E. F. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, Mary L. C. Andrews, Nachman Aronszajn, R. A. Askey, W. F. Atchison, Miriam C. Ayer, J. L. Bagg, W. L. Baily, Jr., Joshua Barlaz, Wallace E. Barnes, J. H. Barrett, J. B. Bartoo, E. H. Batho, J. D. Baum, W. R. Baum, Thomas Bauserman, Helen P. Beard, R. A. Beaumont, E. G. Begle, Richard Bellman, C. P. Benner, A. A. Bennett, A. P. Berens, R. R. Bernard, Dorothy L. Bernstein, R. J. Bickel, R. H. Bing, Garrett Birkhoff, B. H. Bissinger, Warren Blaisdell, I. E. Block, Joseph Blum, J. O. Blumberg, R. O. Blummer, Jr., R. D. Boswell, Jr., S. G. Bourne, J. W. Brace, W. G. Brady, Leila Dragonette Bram, J. L. Brenner, Paul Brock, Foster Brooks, J. L. Brown, Jr., A. M. Bryson, R. C. Buck, R. S. Burington, L. E. Bush, A. T.

Butson, R. K. Butz, J. E. Byrne, E. J. Camp, Dorothy I. Carpenter, W. B. Carver, Maria Castellani, Jeremiah Certaine, J. O. Chellevold, Y. W. Chen, S. S. Chern, T. S. Chihara, J. G. Christiano, C. E. Clark, W. E. Cleland, K. J. Cohen, L. W. Cohen, Geraldine A. Coon, Byron Cosby, Jr., R. R. Coveyou, V. F. Cowling, H. S. M. Coxeter, E. H. Crisler, D. W. Crowe, A. B. Cunningham, J. C. Currie, H. B. Curry, J. H. Curtiss, Elizabeth H. Cuthill, Wayne Dancer, R. B. Davis, R. L. Davis, R. B. Deal, Jr., C. H. Denbow, W. E. Deskins, A. H. Diamond, S. F. Dice, Mary P. Dolciani, Jesse Douglas, T. C. Doyle, W. L. Duren, Jr., W. H. Durfee, P. S. Dwyer, J. C. Eaves, Albert Edrei, P. D. Edwards, R. D. Edwards, Samuel Eilenberg, F. M. Ellis, D. H. Erkiletian, Jr., M. H. M. Esser, H. J. Ettlinger, Trevor Evans, Ky Fan, F. A. Ficken, C. D. Firestone, W. T. Fishback, Gloria C. Ford, M. K. Fort, Jr., J. S. Frame, C. H. Frick, Otha Fuller, Jr., R. E. Fullerton, J. W. Gaddum, T. M. Gallie, Jr., G. N. Garrison, H. M. Gehman, B. H. Gere, K. G. Getman, W. M. Gilbert, Leonard Gillman, Wallace Givens, R. D. Glauz, H. W. Godderz, F. S. Goepper, Jr., Michael Goldberg, Samuel Goldberg, S. I. Goldberg, J. K. Goldhaber, A. W. Goodman, W. O. Gordon, S. H. Gould, Arthur Grad, W. W. Graham, E. L. Grindall, Emil Grosswald, Arnold Grudin, R. R. Gutzman, W. T. Guy, Jr., V. H. Haag, Franklin Haimo, D. W. Hall, Marshall Hall, Jr., H. W. Handsfield, Frank Harary, Gerald Harrison, O. G. Harrold, Jr., H. H. Hartzler, G. A. Hedlund, C. E. Heilman, R. G. Helsel, Melvin Henriksen, R. T. Herbst, Coleman Herpel, I. N. Herstein, Archie Higdon, T. H. Hildebrandt, J. J. L. Hinrichsen, I. I. Hirschman, Jr., J. G. Hocking, L. Aileen Hostinsky, D. B. Houghton, Chuan-Chih Hsiung, G. B. Huff, Ralph Hull, M. Gweneth Humphreys, C. A. Hutchinson, M. A. Hyman, Jane C. Ingersoll, R. Y. Iwanchuk, R. F. Jackson, S. B. Jackson, R. D. James, Evan Johnson, Jr., L. W. Johnson, R. E. Johnson, R. P. Johnson, F. B. Jones, John Jones, Jr., P. S. Jones, Harriett R. Junior, F. E. Justis, Irving Kaplansky, Chosaburo Kato, J. B. Kelly, L. M. Kelly, J. G. Kemeny, J. R. F. Kent, D. E. Kibbey, S. C. Kleene, George Klein, L. C. Knight, Jr., J. C. Knipp, F. T. Kocher, Jr., J. D. E. Konhauser, H. L. Krall, H. C. Kranzer, J. B. Kruskal, Jr., A. H. Kruse, Harold W. Kuhn, Stephen Kulik, R. G. Kuller, R. J. Lambert, E. H. Larguier, George Laush, W. S. Lawton, E. B. Leach, A. B. Lehman, Marguerite Lehr, H. R. Leifer, Walter Leighton, W. W. Leutert, Norman Levine, F. H. Lloyd, R. W. Long, Lee Lorch, D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, G. R. MacLane, Saunders MacLane, H. M. MacNeille, C. G. Maple, R. W. Marsh, Gloria A. Martin, W. T. Martin, K. O. May, John McCarthy, Myles McConnon, Dorothy McCoy, S. W. McCuskey, R. G. McDermot, J. H. McKay, E. J. McShane, A. E. Meder, Jr., R. E. Messick, D. D. Miller, W. I. Miller, Josephine M. Mitchell, E. E. Moise, Deane Montgomery, R. A. Moore, T. W. Moore, C. B. Morrey, Jr., Max Morris, L. T. Moston, B. H. Mount, Jr., C. W. Munshower, W. L. Murdock, F. H. Murphy, J. R. Musselman, E. F. Myers, Zeev Nehari, Morris Newman, J. A. Nohel, E. S. Northam, E. P. Northrop, A. B. J. Novikoff, C. O. Oakley, E. N. Oberg, Ruth E. O'Donnell, M. M. Ohmer, M. W. Oliphant, H. W. Oliver, Emma J. Olson, F. R. Olson, L. A. Ondis, II, J. H. Oppenheim, Morris Ostrofsky, J. C. Oxtoby, J. A. Painter, F. P. Palermo, T. P. Palmer, T. K. Pan, O. O. Pardee, B. C. Patterson, G. W. Patterson, III, P. A. Penzo, F. W. Perkins, W. J. Pervin, I. D. Peters, Mary Pettus, H. R. Phalen, T. J. Pignani, C. F. Pinzka, Everett Pitcher, R. J. Pitts, J. C. Polley, J. W. Popow, J. T. Powers, G. B. Price, R. E. Priest, Valdemars Punga, A. L. Putnam, Tibor Rado, J. F. Randolph, L. T. Ratner, G. E. Raynor, Edgar Reich, Irving Reiner, Haim Reingold, Eric Reissner, A. C. Reynolds, Jr., D. E. Richmond, J. D. Riley, R. F. Rinehart, L. A. Ringenberg, T. J. Rivlin, J. H. Roberts, Fred Robertson, G. deB. Robinson, Louis Robinson, L. V. Robinson, R. A. Rosenbaum, Alex Rosenberg, P. C. Rosenbloom, Louis Ross, J. B. Rosser, J. P. Roth, W. C. Royster, Mary E. Rudin, H. J. Ryser, Louis Sacks, E. A. Saibel, Charles Saltzer, A. C. Schaeffer, R. D. Schafer, Robert Schatten, Edith R. Schneckenburger, K. C. Schraut, B. L. Schwartz, W. R. Scott, C. F. Sebesta, George Seifert, Samuel Selby, D. H. Shaffer, A. L. Shields, R. L. Shively, Marlow Sholander, Edward Silverman, Annette Sinclair, Sister Mary Deborah, Sister Marie Gertrude, M. F. Smiley, E. C. Smith, Jr., W. S. Snyder, E. V. Somers, T. H. Southard, G. L. Spencer, II, Vivian E. Spencer, R. H. Spohn, C. E. Springer, F. H. Steen, H. E. Stelson, Andrew Sterrett, Jr., J. K. Sterrett, Guy Stevenson, F. M. Stewart, R. R. Stoll, M. W. Stone, Alexander Strasser, D. D. Strebe, Nikola Svilokos, J. D. Swift, W. R. Talbot,

T. T. Tanimoto, J. S. Taylor, Mildred E. Taylor, William Clare Taylor, Jean E. Teats, G. B. Thomas, Jr., J. M. Thomas, L. O. Thompson, D. L. Thomsen, Jr., R. M. Thrall, John Todd, M. L. Tomber, Leonard Tornheim, W. R. Transue, Halsey K. Truan, A. W. Tucker, E. P. Vance, R. S. Varga, Mary C. Varnhorn, Bernard Vinograde, S. I. Vrooman, T. L. Wade, Jr., G. L. Walker, Marcelle M. Walker, J. A. Ward, W. H. Warner, G. C. Webber, J. V. Wehausen, Brother Bernard Alfred Welch, C. P. Wells, D. W. Western, F. J. Weyl, C. R. White, P. M. Whitman, G. T. Whyburn, Albert Wilansky, W. L. Williams, R. A. Willoughby, D. M. Young, Jr., J. W. T. Youngs, A. D. Ziebur, J. A. Zilber, R. E. Zindler, R. E. Zink.

Sessions of the Association were held on Thursday morning and afternoon in the Auditorium of the Mellon Institute of the University of Pittsburgh. Vice-President H. S. M. Coxeter presided at the morning session and President E. J. McShane at the afternoon session. The Program Committee for the meeting consisted of Irving Kaplansky, Chairman; H. L. Meyer, Jr., and P. C. Rosenbloom.

FIRST SESSION OF THE ASSOCIATION

Session on Foundations of Geometry

"Geometries and incidence matrices," by Professor H. J. Ryser, Ohio State University.

"Finite projective planes," by Professor Marshall Hall, Jr., Ohio State University.

"The geometries of Hjelmslev," by Professor Reinhold Baer, University of Illinois.

SECOND SESSION OF THE ASSOCIATION

"Classroom notes," by Professor M. F. Smiley, State University of Iowa.

"Mathematics on television (demonstration and discussion)," by Professor P. S. Jones, University of Michigan. Discussants: Professors F. G. Fender, Rutgers University, Marguerite Lehr, Bryn Mawr College, and Fred Robertson, Iowa State College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Wednesday morning in the Lounge of the Hotel Webster Hall in Pittsburgh, with twenty-one members present. Among the more important items of business transacted were the following:

Professor G. B. Price of the University of Kansas was elected Second Vice-President for 1955–1956.

Professor D. E. Richmond of Williams College was elected Representative on the Policy Committee for Mathematics for 1955–1957.

Approval was given by the Board to the appointment by President McShane of the following Nominating Committee for 1955: A. W. Tucker, Chairman; Ralph Hull, and D. E. Richmond.

It was voted to hold the Thirty-seventh Summer Meeting at the University

of Washington, Seattle, Washington, on August 20–21, 1956, and the Fortieth Annual Meeting at the University of Rochester, Rochester, New York, on December 29, 1956. The Board also voted to approve the establishment of a joint committee with the American Mathematical Society to study a possible change in the time of holding the annual meetings of the two organizations.

In order to decrease the back-log of unpublished papers now in the hands of the Editor, the Board voted to authorize the printing of thirty-two additional pages of the MONTHLY during 1955.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, December 30, 1954, at 2:00 p.m., in the Auditorium of the Mellon Institute of the University of Pittsburgh, Pittsburgh, Pennsylvania. President E. J. McShane presided.

The Secretary announced the results of the balloting for officers, in which 1299 votes were cast: W. L. Duren, Jr., of Tulane University was elected President for the two-year term 1955–1956. H. W. Brinkmann of Swarthmore College and M. A. Zorn of Indiana University were elected Governors for the three-year term 1955–1957.

MEETINGS OF OTHER ORGANIZATIONS

Sessions of the American Mathematical Society began on Monday, December 27 and continued through Wednesday afternoon. The Gibbs lecturer was Professor K. O. Friedrichs, and invited addresses were delivered by Professors Samuel Eilenberg and Lipman Bers.

The Association for Symbolic Logic met on Wednesday morning and afternoon, with an invited address by Professor G. H. von Wright.

A meeting of the Society for Industrial and Applied Mathematics was held on Tuesday evening.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: J. C. Knipp, Chairman; J. O. Blumberg, A. M. Bryson, L. W. Cohen, H. M. Gehman, George Laush, Norman Levine, E. F. Myers, J. S. Taylor, Jean Teats.

Registration headquarters was in the lobby of Hotel Webster Hall. Persons attending the meeting were housed in that hotel and other Pittsburgh hotels. Lunch was served in the Faculty Club in the Cathedral of Learning and the Tuck Shop was available for light refreshments.

Tea was served in the Faculty Club on Monday afternoon by the ladies of the Mathematics Department. The Nationality Classrooms were open for inspection during the tea. A conducted tour of the Mellon Institute was held on Tuesday afternoon. A three-hour tour of the city of Pittsburgh took place on Wednesday morning.

A banquet for members of the mathematical organizations and their guests was held on Wednesday evening at Hotel Webster Hall. Professor J. S. Taylor was toastmaster. Speakers were Dean S. C. Crawford of the University of Pittsburgh; President H. W. Kuhn of the Society for Industrial and Applied Mathematics; President W. V. Quine of the Association for Symbolic Logic; President E. J. McShane of the Mathematical Association of America, and Associate Secretary L. W. Cohen of the American Mathematical Society.

At the banquet, Professor C. B. Allendoerfer presented a resolution of thanks to our host, the University of Pittsburgh, on behalf of the four mathematical organizations, which was adopted by a rising vote. Particular thanks are to be given to the local members of the Committee on Arrangements which did such an excellent job in organizing this interesting and valuable meeting.

H. M. GEHMAN, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held on October 29, 1954, at Oklahoma City University, Oklahoma City, Oklahoma.

There were one hundred thirty-eight in attendance, including the following fifty members of the Association:

R. V. Andree, Arthur Bernhart, E. N. Brandt, Jr., J. C. Brixey, H. N. Carter, N. A. Court, R. B. Deal, Jr., R. C. Dragoo, I. E. Glover, B. T. Goldbeck, Jr., E. V. Greer, O. H. Hamilton, T. J. Head, E. E. Heimann, J. E. Hoffman, W. N. Huff, R. A. Hultquist, P. W. M. John, L. W. Johnson, J. T. Krattiger, J. E. LaFon, W. T. Lee, Gene Levy, H. W. Linscheid, Dora McFarland, B. L. Mackin, G. E. Meador, Dorothea Meagher, R. R. Murphy, Leonard Nichols, F. J. Palas, T. K. Pan, D. L. Patten, C. M. Pirrong, Jr., E. C. Rice, W. A. Rutledge, Herbert Scholz, Jr., J. W. Sehestedt, D. R. Shreve, M. G. Shults, H. W. Smith, O. S. Spears, C. E. Springer, Vivian Spurgeon, Margaret O. Taylor, J. D. Thomas, R. W. Veatch, G. R. Vick, Wesley Whittlesey, II, J. H. Zant.

At the business meeting, it was decided that the present arrangement of holding meetings of the Oklahoma Section of the Mathematical Association of America in conjunction with the annual meeting of the Mathematics Section of the Oklahoma Education Association and the National Council of Teachers of Mathematics did not provide enough time for presented papers. The members voted to continue holding a meeting in conjunction with the O.E.A., but to restrict papers at this meeting to those of reasonable interest to high school teachers, and to hold a second meeting on a different date at which research papers would be welcomed.

The following officers were elected for the year 1954–1955: Chairman, Professor R. R. Murphy, Panhandle Agricultural and Mechanical College; Vice-Chairman, Professor C. M. Pirrong, Oklahoma City University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The following papers were presented:

1. *Series related to the Fourier coefficients*, by Professor Simon Green, University of Tulsa.

If $f(t)$ is Lebesgue integrable $0 \leqq t \leqq 2\pi$, then the series $\sum a_n/n$ converges where a_n is the Fourier coefficient of $f(t)$ with respect to the function $\pi^{-1/2} \sin nt$. It is required, when n is a fixed number, that the coefficients ξ_i of the Fourier series are determined in such a way that $s_n(t)$ in the space L^2 approaches as close as possible $f(t)$, which leads to Bessel's inequality:

$$\sum_{i=1}^{\infty} c_i^2 \leqq \int_a^b f^2(t)dt;$$

therefore $\sum c_i^2$ is convergent. It is shown that $\sum_{i=1}^{\infty} \sqrt{\bar{a}_i \bar{b}_i}$ is convergent if $\sum \bar{a}_i$ and $\sum \bar{b}_i$ are convergent. By replacing $\bar{a}_i$ by a_i^2 and $\bar{b}_i$ by $1/n^2$ it is concluded that $\sum_{n=1}^{\infty} a_n/n$ is convergent.

2. *On the combination of two bi-variate normal dispersion patterns relating to artillery fire*, by Major O. S. Spears, Combat Development Department, The Artillery School, Fort Sill, Oklahoma.

An artillery weapon is pointed at point O_1, a point in the target area which is the center of impact for a very large number of projectiles. The dispersion pattern of the weapon around this origin O_1 is a normal, bi-variate circular pattern; *i.e.*, the standard deviation of the fall of projectiles along any given line through O_1 is the same as that for any other line through the point. When each projectile explodes, the fragment spray is such that the probability-of-casualty function is also a bi-variate, normal distribution, with center at O_{2i}, the horizontal projection of the burst center for the i-th projectile. A method is developed for determining the expected fraction of casualties for n projectiles, and for a stipulated target area, when the standard deviations of the two previously mentioned bivariate probability density functions are given.

3. *The price of consistency*, by Professor J. B. Giever, University of Oklahoma, introduced by the Secretary.

The paper is a semi-expository discussion of the theorem that every field has an algebraically closed extension. A very simple, but essentially incorrect, proof is presented and several other proofs are outlined and considered from the point of view that the additional complications may be considered the price paid in order to work in a consistent mathematics. A proof is given which approximates the simplicity of the incorrect proof. Some ways of correcting the first proof are considered.

4. *Some interesting indecomposable continua*, by Professor O. H. Hamilton, Oklahoma Agricultural and Mechanical College.

The standard definition of an indecomposable continuum is presented and some of the properties of indecomposable continua derived by Kuratowski, Janizewski, and Moore are discussed. A special type of indecomposable continuum, the pseudo-arc as defined by Moise, is examined in detail and its property of being homogeneous, as discovered by Bing, is discussed.

5. *Concavity of curves relative to a point*, by Miss Ella L. Clement, Douglas High School, Oklahoma City, Oklahoma, introduced by the Secretary.

The intuitive notion of concavity, inward or outward, of an arc of a curve relative to a point is formulated mathematically in a manner analogous to the mathematical formulation of the notion of concavity, upward or downward, of an arc of a curve relative to a line. A theory of such concavity analogous to that employed in studying concavity of curves relative to a line is developed. Necessary and sufficient conditions for an arc of the graph of a function of a somewhat restricted class of functions $r=f(\theta)$ to be concave inward or outward in a θ-interval are established.

Necessary conditions are established as well as sufficient conditions. The theory is then employed to simplify the study of relative maxima and minima, points of inflection and curvature of polar loci. As an incident to the investigation, necessary and sufficient conditions that any of the polar coordinates of a point on the locus of $r=f(\theta)$ satisfy the equation are established.

6. *Similar triangles represented by points*, by Professor Arthur Bernhart, University of Oklahoma.

A triangle with three degrees of freedom may be represented by a point. Several such correspondences are considered. Triangles of a given shape generate simple point loci.

7. *A report on the 1954 Summer Conference in Collegiate Mathematics, Chapel Hill, North Carolina*, by Professor J. H. Zant, Oklahoma Agricultural and Mechanical College.

The conference heard Professor E. Artin lecture on Abstract Algebra and Professor T. Rado on Rigid Surfaces; Professors Tucker, Thrall, and Griffin conducted discussion groups. The aim to foster the improvement of undergraduate mathematics by discussing the meaning and development of modern mathematics implies that its basic concepts should be included in the undergraduate curriculum. Such a need is recognized by all in graduate school mathematics. The conference did not spell out specifically in even one instance or course how this should be done. This must be done either by future conferences or by professors in the colleges. Many will need help in doing this.

8. *What mathematics is wanted by the petroleum industry*, by Dr. D. R. Shreve, The Carter Oil Company, Research Laboratory, Tulsa, Oklahoma.

The mathematics used in economics, transportation, refining, exploration, and production problems in the petroleum industry was discussed. Emphasis was primarily on numerical mathematics of use in routine calculations and on the more modern numerical analysis required with high speed electronic computers. The need for mass production techniques, greater knowledge of numerical integration procedures, and understanding of computational stability was discussed.

R. V. ANDREE, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C. on December 4, 1954. Professor C. H. Frick, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred thirty persons in attendance, including the following eighty-two members of the Association:

J. C. Abbott, D. F. Atkins, R. P. Bailey, M. P. Berri, Evelyn Boyd, J. W. Brace, J. L. Brenner, B. H. Buikstra, G. H. Butcher, W. E. Byrne, H. H. Campaigne, J. F. Canu, G. R. Clements, E. W. Coffin, L. W. Cohen, C. H. Cook, Elizabeth H. Cuthill, J. A. Duerksen, R. P. Eddy, P. J. Federico, E. J. Finan, Gloria C. Ford, C. H. Frick, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, D. W. Hall, J. E. Houle, Jr., M. Gweneth Humphreys, Louise S. Hunter, J. E. Ikenberry, Jane C. Ingersoll, S. B. Jackson, F. E. Johnston, L. M. Kells, C. F. Koehler, A. E. Landry, D. B. Lloyd, J. J. MacDonnell, G. J. Mann, Ella C. Marth, M. H. Martin, R. M. Mason, E. S. Mayer, Carol V. McCamman, E. J. McShane, Florence M. Mears, Joseph Milkman, L. I. Mishoe, A. K. Mitchell, R. W. Moller, T. W. Moore, W. H. Norris, E. P. Northrop, M. W. Oliphant, W. W.

Proctor, O. J. Ramler, R. W. Rector, J. N. Rice, Sara L. Ripy, W. G. Saunders, J. W. Sawyer, Veryl G. Schult, Paul Shapiro, W. F. Shenton, C. H. Sisam, W. S. Soar, G. L. Spencer, II, C. F. Stephens, W. J. Strange, Choy-Tak Taam, Feodor Theilheimer, J. A. Tierney, John Todd, Marian M. Torrey, John Tyler, C. H. Wheeler, III, P. M. Whitman, G. T. Whyburn, G. T. Williams, D. M. Young, Jr.

The Chairman announced the appointment of the High School Contest Prize Committee: Mr. P. L. Chessin, Mr. R. O. Blummer, and Professor C. H. Wheeler, III. The Secretary reported the addition of four names to the list of volunteer lecturers in the section's Undergraduate Lecture Program: Professor R. C. Yates, Virginia Polytechnic Institute; Professor W. K. Morrill, Johns Hopkins University; Dr. W. W. Leutert, Ballistics Research Laboratory, Aberdeen Proving Ground; and Dr. E. P. Northrop, National Science Foundation. The Section, by a unanimous vote, expressed its appreciation to the members of the High School Contest Committee (Chairman, Professor W. H. Norris) for their excellent work.

The following papers were presented:

1. (A) *Evaluation of a certain finite sum;* (B) *A case of superiority of numerical integration over integration in closed form*, by Professor C. L. Beckel, Georgetown University, introduced by Professor Oliphant.

(A) The sum $f(f-1)\cdots(f-m)\sum_{k=0}^{m}(-1)^k(m/k)(f-k)^{-1}$, arising in a problem in molecular structure, was considered. It was shown that the sum is $(-1)^m m!$, independent of f.

(B) A certain set of definite integrals was encountered in the same molecular problem. Performing the integrations led to a double sum of terms in which the signs alternated. In the process of summing all significant figures might be lost. However, a new formulation led to numerical integrations giving answers having the desired degree of precision.

2. *On the uniform convergence of a certain eigenfunction series*, by Professors L. I. Mishoe and Gloria C. Ford, Morgan State College, presented by Miss Ford.

The problem of expanding a function $F(x)$ of bounded variation on $(0, 1)$ in terms of the eigenfunctions $u_n(x)$ of the equation $u''+q(x)u+(p(x)u-u')=0$, satisfying the boundary conditions $u(0)=u(1)=0$, was considered by Friedman and Mishoe. In the present paper it was proved that the series $\sum_{-\infty}^{\infty} a_n u_n$ converges uniformly to $F(x)$ in the open interval $(0, 1)$ provided $F(0)=F(1)=0$, and $F(x)$ has a continuous first derivative on $(0, 1)$. The condition is sufficient but not necessary.

3. *On certain solutions of* $B(y)u_{xx}+u_{yy}=0$, by Professor E. C. Watters, United States Naval Academy, introduced by the Secretary.

Consideration of certain problems in fluid dynamics led to the study of solutions to the partial differential equation

$$B(y)u_{xx} + u_{yy} = 0, \qquad B(y) \not\equiv \text{constant},$$

defined implicitly by $U(u)=X(x)+Y(y)$. For solutions of this type to exist which are not obtainable by ordinary separation of variables, it is necessary that $B(y)$ satisfy a certain differential equation. Conversely, given $B(y)$ satisfying the latter differential equation, a method for obtaining the solutions of the partial differential equation is outlined. The solution of this problem depends partially on results obtained by M. H. Martin.

4. *Hitting the mark*, by Mr. G. T. Williams, Operations Research Office, Johns Hopkins University.

A marksman fires at a target, with a constant probability p of hitting it. It takes him time t' between rounds, to aim and fire. If this time is distributed according to a known function $f(t)$, and $F(t)$ gives the distribution of time it actually takes to hit the target, it was found that

$$F(t) = pf(t) + (1 - p) \int_0^t f(\tau)F(t - \tau)d\tau.$$

A simple calculation then showed that

$$pM_n = \mu_n + (1 - p)\sum_{k=1}^{n-1} \binom{n}{k} \mu_k M_{n-k},$$

where $M_n = \int_0^\infty t^n F(t)dt$ and $\mu_n = \int_0^\infty t^n f(t)dt$. It was pointed out that the case $n=1$ is of particular interest, since the conclusion may be drawn that the mean time it takes to hit the target is equal to the mean number of rounds fired multiplied by the mean length of the rounds. A brief discussion was also given of the explicit solution of the cognate problem where $f(t)$ is the normal distribution.

5. *The discriminant of a polynomial of special type*, by Professor J. E. Houle, Georgetown University.

Given $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$, with the a_i in an algebraic extension field $F(\theta)$ where the defining equation of θ is of degree m, Finan (this MONTHLY, vol. 45) discussed a matrix method of obtaining an equation $h(x) = 0$ over F among whose roots occur all the roots of $f(x) = 0$. Let C_f be the companion matrix of $f(x)$, and let B denote the compound matrix obtained from the matrix $A = f'(C_f)$ by a transformation similar to Finan's. It was shown that the discriminant of $h(x) = 0$ is $(-1)nm(nm+1)|B|a^2/2$, where a is in $F(\theta)$. This involves extending certain results due to MacDuffee (this MONTHLY, vol. 57).

6. *Report on the High School Mathematics Contest*, by Mr. W. H. Norris, Maury High School, Norfolk, Virginia.

Mr. Morris, Chairman of the High School Contest Committee, reported the results of the first Contest, held in May 1954. Dr. D. B. Lloyd served as Chairman of the Test Committee, Dr. F. E. Johnston served as Chairman of the Board of Reviewers. About 1800 students in more than 40 high schools participated. The following results were announced: *First Place,* Thomas B. Ballard, Baltimore Polytechnic Institute; *Second Place,* Courtney W. Doyle, Baltimore Polytechnic Institute; *Regional Winners: Maryland,* Thomas B. Ballard, Baltimore Polytechnic Institute; *District of Columbia:* Frederick Hoffman, Calvin Coolidge High School, Washington, D. C.; *Virginia,* Paul E. Heath, Jr., Norview High School, Norfolk, Virginia.

7. *The circulation index in topological analysis*, by Professor G. T. Whyburn, University of Virginia.

Professor Whyburn discussed recent results and methods in the program of developing and proving by topological methods those theorems in analysis, particularly the ones concerned with functions of a complex variable, which are essentially topological in character. A procedure was outlined for obtaining, with the aid of the circulation index of a mapping about a point, the fundamental properties of lightness and openness for a non-constant analytic function as a special case of the theorem which asserts that these properties belong to any non-constant function which is the limit of a uniformly convergent sequence of differentiable functions.

R. P. BAILEY, *Secretary*

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PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF JANUARY 1, 1955

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1947. For information about preceding years, consult the American Mathematical Monthly for December 1951.)

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M. R. HESTENES	1950–1952	F. B. WILEY	1949–1952
E. H. C. HILDEBRANDT	1947–1950	K. P. WILLIAMS	1945–1947
D. L. HOLL	1953–1954	W. L. WILLIAMS	1946–1949

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 23, 1955.

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, April 22–23, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April 22–23, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

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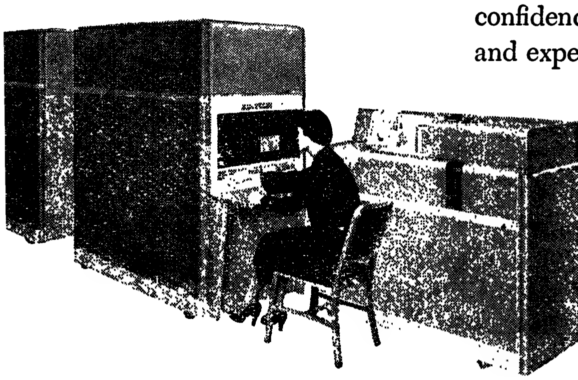

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MEAN VALUE THEOREMS AND LINEAR OPERATORS

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The object of this note is to isolate and systematize a procedure occurring in some apparently unrelated problems, such as the proof of the mean value theorem of differential calculus and Pólya's generalization [8] of it, and Zaremba's treatment [11] of solutions of the Laplace equation. As immediate consequences (without the use of the existence or positivity of the Green kernel, in which regard *cf.* [2]), there will result the mean value theorems of Blaschke [1] for the Laplacian, as well as mean value theorems for the heat equation operator and for other differential and "generalized" differential operators. The procedure to be dealt with has already been subject to some systematization in a paper of Pólya [9].

Let D and E be fixed subsets of a topological space S of points x and let D be dense in S. Let $\mathfrak{M}$ be a fixed linear manifold of (real- or complex-valued) functions $u=u(x)$ which are defined and continuous on S, and let $\mathfrak{N}$ be a fixed linear submanifold of $\mathfrak{M}$. Finally, for every u in $\mathfrak{M}$, let $L(u)$ be a function (of the position x) on the subset D of S. The continuity of the function $L(u)$ with respect to x is not assumed, nor is there any assumption of continuity for the operator L with respect to u.

In the applications below, D will be either an open set or an open set and part of its boundary (in an n-dimensioned Euclidean space), S the closure of D and E either the interior of D or D itself, L a differential or "generalized" differential operator, $\mathfrak{M}$ the set of continuous functions u on S for which $L(u)$ is defined (not necessarily continuous) on D, finally $\mathfrak{N}$ the set of those functions $u\in\mathfrak{M}$ which satisfy a certain boundary condition, that is, functions u possessing certain derivatives which, together with u, vanish on a part of the boundary of D.

The main theorem to be proved is as follows:

(*) *Let the operator L satisfy the following conditions*:

(i) *$L(u)$ is linear on $\mathfrak{M}$, that is,*

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2); \tag{1}$$

(ii) *there exists a solution $u = U_0$ of the boundary value problem*

$$L(U_0) = 1 \text{ on } D, \qquad \text{where } U_0 \in \mathfrak{N}; \tag{2}$$

(iii) *if $u\in\mathfrak{N}$ and $L(u)\neq 0$ on D, then $u\neq 0$ on E.*

Further, let u be any function of $\mathfrak{M}$ for which there exists a solution U of the boundary value problem

$$L(U) = 0 \text{ on } D, \qquad \text{where } U - u \in \mathfrak{N}. \tag{3}$$

Then, corresponding to every $x\in E$, there exists at least one point $\theta=\theta(x)$ of D satisfying

$$u(x) = U(x) + U_0(x)L(u(\theta)). \tag{4}$$

It is understood that $L(u(\theta))$ means the value of the function $L(u)$ at the point θ of D.

The assumption (iii) is analogous to Rolle's theorem and the assertion (4) to the mean value theorem in differential calculus.

Proof. Since (iii) implies that the solution U_0 of (2) does not vanish on E, it follows that, corresponding to a given point x of E, there exists a number A satisfying

$$u(x) - U(x) - U_0(x)A = 0.$$

On the other hand, the function $u-U-U_0A$ is in $\mathfrak{N}$, by (2) and (3), and vanishes at the point x of E. Hence (iii) assures that $L(u-U-U_0A)$ vanishes at some point θ of D. But the linearity of L and $L(U)=0$, $L(U_0)=1$ imply that $L(u(\theta))-A=0$. Hence (4) follows from the last formula line, and so (*) is proved.

(A) If L satisfies conditions (i)–(iii) of (*), then the following uniqueness theorem holds:

(iii *bis*) *Any solution u of the boundary value problem*

$$L(u) = 0 \text{ on } D, \qquad \text{where } u \in \mathfrak{N}, \tag{5}$$

satisfies $u\equiv 0$ on E.

For suppose that $u=v$ is a solution of (5) and that $v(x_0)\neq 0$ at some $x=x_0\in E$. Then, by (2) and (5), $U_0+cv\in\mathfrak{N}$ and $L(U_0+cv)=1\neq 0$ on D for any constant c. Since c can be chosen so that $u=U_0+cv=0$ at $x=x_0$, there results a contradiction to (iii).

On the other hand, (i), (ii) and (iii *bis*) do not imply (i)–(iii). In order to see this, let $D=E$ be the 1-dimensional open interval $0<x<5\pi/2$; S the closure of D; $\mathfrak{M}$ the set of continuous functions u on S having a continuous second derivative on D; $\mathfrak{N}$ the set of functions $u\in\mathfrak{M}$ satisfying $u(0)=u(5\pi/2)=0$; finally, $L(u)=d^2u/dx^2+u$. Then the solution of (5) is $u\equiv 0$. But the (only) solution of (2) is $U_0=1-2^{1/2}\sin(x+\pi/4)$, which vanishes at the point $x=\frac{1}{2}\pi$ of $E(=D)$.

Although (iii *bis*) does not imply (iii), one has:

(B) If L satisfies (i), (ii) and the following *relaxed* form of the *weak* maximum principle (iii *tre*), then L satisfies (iii).

(iii *tre*) *Let S be compact, let D be connected and let $u\in\mathfrak{N}$ imply that u vanishes on the complement $S-E$ of E. Then, if $L(u)>0$ on D, the function u has no maximum on E.*

Whenever (iii *tre*) is used, it is understood that the functions $u\in\mathfrak{M}$ are assumed to be real-valued. In order to see that (iii *tre*) implies (iii), let a $u\in\mathfrak{N}$ satisfy $L(u)\neq 0$ on D. It can be supposed that $L(u)>0$, for otherwise u could be replaced by $-u$. If (iii) does not hold, so that $u(x_0)=0$ for some $x_0\in E$, then $u(x_0)$

is either a maximum value of u or u has a positive maximum at some other point of E. Since this contradicts (iii *tre*), it follows that (iii *tre*) implies (iii).

Several illustrations of (*) will now be given. The functions u will always be assumed to be real-valued.

(I) Taylor's formula is valid in the following form: Let D be the 1-dimensional open interval $0<x<h$. Let $\mathfrak{M}$ be the continuous functions u on $0\leqq x\leqq h$ which possess continuous derivatives $du/dx, \cdots, du^{m-1}/dx^{m-1}$ on the half-open interval $0\leqq x<h$ and an mth order (not necessarily continuous) derivative $d^m u/dx^m$ on the open interval $0<x<h$. Then, if $u\in\mathfrak{M}$, there exists at least one number θ satisfying $0<\theta<h$ and

$$u(h) = \sum_{k=0}^{m-1} u^{(k)}(0)h^k/k! + u^{(m)}(\theta)h^m/m!.$$

This fact (I), proved in [4] directly, follows from (*) by the choices $L(u) = du^m/dx^m$, $E=(0<x\leqq h)$ and $\mathfrak{N}$ the set of functions $u\in\mathfrak{M}$ satisfying $u(0) = u^{(1)}(0) = \cdots = u^{(m-1)}(0) = 0$.

(II) Pólya's extension [8] of Rolle's theorem is the case where $L(u)$ is an ordinary differential operator of the mth order with continuous coefficients on the closed interval $E=D=(0\leqq x\leqq 1)$; $\mathfrak{M}$ is the set of functions u possessing a continuous mth derivative on D; B denotes a set of m (not necessarily distinct) points $a_1\leqq a_2\leqq \cdots \leqq a_m$ of $S=(0\leqq x\leqq 1)$; finally, $u\in\mathfrak{N}$ means that u vanishes in the kth order (at least) at $x=a$, if $x=a$ is a point of B counted with multiplicity k. In effect, Pólya shows that if L has property (iii *bis*) for every choice of the m points of B, then L has properties (ii) and (iii) for every choice of B; that is, if the only solution of $L(u)=0$ having m zeros on $0\leqq x\leqq 1$ is $u\equiv 0$, then L has properties (ii) and (iii) for every choice of the m points of B.

(III) Let $D(=E)$ be the interior of the circle $x^2+y^2<1$ and B its boundary $x^2+y^2=1$. Let $u\in\mathfrak{M}$ mean that u is continuous on $S=D+B$ and that

$$L(u) = \lim_{h\to 0} \{f(x-h, y-h) + f(x+h, y-h) + f(x+h, y+h) + f(x-h, y+h) - 4f(x, y)\}/h^2$$

exists at $(x, y)\in D$, and let $u\in\mathfrak{N}$ mean that $u\in\mathfrak{M}$ and $u=0$ on B. Clearly, L has property (i), as well as (iii *tre*), hence (iii). Finally, property (ii) follows from the fact that if $u(x, y)$ has the second order partial derivatives u_{xx}, u_{yy} at a point, then $L(u)$ exists at that point and

$$L(u) = \Delta u, \quad \text{where} \quad \Delta u = u_{xx} + u_{yy} \tag{6}$$

(*cf.* [11], p. 174); so that the solution $u=U_0$ of (2) exists and is

$$U = \tfrac{1}{4}(x^2 + y^2 - 1). \tag{7}$$

If u is continuous on B, there exists a solution U of (3); in fact, U is the function which is harmonic in D and assumes the values of u on the boundary.

It is now easy to conclude Zaremba's theorem [11], which states that if u is continuous on $S=D+B$ and if $L(u)=0$ on D, then u is harmonic on D. In fact, this is an obvious consequence of the statement (4), which indeed shows that $u\equiv U$. Similarly, if $L(u)=2\pi f$ exists (and is continuous), then u is, up to an additive harmonic function, the logarithmic potential with density f; *cf.* [10], p. 735.

If, in Zaremba's theorem, the function $u(x, y)$ is chosen to be independent of y, there results a theorem of Hölder ([6], pp. 183–185) which states that if

$$\{u(x+h)-2u(x)+u(x-h)\}/h^2\to 0, \qquad h\to 0,$$

for every point x of an interval, then u is a linear function.

(IV) It is clear that the considerations of (III) can be repeated if the unit circle $x^2+y^2<1$ is replaced by any Jordan domain D. Similarly, if $L(u)$ is defined by (6) for functions u continuous on $D+B$ and having second order partial derivatives u_{xx}, u_{yy} on D, then, corresponding to any $(x, y)\in D$, there exists an $(\alpha, \beta)\in D$ satisfying

$$u(x, y) = U(x, y) + U_0(x, y)\Delta u(\alpha, \beta), \tag{8}$$

where U is harmonic in D and satisfies $U=u$ on the boundary B of D, while $\Delta U_0=1$ on D and $U_0=0$ on B. Note that the continuity of u_{xx} or u_{yy} is not assumed, nor is the existence of u_{xy}. The mean value theorem (8) is given by Pólya [9] under the assumption that u has continuous second order partial derivatives on D.

If D is particularized again, this time to $D=(x^2+y^2<r^2)$, then $U_0=\frac{1}{4}(x^2+y^2-r^2)$; *cf.* (7). Let

$$\mu_r(u) = (2\pi)^{-1}\int_0^{2\pi} u(r\cos\phi, r\sin\phi)d\phi. \tag{9}$$

Then $U(0, 0)=\mu_r(U)=\mu_r(u)$. Hence (8) becomes, for $(x, y)=(0, 0)$,

$$\mu_r(u) - u(0, 0) = r^2\Delta u(\alpha, \beta)/4, \quad \text{where} \quad \alpha^2+\beta^2<r^2. \tag{10}$$

This corresponds to a mean value theorem of Blaschke [1] who, however, imposes more stringent conditions on u.

Let $0<s<r$ and let r be replaced by s in (10). Then $\alpha=\alpha(s)$, $\beta=\beta(s)$ and $\alpha^2(s)+\beta^2(s)<s^2$. It is clear from (10) that $\Delta u(\alpha(s)), \beta(s))$ is a continuous function of $s(>0)$. Thus, if (10) is multiplied by sds, integration over $0<s<r$ gives, for some (α, β),

$$\lambda_r(u) - u(0, 0) = r^2\Delta u(\alpha, \beta)/8, \qquad \alpha^2+\beta^2<r^2, \tag{11}$$

where

$$\lambda_r(u) = \int\int_{x^2+y^2<r^2} u(x, y)dxdy/(\pi r^2) = 2\int_0^r \mu_s(u)sds/r^2. \tag{12}$$

The relation (11) corresponds to another mean value theorem of Blaschke [1].

Let D be the unit circle $x^2+y^2<1$ and B its boundary. Let $u \in M$ mean that u is continuous on $S=D+B$ and that either

$$L(u) = \lim_{s\to 0} 4\{\mu_s(u(x+x_0,\ y+y_0)) - u(x_0,\ y_0)\}/s^2$$

or

$$L(u) = \lim_{s\to 0} 8\{\lambda_s(u(x+x_0,\ y+y_0)) - u(x_0,\ y_0)\}/s^2$$

exists at every point (x_0, y_0) of D. Finally, let $u \in \mathfrak{N}$ mean that $u \in \mathfrak{M}$ and that $u=0$ on B. By the procedure of (III), it is clear that if $u \in \mathfrak{M}$ and if $L(u)=0$ on D, then u is harmonic on D; *cf.* [1].

The preceding theorems have obvious extensions to spaces of dimensions $n>2$. Furthermore, Δu (and its generalizations) can be replaced by other elliptic differential operators; for example, by $\Delta u+\text{const. } u$, where the constant is negative and D is a Jordan domain (with sufficiently smooth boundary) or by $\Delta u+\text{const. } u$, where the constant is positive but D is sufficiently small; *cf.* [7], pp. 217–219.

(V) Consider the case of a suitable parabolic operator $L(u)$. Let $D(=E)$ be the square $(0<x<1,\ 0<y\leqq 1)$ and let $S=D+B$, where B is the sum of the lower $(y=0,\ 0\leqq x\leqq 1)$ and lateral $(x=0$ and $x=1,\ 0\leqq y\leqq 1)$ boundaries of D. Let $u \in \mathfrak{M}$ mean that u is continuous on the closure $D+B$ of D and that u_x, u_y, u_{xx} exist on D, and define $L(u)$ by

$$L(u) = \partial u, \quad \text{where} \quad \partial u = u_{xx} - u_y. \tag{13}$$

Let $u \in \mathfrak{N}$ mean that $u \in \mathfrak{M}$ and that $u=0$ on B. Since (i) is obvious, as is the maximum principle (iii *tre*), and since (ii) is easily verified, (*) implies the mean value theorem

$$u(x, y) = U(x, y) + U_0(x, y)\partial u(\alpha, \beta),$$

where $(x, y) \in D$, $(\alpha, \beta) \in D$, U satisfies $\partial U=0$ on D and $u=U$ on B, while U_0 satisfies $\partial U_0=1$ on D and $U_0=0$ on B; *cf.* Pólya [9].

As in (III), if (12) is replaced by

$$L(u) = \lim_{h\to 0} \{u(x+h,\ y) + u(x-h,\ y) + u(x,\ y-h^2) - 3u(x,\ y)\}/h^2$$

(*cf.* [5], pp. 598–601), it can be shown that if u is continuous on $D+B$ and if $L(u)=0$ on D, then the function u must have derivatives of arbitrarily high order, and satisfy $\partial u=0$, on D. Correspondingly, the last definition of $L(u)$ can be replaced by

$$L(u) = \lim_{h\to 0} \{[u_x(x+h,\ y) - u_x(x,\ y)]/h - [u(x,\ y) - u(x,\ y-h^2)]/h^2\},$$

where it is assumed that u is continuous on $D+B$ and that u_x and $L(u)$ exist on D; *cf.* Gevrey [3], p. 363 and p. 369.

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CRITERIA FOR IRRATIONALITY OF CERTAIN CLASSES OF NUMBERS. II

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Introduction. In a paper with the same title published recently in this MONTHLY,* criteria were obtained for the irrationality of the number x given by the series (1) wherein the a_i are integers and the b_i are positive integers. In particular were considered the series (1) where a_i, b_i satisfy (2) and (3) below. These criteria depended on the limits of the sequence $c_i=a_i/b_i$. But the criteria there found did not cover the case when each of the limits of the sequence c_i is rational and not 0 and not 1. Theorem A below deals with this case.

The series (1) may be called Cantor's series since Cantor [1] showed that for given integers $b_i\geqq 2$, a real number x can be expanded uniquely (if irrational) in the form (1) with the a_i satisfying (3). (If x is rational two expansions are possible). Cantor showed that if the b_i satisfy the condition that the product $b_1b_2 \cdots b_i$ is divisible by an arbitrary integer q for some sufficiently large i (and

* This MONTHLY, vol. 61, 1954, pp. 235–241.

therefore for all greater i), then x is irrational unless $a_i=0$ (all large i) or $a_i=b_i-1$ (all large i) in which cases x is rational. He also showed that when the b_i are ultimately periodic, x is rational if and only if the a_i are ultimately periodic.

In Theorem B below we give a criterion which includes both these criteria of Cantor.

1. We confine ourselves for simplicity to the series (1) subject to the conditions (2) and (3).

$$x = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots, \tag{1}$$

$$b_i \geqq 2, \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

$$0 \leqq a_i \leqq b_i - 1, \qquad (i = 1, 2, \cdots). \tag{3}$$

Criteria for irrationality were obtained which depended on the limits of the sequence

$$c_i = a_i/b_i$$

when $i \to \infty$. It was proved that x is irrational in each of the following cases: for some subsequence (i_n)

$c_{i_n} \to \xi$	(irrational),
$c_{i_n} \to 1$	provided $a_i < b_i - 1$ infinitely often,
$c_{i_n} \to 0$	provided $b_{i_n} \to \infty$ and $a_i > 0$ infinitely often.

The theorem which follows covers the remaining case when *every* limit of the sequence (c_i) is a rational number h/k where $0<h<k$, $(h, k)=1$.

THEOREM A. *If every limit of $c_i=a_i/b_i$ is a rational number h/k where $0<h<k$, $(h, k)=1$, then x is irrational except possibly when*

$$a_i = [hb_i/k]$$

for all large i in the subsequence for which $c_i \to h/k$.

In the excepted case x may be rational or irrational.

Proof. By the hypothesis there exists a subsequence (i_n) such that $c_i \to h/k$ and either

$$\frac{a_i}{b_i} > \frac{h}{k} \qquad \text{infinitely often,}$$

or

$$\frac{a_i + 1}{b_i} \leqq \frac{h}{k} \qquad \text{infinitely often}$$

in this subsequence. Hence $b_{i_n} \to \infty$.

Recalling that, by Lemma 2,

$$\frac{a_i}{b_i} \leqq x_i < \frac{a_i + 1}{b_i},$$

(for $a_i < b_i - 1$ infinitely often) we see that, for this subsequence,

$$x_i \to h/k, \qquad x_i \neq h/k.$$

If x is a rational number with denominator q, then each x_i is a rational number with denominator q. But a rational number with bounded denominator cannot tend to a rational h/k except by equality. The contradiction shows that x is irrational.

The excepted case is illustrated by two examples.

(i) $$a_i = i, \; b_i = 2i + 1 \; (i = 1, 2, \cdots): \; a_i/b_i \to \tfrac{1}{2}.$$

Here $x = \frac{1}{2}$; $a_i = [\frac{1}{2} b_i]$.

(ii) By Theorem 4 the number

$$y = \sum_{i=1}^{\infty} 1/b_1 \cdots b_i, \qquad (b_i = 3i + 2),$$

is irrational. Take $a_i = i$. Then

$$3x = \sum_{1}^{\infty} \frac{3i}{5 \cdots (3i + 2)} = 1 - y$$

is irrational so that x is irrational. Here plainly

$$c_i = a_i/b_i \to 1/3, \qquad a_i = [b_i/3].$$

2. Condensation. From the series (1), a series of precisely the same form may be obtained by grouping terms together thus:

$$\frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \cdots + \frac{a_{i_1}}{b_1 \cdots b_{i_1}} = \frac{A_1}{B_1}$$

where $B_1 = b_1 b_2 \cdots b_{i_1}$, $0 \leqq A_1 \leqq B_1 - 1$, these inequalities following at once from (3). By this process of condensation as it may be called we are led to

$$x = X = A_0 + \frac{A_1}{B_1} + \frac{A_2}{B_1 B_2} + \frac{A_3}{B_1 B_2 B_3} + \cdots$$

where

$$A_0 = a_0, \; B_1 = b_1 \cdots b_{i_1}, \; B_2 = b_{i_1+1} \cdots b_{i_1+i_2}, \cdots, \; B_i \geqq 2, \; 0 \leqq A_i \leqq B_i - 1,$$

a series of precisely the same form as (1).

THEOREM B. *A necessary and sufficient condition that x given by* (1), *subject to* (2) *and* (3), *shall be rational is this: coprime integers h, k, $0 \leqq h \leqq k$, an integer N and a condensation shall exist such that*

$$A_i = \frac{h}{k}(B_i - 1)$$

for all $i \geqq N$.

The sufficiency of the condition is obvious. That it is necessary may be seen as follows. If x is rational with denominator $q \geqq 1$, every x_i is rational with denominator $\leqq q$. Hence there exists a subsequence (j_n) such that

$$x_{j_n} = h/k, \qquad (0 \leqq h \leqq k,\ k \geqq 1,\ (h, k) = 1),$$

since $0 \leqq x_i \leqq 1$.

Use the condensation defined by

$$i_1 = j_1 - 1,\ i_1 + i_2 - 1 = j_2,\ i_1 + i_2 + i_3 - 1 = j_3, \cdots .$$

For this condensation

$$x_{j_1} = X_2, \qquad x_{j_2} = X_3, \cdots ,$$
$$B_r X_r = A_r + X_{r+1},$$

so that

$$A_r = X_r(B_r - 1) = \frac{h}{k}(B_r - 1), \qquad (r = 2, 3, \cdots),$$

which plainly implies that $k \mid B_r - 1$. Thus the condition is necessary.

The criterion given in Theorem B is interesting since it includes the two criteria given by Cantor: (i) when the b_i are such that, for every integer q, $q \mid b_1 b_2 \cdots b_n$ for all large n, (ii) when the b_i are periodic.

In the first case the criterion is this: x is rational if and only if $a_i = 0$ $(i > i_0)$ or if $a_i = b_i - 1$ (all $i > i_0$).

In the second case the criterion is this: x is rational if and only if the a_i are ultimately periodic.

It is unnecessary to give the details of the deduction of these criteria from Theorem B.

Reference

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SOME INEQUALITIES ARISING FROM A GENERALIZED MEAN VALUE THEOREM

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1. Introduction. Let $f(x)$ be a given real function, defined and continuous on the closed interval $c \leqq x \leqq c+h$ (where $h>0$); suppose also that $f(x)$ is differentiable in the open interval $c<x<c+h$. Then the Mean Value Theorem asserts the existence of a real number ξ satisfying

$$f(c+h)-f(c)=hf'(c+\xi), \qquad 0<\xi<h.$$

In this note we shall first establish an easy extension of this for higher derivatives; such extensions are presumably familiar (in one form or another) to many mathematicians, but the writer has not been able to trace any general statement in the literature. Next, temporarily abandoning the continuous variable and turning to the discrete case, we obtain a combinatorial identity (to which similar remarks apply) involving repeated differences of an arbitrary sequence.

These two results have some immediate corollaries of a rather more qualitative type. In particular, we use them to establish a class of inequalities which do not seem to be known, and which do not yield easily to the usual methods.

All functions considered and constructed below are to be understood as real functions of the variable x (defined over appropriate intervals of the real axis); we shall not always trouble to repeat this at each stage.

Given any non-zero h, we shall write

$$\Delta_h f(x)=\{f(x+h)-f(x)\}/h;$$

we shall in fact be concerned exclusively with *positive* h, but there is no difficulty in extending our results to the general case.

2. The Generalized Mean Value Theorem. Our point of departure is the following Theorem:

THEOREM 1. *Let $f(x)$ be continuous for $a \leqq x \leqq b$, and suppose that $f^{(n)}(x)$ exists in the open interval $a<x<b$ (where $a<b$ and n is some given positive integer). Then, given any n positive numbers $h_1, \cdots, h_n$, there is a sequence of real (but not necessarily continuous) functions $\xi_k(x)$ $(k=1, \cdots, n)$, respectively defined on the interval $a \leqq x \leqq b-h_1-\cdots-h_k$, and such that*

$$\left.\begin{aligned} &(1) \quad 0<\xi_k(x)<h_1+\cdots+h_k \\ &(2) \quad \Delta_{h_k}\cdots\Delta_{h_1}f(x)=f^{(k)}(x+\xi_k(x)) \end{aligned}\right\} \qquad \begin{aligned} &(a \leqq x \leqq b-h_1-\cdots-h_k; \\ &k=1,\cdots,n). \end{aligned}$$

Proof. The result for $n=1$ is just the statement of the Mean Value Theorem (as applied to $f(y)$ in the intervals $x \leqq y \leqq x+h_1$). Working by induction on n, we suppose the result proved for $n-1$ (and for *all* permissible $f, a, b, h_1, \cdots, h_{n-1}$). It will then be enough to suppose $\xi_1(x), \cdots, \xi_{n-1}(x)$ already defined (over their

respective intervals) satisfying (1), (2), and to find, for each x in the interval

$$I:\ a \leqq x \leqq b - h_1 - \cdots - h_n,$$

a number $\xi_n(x)$ satisfying

$$0 < \xi_n(x) < h_1 + \cdots + h_n,$$
$$\Delta_{h_n} \cdots \Delta_{h_1} f(x) = f^{(n)}(x + \xi_n(x)).$$

For any $x \in I$ the Mean Value Theorem, applied to the function

$$g(y) = \Delta_{h_{n-1}} \cdots \Delta_{h_1} f(y)$$

in the interval $x \leqq y \leqq x + h_n$, assures us of the existence of a number $\eta(x)$ satisfying

$$0 < \eta(x) < h_n,$$
$$\Delta_{h_n} g(y) \big|_{y=x} = g'(y) \big|_{y=x+\eta(x)} = \Delta_{h_{n-1}} \cdots \Delta_{h_1} f'(y) \big|_{y=x+\eta(x)}.$$

Also, by our induction hypothesis, applied to $f'(y)$ in the closed interval $x+\eta(x) \leqq y \leqq x + \eta(x) + h_1 + \cdots + h_{n-1}$ (which lies strictly interior to the open interval $a < y < b$), there is a number $\zeta = \zeta_{n-1}(x + \eta(x))$ satisfying

$$0 < \zeta < h_1 + \cdots + h_{n-1},$$
$$\Delta_{h_{n-1}} \cdots \Delta_{h_1} f'(y) \big|_{y=x+\eta(x)} = f^{(n)}(x) + \eta(x) + \zeta).$$

Hence

$$\Delta_{h_n} \cdots \Delta_{h_1} f(x) = \Delta_{h_n} g(y) \big|_{y=x} = f^{(n)}(x + \eta(x) + \zeta),$$

and the theorem follows at once on taking $\xi_n(x) = \eta(x) + \zeta$.

COROLLARY. *If, in Theorem 1, a particular derivative $f^{(k)}(x)$ has constant sign in the open interval $a < x < b$, then $\Delta_{h_k} \cdots \Delta_{h_1} f(x)$ has the same constant sign throughout its range of definition (i.e. for $a \leqq x \leqq b - h_1 - \cdots - h_k$).*

Unfortunately, even for $n = 1$, there is no simple converse of this Corollary (deducing a property of $f^{(k)}(x)$ from a hypothesis about the sign of $\Delta_{h_k} \cdots \Delta_{h_1} f(x)$, where $h_1, \cdots, h_n$ are of course always regarded as fixed). For we have no guarantee that the points $z = x + \xi_k(x)$ can be chosen so as to exhaust the interval $a < y < b$. It is easy, for the case $n = k = 1$, to give a precise statement about the various possibilities, as folllows:

THEOREM 2. *Let $f(x)$ be continuous for $a \leqq x \leqq b$, and differentiable for $a < x < b$. Given any h with $0 < h < b - a$, let $Z = Z(f, a, b, h)$ denote the set of all points z satisfying*

$$x < z < x + h, \qquad f(x + h) - f(x) = hf'(z)$$

for some x in the range $a \leqq x \leqq b - h$. Then Z must contain at least two distinct points z; also Z can contain exactly two points for suitably chosen f, a, b, h.

Proof. Let $P_x = (x, f(x))$ denote the point of the graph of $f(x)$ with ordinate

x, and consider the chord P_aP_{a+h} joining the points with ordinates a, $a+h$. There are only two possibilities: either (i) this chord contains in its interior a third point (P_c, say) of the graph, or (ii) it does not.

In case (i), by applying the Mean Value Theorem to the intervals

$$a \leqq x \leqq c, \qquad c \leqq x \leqq a + h,$$

we see that there are points z_1, z_2 satisfying

$$(1) \qquad a < z_1 < c, \qquad\qquad c < z_2 < a + h,$$

$$(2) \qquad f(c) - f(a) = (c - a)f'(z_1), \qquad f(a + h) - f(c) = (a + h - c)f'(z_2).$$

But, since P_c lies on the segment P_aP_{a+h}, the gradients $\{f(c)-f(a)\}/(c-a)$, $\{f(a+h)-f(c)\}/(a+h-c)$ of the subsegments P_aP_c, P_cP_{a+h} are both equal to the gradient $\{f(a+h)-f(a)\}/h$ of the full segment. Thus we may rewrite (2) as

$$f(a + h) - f(a) = hf'(z_1) = hf'(z_2);$$

also, by (1), z_1, z_2 are distinct points of the open interval $a<x<a+h$. In other words, Z contains the two distinct points z_1, z_2.

Suppose next that (ii) holds, *i.e.*, (by the continuity of $f(x)$) that the graph of $f(x)$ lies strictly to one side of the open segment P_aP_{a+h} for $a<x<a+h$. We may suppose without loss that the graph lies *above* the chord. By the Mean Value Theorem, Z contains at least one point in the interior of every subinterval of $[a, b]$ having length h. Suppose, if possible, that Z contains only one point, say z_0. This would imply, for *all* x with $a \leqq x \leqq b-h$, that

$$x < z_0 < x + h, \qquad f(x + h) - f(x) = hf'(z_0),$$

i.e., that

$$(3) \qquad b - h < z_0 < a + h,$$

and

$$(4) \qquad f(x + h) - f(x) = hf'(z_0) \qquad (a \leqq x \leqq b - h).$$

This is clearly absurd if $[a, b]$ has length $2h$ or more. But, in any case, substituting successively $x=a$, $x=a+\epsilon$ in (4) and subtracting, we deduce, for $0<\epsilon \leqq \min(b-a-h, h)$, that

$$f(a + h + \epsilon) - f(a + h) = f(a + \epsilon) - f(a) > \epsilon\{f(a + h) - f(a)\}/h,$$

since the graph lies above the chord. On letting $\epsilon \to 0$, it follows that

$$f'(a + h) \geqq \{f(a + h) - f(a)\}/h.$$

But also, since the graph of $f(x)$ lies above P_aP_{a+h} in every sufficiently small left neighborhood of $x=a+h$, we have

$$f'(a + h) \leqq \{f(a + h) - f(a)\}/h.$$

Hence in fact

$$f'(a+h) = \{f(a+h) - f(a)\}/h,$$

and so, taking $x=a$ in (4), we see that $f'(z_0)=f'(a+h)$; thus (4) may be rewritten

$$f(x+h) - f(x) = hf'(a+h), \qquad (a \leqq x \leqq b-h).$$

In particular, for any δ with $0<\delta<\min(b-a-h, h)$,

$$f(a+h+\delta) - f(a+\delta) = hf'(a+h),$$

and so Z contains the point $a+h$; also, by (3), $a+h\neq z_0$, so we have the desired contradiction.

We have now shown (directly in case (i), and by a *reductio ad absurdum* in case (ii)) that Z must always contain at least two points. Finally, our argument makes it clear that Z need not contain more than two points; a simple concrete example is provided by the function $f(x)=\sin x$, with $a=0$, $b=7$, $h=2\pi$.

3. An Identity. Given any sequence of numbers a_t $(t=0, 1, 2, \cdots)$, we write

$$\Delta^0 a_t = a_t, \qquad \Delta^1 a_t = \Delta a_t = a_{t+1} - a_t,$$

and define inductively

$$\Delta^k a_t = \Delta(\Delta^{k-1}a_t) \qquad (k = 2, 3, \cdots; t = 0, 1, \cdots).$$

THEOREM 3. *For any given sequence a_t, any given y, and any non-negative integer n, we have the identities*

$$I_{n,k}\colon\ (-1)^k \sum_{r=0}^{n-k} \binom{n-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s} = \sum_{t=0}^{n} \binom{n}{t} y^t a_t,$$
$$(k = 0, 1, \cdots, n).$$

Proof. The identities can of course be verified by comparing coefficients of the powers of y and expanding the terms $\Delta^{k-s} a_{r+s}$ by Newton's formula. However, we shall give a proof by induction (the case $n=0$ being trivial).

We note first that, for $k=1, \cdots, n$,

$$\begin{aligned}
&\sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s} \\
&= \sum_{s=0}^{k} \left\{ \binom{k-1}{s} + \binom{k-1}{s-1} \right\} (-1-y)^s \Delta^{k-s} a_{r+s} \\
&= \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \Delta^{k-s} a_{r+s} + \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^{s+1} \Delta^{k-s-1} a_{r+s+1} \\
&= \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \left\{ \Delta^{k-s} a_{r+s} - (1+y) \Delta^{k-s-1} a_{r+s+1} \right\},
\end{aligned}$$

while also

$$\Delta^{k-s}a_{r+s} - (1+y)\Delta^{k-s-1}a_{r+s+1} = \Delta^{k-1-s}\{\Delta a_{r+s} - (1+y)a_{r+s+1}\}$$
$$= -\Delta^{k-1-s}(a_{r+s} + ya_{r+s+1});$$

hence

$$(-1)^k \sum_{r=0}^{n+1-k} \binom{n+1-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s}$$
$$= -(-1)^k \sum_{r=0}^{n-(k-1)} \binom{n-(k-1)}{r} y^r$$
$$\cdot \sum_{s=0}^{k-1} \binom{k-1}{s} (-1-y)^s \Delta^{k-1-s}(a_{r+s} + ya_{r+s+1}).$$

Suppose now that $I_{n,k}$ $(k=0, 1, \cdots, n)$ hold for a given n (and for *all* sequences), and consider $I_{n+1,k}$ $(k=0, 1, \cdots, n+1)$. Of these, $I_{n+1,0}$ is trivial, while for $k \geqq 1$, by applying $I_{n,k-1}$ (for the sequence $a_t + ya_{t+1}$) to the expression on the right of the identity we have just obtained, we find

$$(-1)^k \sum_{r=0}^{(n+1)-k} \binom{(n+1)-k}{r} y^r \sum_{s=0}^{k} \binom{k}{s} (-1-y)^s \Delta^{k-s} a_{r+s}$$
$$= \sum_{t=0}^{n} \binom{n}{t} y^t (a_t + ya_{t+1}) = \sum_{t=0}^{n+1} \binom{n+1}{t} y^t a_t$$

by an argument similar to that used above; in other words, $I_{n+1,k}$ holds.

The result now follows generally by induction.*

We shall not require Theorem 3 in its full generality, but only in the special case $k=n$, which it will be convenient to state separately:

THEOREM 4. *For any sequence a_t, and any y,*

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t = (-1)^n \sum_{s=0}^{n} \binom{n}{s} (-1-y)^s \Delta^{n-s} a_s \qquad (n = 0, 1, \cdots).$$

We note, in passing, a simple consequence of this. Applying Theorem 4 twice in succession, first for a_t, y, and then for $\Delta^{n-t}a_t$, $-1-y$, we obtain

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t = (-1)^n \sum_{s=0}^{n} \binom{n}{s} (-1-y)^s \Delta^{n-s} a_s$$
$$= (-1)^n \cdot (-1)^n \sum_{r=0}^{n} \binom{n}{r} \{-1-(-1-y)\}^r \Delta^{n-r}(\Delta^{n-r} a_r)$$

* It is clear that the properties of the real number system are only partially relevant to the argument; in fact the result still holds for sequences from any additive group which admits y as an operator.

$$= \sum_{r=0}^{n} \binom{n}{r} y^r \Delta^{n-r}(\Delta^{n-r}a_r).$$

Since this holds for all y, comparison of coefficients of powers of y gives:

COROLLARY. *For any sequence* a_t,

$$\Delta^{n-t}(\Delta^{n-t}a_t) = a_t \qquad (0 \leqq t \leqq n).$$

This result is perhaps slightly surprising at first sight, but it becomes intuitively clear from a consideration of the formal difference table of the a_t; alternatively, a more pedestrian check can be made by a direct application of Newton's formula (using the fact that, for fixed u,

$$\Delta^{n-t}\binom{n-t}{u} = (-1)^{n-t} \text{ if } u = n-t,$$

and is otherwise zero).

4. Some Inequalities. We shall now briefly discuss some joint consequences of the results of the two previous sections. First of all, Theorem 4 leads at once to

THEOREM 5. (i) *If* $(-1)^{n-s}\Delta^{n-s}a_s \geqq 0$ $(s=0, 1, \cdots, n)$, *with at least one inequality, then*

$$\sum_{t=0}^{n} \binom{n}{t} y^t a_t > 0 \qquad (y > -1).$$

(ii) *If* $\Delta^{n-s}a_s \geqq 0$ $(s=0, 1, \cdots, n)$, *with at least one inequality, then*

$$(-1)^n \sum_{t=0}^{n} \binom{n}{t} y^t a_t > 0. \qquad (y < -1).*$$

We shall concern ourselves only with results of the type (i); there are precisely analogous results of type (ii). We shall also now consider only the case in which the a_t arise as the values at integer points of a given (and appropriately differentiable) function $f(x)$; but, again, a result similar to Theorem 7 below holds good for arbitrary discrete sequences (and can be similarly proved directly, or deduced as a corollary).

By Theorems 1 (with $h_1 = \cdots = h_n = 1$) and 4, we have

THEOREM 6. *Let* $f(x)$ *be a given function continuous over the interval* $0 \leqq x \leqq n$, *and suppose also that* $f^{(n)}(x)$ *exists in the open interval. Then there are numbers* $x_0, x_1, \cdots, x_n$, *with* $s < x_s < n$ $(s=0, 1, \cdots, n-1)$, $x_n = n$, *such that, for all* y,

* If we do not demand at least one inequality in (i) and (ii), then (even if we also allow $y=-1$) we of course have parallel limiting results stating that the sums are non-negative. Also, if we insist on having inequality in the hypothesis for $s=0$, we can allow $y=-1$ in each case and still be sure of having strict inequality in the conclusions. Similar remarks apply to Theorem 7 below.

$$\sum_{t=0}^{n}\binom{n}{t}y^t f(t) = (-1)^n \sum_{s=0}^{n}\binom{n}{s}(-1-y)^s f^{(n-s)}(x_s).$$

In analogy with Theorem 5(i), if $(-1)^{n-s}f^{(n-s)}(x) \geqq 0$ $(s=0, 1, \cdots, n)$, then we can deduce that the sum on the left is non-negative for $y \geqq -1$. A little more generally, we have:

THEOREM 7. *Let $f_1(x), \cdots, f_q(x)$ be given continuous functions over the interval $0 \leqq x \leqq n$, each differentiable n times in the open interval and positive for $x=n$; suppose also that*

$$(-1)^k f_p^{(k)}(x) \geqq 0 \qquad (p = 1, \cdots, q;\ k = 1, \cdots, n;\ n-k < x < n).$$

Then

$$\sum_{t=0}^{n}\binom{n}{t}y^t \prod_{p=1}^{q} f_p(t) > 0 \qquad (y > -1).$$

Proof. On writing $f(x) = \prod_{p=1}^{q} f_p(x)$, then $f(x)$ satisfies the conditions of Theorem 6. Also $f(n) > 0$, so, by Theorem 6, it is enough to show that

$$(-1)^{n-s}f^{(n-s)}(x) \geqq 0 \qquad (s = 0, 1, \cdots, n-1;\ s < x < n),$$

or, equivalently,

$$(-1)^k f^{(k)}(x) \geqq 0 \qquad (k = 1, \cdots, n;\ n-k < x < n);$$

and this follows immediately from the identity

$$(-1)^k f^{(k)}(x) = \sum_{\substack{k_1+\cdots+k_q=k \\ k_1 \geqq 0, \cdots, k_q \geqq 0}} \frac{k!}{k_1! \cdots k_q!}(-1)^{k_1}f_1^{(k_1)} \cdots (-1)^{k_q}f_q^{(k_q)}.$$

As an illustration of Theorem 7, one set of functions satisfying the conditions is

$$f_p(x) = (1 + \alpha_p x)^{-\beta_p} \qquad (p = 1, \cdots, q),$$

where the α_p, β_p can be any non-negative numbers. The writer is indebted to Mr. E. M. L. Beale for bringing to his notice the inequality corresponding to this choice of the f_p (for $\beta_1 = \cdots = \beta_q = \frac{1}{2}$); Beale had obtained an *ad hoc* proof for this particular case, not easily capable of extension to cover anything appreciably more general, and it was the writer's attempt to produce a more satisfying proof which led to this note.

POSTULATES AND MATHEMATICS

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The common denominator of the views held by most mathematicians, with regard to the nature and authenticity of mathematical findings, seems to be the conclusion that mathematics has nothing to do with meaning or with truth, but is concerned only with validity. This view is so far removed from the convictions of most students and most laymen that a very difficult communication problem arises—one which concerns those who wish to improve mathematical education and also those who regret any fragmentation of knowledge into exclusive and mutually incomprehensible specialties.

Teachers of secondary or undergraduate college courses face clients who, like laymen in general, believe that mathematics is the sure method of arriving at infallible and eternal truths. For example, it is generally believed that the armchair methods of euclidean geometry are powerful enough to demonstrate the necessary behavior of distance, of congruent figures, *etc.*, beyond the outermost of the galaxies. Though this view has been obsolete for at least a century, it is well to recall here that thinkers of this persuasion once established the intellectual patterns of western civilization. For more than two thousand years believers in the methods of Euclid formulated the "axioms" of other fields, and from them attempted to deduce absolute truths and final systems in natural science, political science, and in ethics. Mathematics was considered, in that period, as the brightest star in the constellation of deductive knowledge.

We are especially concerned here to note the divergent paths followed by science and by mathematics after this period of rampant rationalism ended; after a long list of discoveries, including that of non-euclidean geometries, gave evidence that deduction from axioms was an untenable method of achieving absolute knowledge in mathematics or in science. In science the concept of "absolute truth" has since been replaced by that of "probable truth." Moreover, rationalism has been relegated to the status of being one of the many tools of science, together with experimentation, hypothesizing, *etc.*

It is not surprising that mathematicians should choose a far different solution to the problem posed by the downfall of the "absolute truth" philosophy. They were mindful that it was the inner harmonies of their subject, rather than experimental data or physical problems, which led, for example, to the discovery of quaternions and to the theory of groups. The belief in the absolute truth of euclidean geometry seems to have been secondary, in their thinking, to their pride in the logical beauty and consistency of geometry; attributes which passed unscathed through Lobachevsky's onslaught. In the emergent philosophy of mathematics they "took the cash" of inner harmony and beauty and "let the credit go" of conquering reality. Guided by the concepts of the postulational approach, they developed new sources of mathematical power and discovered many hidden treasures—and at the same time their aims and motivations, even

more than their findings, became increasingly inexplicable to the general public who are their sponsors.

The mere existence of non-euclidean geometry and abstract number systems can be used, of course, to shatter the faith of our clients in the absolute truth view of mathematics. Unfortunately, however, extremely few of them are mathematically mature enough to appreciate the abstract postulational approach as a substitute view. This great divergence in outlook creates frustration in students and/or teachers, especially when the latter adopt an uncompromising "this or nothing" attitude. A young college instructor, thoroughly imbued with the postulational philosophy, may find, for example, that his freshmen do not understand his remarks on irrational numbers. His premises about the nature of mathematics then force him to draw one of three conclusions. Either he has failed to state his postulates with precision, or his development of, say, Cantor sequences lacked logical rigor, or else his students are simply incapable of learning the topic in question. His presuppositions allow him no other analysis of his pedagogical failure, and preclude him from experimenting with other methods of presentation.

The extreme form of "postulationism" not only can lead to an educationally disastrous breakdown of communication between specialist and layman; it also has certain intrinsic difficulties. For example, deductions in formal systems abound in combinatorial arguments. If two of the three roots of an equation have been found it is usually taken as a fact (not as a deduction from purely arbitrary postulates), that there is one more. The number of arrangements of three objects is taken as six, without stating that this is merely a valid deduction in a certain postulational system. The group of symmetries of a square is an eight group and is not cyclic. . . . If it is argued that all results of this kind should be labeled "metamathematical" then our thesis can also be restated; it is that a philosophy of mathematics is needed, for undergraduates, in which metamathematics finds a more natural and more significant place.

Let us review, briefly, the reasons for adopting the abstract postulational method. The historical development of this method was influenced by the realization that it is frequently found more efficient to study a system which has been formalized and divorced from meaning. In addition to arguments of efficiency, however, there is a more compelling reason. It is simply the fact that, in a logical system, it is impossible to prove each statement; since such a proof would require the use of still other statements. This argument, we repeat, is based on the decision to define mathematics as a pure logical system. It is our thesis that, from the view of the secondary teacher and the teacher of beginners in college, this decision is neither necessary nor wise. The desirable alternative seems to be to adopt the solution of the scientists: to study mathematics as a system partly logical and partly empirical. This solution is far less drastic than it first appears to be, due to the highly specialized nature of mathematical subject matter. As an illustration of this specialized subject matter and its mathe-

matical analysis let us consider, not "numbers as things in themselves," but number operations, such as addition.

In prehistoric times numbers doubtless originated from tally marks, and the operation of adding 3 and 4 to get 7 was, at first, a symbolic statement of a fairly general scientific fact—a fact, however, not about tally marks, but about a certain physical operation on objects "named" by the tallies. That is to say, while the nature of the tally marks is unimportant, (philosophical researches into the meaning of number are irrelevant here), the operation "$+$" has a real physical meaning and from this viewpoint $3+4=7$ is empirically verifiable.* The symbolic language need not deceive us; $3+4=7$ is a scientific fact about many objects of experience, and like most scientific generalizations must be applied properly. In an air compressor 3 quarts of air and 4 quarts of air may give 1 quart of air.

Any science proceeds from its basic facts by using induction. Not all possible cases of the laws of physics can be tested, so inductive generalizations are made, and experimentally unprovable assumptions such as the "uniformity of nature" or the "conservation of energy." Mathematics follows the same general procedure. For small "numbers" (*i.e.*, sets of objects), the associative laws, the distributive law, *etc.*, can be proved empirically, but it is more difficult to prove in this way that $10^{1000}+10^{1000}=2\cdot 10^{1000}$. Hence suitable assumptions about the behavior of all "positive integers" are made, the most striking of which is, perhaps, that their generation is a certain type of never-ending process.

In addition to the operations of "addition and multiplication of integers" modern mathematics studies, of course, many other pairing processes; for example, addition modulo 2, in which 1 and 1 are paired with 0; the logical sum of classes; addition of endomorphisms of a ring; *etc.* These and other considerations lead to a definition of mathematics as the science which studies all possible types of pairings, (relations, operations, correspondences). In practice the restriction seems to be added, by tacit consent, that a study is not to be called mathematics unless it is conducted with the aim of analyzing the logical structure and implications of the pairing operations. When, in theoretical mechanics, the pairing of force with mass and acceleration is extensively studied, but with a somewhat different aim, the result is likely to be dubbed "physics" or perhaps "applied mathematics." Furthermore, there seems to be a tendency not to classify a subject as mathematics unless its experimental component (a) is very small or indirect, as in the theory of irrational numbers, and in the traditional concept of geometry, or (b) involves only the most elementary and common experiences of mankind, so that actual experiments can be replaced by "thought experiments," as in the theory of permutations or the theory of probability.

Several of the areas of study just mentioned illustrate the point that the

* In the addition of perpendicular components of vectors, $3+4$ does not equal 7, but 5; but this is merely one of the many additional meanings which has been given to "$+$" in the course of time, and which we need not consider here.

"elementary" portion, at least, of mathematics is an intimate mixture of nature and man, a blend of mental power and physical reality. Its basic ideas have grown out of experience. Such concepts as number, distance, position, recurrence, continuity, are probably man's earliest, and, in a sense, deepest insights into the orderliness of events; his most basic scientific achievements. Their importance is reflected in the fact that the branches of mathematics which deal with these concepts are growing in number, with lattice theory, the study of metrics, the algebra of classes, the study of continuity, and many others, having been recently added as analyses of the concepts of order, distance, belonging to one or more sets, *etc.* It is significant that in each of these studies experiment alone is inadequate; that the analysis is conducted in terms of (pairing) operations and formally based on postulates. This process is one of the important methods of scientific analysis; it seems to be related to the mental ability to form "models" or "idealizations" from experience. Engineers lean heavily on the idealizations known as "rigid bodies," "particles," *etc.*; and is not the set of symbols, "one, two, three," *etc.*, itself an idealization of a set of similar but distinct objects or events?

To return to our clients. If we adopt the viewpoint toward mathematics which is implicit in the definition above, we need not try to persuade them to reject, in mathematics, all sense evidence such as that used extensively in the primary grades. Instead, we can demonstrate, even to beginners, the necessity for some assumptions about the behavior of large numbers, and thus lay the foundation for critical thinking. We can show them that in some branches of mathematics the "postulational component" is of less importance, (as in the arithmetic of small numbers, or in the theory of finite groups, reckoning with Cayley's Theorem); whereas in other branches it is found fruitful to study the consequences of postulates without regard, at the moment, to the question of whether any entities exist which can be paired in the ways specified by the postulates. We can explain that, by their very nature, the postulates of geometry are not susceptible of a laboratory proof, and that this is a sufficient reason, even if there were no others, for studying the consequences of other sets of geometric assumptions.

Perhaps this "compromise view" of the nature of mathematics will help us to re-establish some needed rapport with our "public," (which is puzzled by the boast that mathematics is now completely divorced from reality), especially if we point out that physics and chemistry, for example, with their theories of "point particles," "frictionless machines," "ideal gases," *etc.*, also have a strong postulational component.

THE ALGEBRA OF SEMI-MAGIC SQUARES

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1. Introduction. A semi-magic square is a square array of n^2 numbers for which the sum of the n numbers in any row or column is a constant S. If the sum of each diagonal is also equal to S, the array is referred to as a magic square. The objects referred to as numbers will, for the purposes of this paper, be taken to be the elements of a field F whose characteristic is prime to n. Any such semi-magic square may be considered as an n by n matrix and will hereafter be called an S-matrix of order n according to the following

DEFINITION. *An S-matrix $A=(a_{ij})$ of order n is an n by n matrix for which $\sum_{i=1}^n a_{ij}=\sum_{i=1}^n a_{ji}=S(A)$ for every j.*

THEOREM 1. *The set of all S-matrices of order n forms a subalgebra R_n of the total matric algebra of degree n which satisfies the following conditions:*

$$\text{(i)} \qquad S(A+B)=S(A)+S(B),$$

$$\text{(ii)} \qquad S(kA)=kS(A) \qquad \text{for } k \text{ in } F,$$

$$\text{(iii)} \qquad S(AB)=S(A)S(B).$$

Proof. Let $A=(a_{ij})$ and $B=(b_{ij})$ be S-matrices of order n, and $C=(c_{ij})=A+B$. Then $c_{ij}=a_{ij}+b_{ij}$, and

$$\sum_{i=1}^n c_{ij}=\sum_{i=1}^n (a_{ij}+b_{ij})=\sum_{i=1}^n a_{ij}+\sum_{i=1}^n b_{ij}=S(A)+S(B).$$

The same holds for the rows of C. The matrix kA has elements ka_{ij}, and

$$\sum_{i=1}^n ka_{ij}=k\sum_{i=1}^n a_{ij}=k\sum_{i=1}^n a_{ji}=kS(A)$$

for every j. Next let $D=(d_{ij})=AB$. Then

$$d_{ij}=\sum_{k=1}^n a_{ik}b_{kj},$$

and

$$\sum_{i=1}^n d_{ij}=\sum_{i=1}^n\sum_{k=1}^n a_{ik}b_{kj}=\sum_{k=1}^n\sum_{i=1}^n a_{ik}b_{kj}=\sum_{k=1}^n b_{kj}\sum_{i=1}^n a_{ik}=\sum_{k=1}^n b_{kj}S(A)=S(B)S(A).$$

A similar computation holds for the rows of D.

Unfortunately, this result cannot be extended to magic squares as is shown by the following counterexample:

$$A=\begin{bmatrix}8&1&6\\3&5&7\\4&9&2\end{bmatrix},\qquad B=\begin{bmatrix}6&1&8\\7&5&3\\2&9&4\end{bmatrix},\qquad AB=\begin{bmatrix}67&67&91\\67&91&67\\91&67&67\end{bmatrix}.$$

The set of all magic squares of order n, however, does form a linear subspace of the algebra R_n.

The remainder of this paper will be devoted to the determination of the structure of the algebra R_n.

2. The decomposition theorem. Unless otherwise stated, the order of the matrices comprising the algebra R_n will be considered fixed throughout this section, and so the subscript n will be dropped in this and the following section.

We shall first exhibit two ideals of the algebra R. Let M be the set of all S-matrices A for which the entries a_{ij} are all equal to a fixed number a, and let N be the set of all S-matrices A for which $S(A)=0$. It is easy to see that M and N are both ideals of R. If $A=(a_{ij})=(a)$ belongs to M and $B=(b_{ij})$ belongs to R, then the element in the ith row and jth column of $AB=a\sum_{k=1}^{n} b_{kj}=aS(B)$ which indicates that AB belongs to M. Similarly, BA belongs to M. If, on the other hand, A belongs to N and B belongs to R, then $S(AB)=S(BA)=S(A)S(B)=0$.

Next let A be any matrix belonging to R, and let $B=A-C$ where C is the matrix all of whose entries are equal to $S(A)/n$. Then C belongs to M, $S(C)=nS(A)/n=S(A)$, $S(B)=S(A)-S(C)=0$, and B belongs to N. Since $A=B+C$, we see that any element of R is expressible as the sum of an element of M and an element of N; that is, the algebra R is the sum of the algebras M and N. This sum is actually a direct sum; for if A belongs to both M and N, A has a fixed entry a in each place, $S(A)=na=0$, $a=0$, and A is the zero matrix of R. We have proved

THEOREM 2. *The algebra R is the direct sum of the one dimensional ideal M and the ideal N.*

3. The structure of the algebra N. Define A_{ij} to be the matrix which has the number 1 in the first row and first column, the number -1 in the first row and jth column, the number -1 in the ith row and first column, the number 1 in the ith row and jth column, and zeros elsewhere for $i, j=2, 3, \cdots, n$.

THEOREM 3. *Let the S-matrices A_{ij} be as defined above for $n>1$, and $i, j=2, 3, \cdots, n$. Then the algebra N has dimension $(n-1)^2$, and the matrices A_{ij} form a basis for N.*

Proof. Assume first that we have a relationship of the form $\sum_{i,j=2}^{n} a_{ij}A_{ij}=0$. When $2\leqq p$, $q\leqq n$, $a_{pq}A_{pq}$ is the only component of this sum which contributes anything to the entry in the pth row and qth column which is zero. It follows that $a_{pq}=0$ for $p, q=2, 3, \cdots, n$, and the A_{ij} are linearly independent.

Next let $B=(b_{ij})$ be any S-matrix belonging to N, and consider the S-matrix

$\sum_{i,j=2}^{n} b_{ij}A_{ij}$. This matrix is equal to B since when $2 \leqq p,\ q \leqq n$, the element in the pth row and qth column is b_{pq}, the element in the first row and qth column is $-\sum_{i=2}^{n} b_{iq} = b_{1q}$, the element in the pth row and first column is $-\sum_{j=2}^{n} b_{pj} = b_{p1}$, and the element in the first row and first column is $\sum_{j=2}^{n} \sum_{i=2}^{n} b_{ij} = -\sum_{j=2}^{n} b_{1j} = b_{11}$. This shows that the S-matrices A_{ij} span N and completes the proof of the theorem.

ON THE EULER CHARACTERISTIC OF A LIE ALGEBRA

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It is well-known that the Euler characteristic $\chi(G)$ of a compact Lie group G is zero. Using H. Hopf's theorem [2] on the cohomology ring of a compact Lie group Chevalley and Eilenberg [1] have shown that if L is a semi-simple Lie algebra over a field of characteristic 0 then $\chi(L)=0$. In this note we generalize this fact to the case of any Lie algebra over an arbitrary field F, and furthermore, we give an elementary proof of this result.*

Let L be a Lie algebra of dimension n defined over F and denote by $C^q(L)$ $(q>0)$ the space of q-linear alternating functions in L with values in F, that is, $C^q(L)$ is the space of those functions f, henceforth called cochains, having the properties:

(a) f is linear in each argument, and

(b) $f(x_{\pi(1)}, \cdots, x_{\pi(q)}) = (\operatorname{sgn} \pi) f(x_1, \cdots, x_q)$

where π is a permutation of the indices $1, \cdots, q$ and $\operatorname{sgn} \pi$ denotes its sign. For $q=0$, $C^q(L)=F$, by definition. Furthermore, $\dim C^q(L) = \binom{n}{q}$, the binomial coefficient. It is this fact which is the crux of the whole proof.

To each element $f \in C^q(L)$ an element $d_q f \in C^{q+1}(L)$ is defined as follows:

$$(1)\quad (d_q f)(x_1, \cdots, x_{q+1}) = \sum_{i<j} (-1)^{i+j+1} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1})$$

where $[x_i, x_j]$ denotes the multiplication in the Lie algebra and the 'roof' over an element $x \in L$ implies omission of the argument x. For $q=0$, we set $d_q f=0$. Using induction, it follows easily that $d_{q+1}d_q f=0$ for $f \in C^q(L)$ and $q=0, 1, \cdots, n$.

A cochain f is a 'cocycle' provided $d_q f=0$. The cocycles of dimension q form

* The author wishes to thank the referee for pointing out that the Euler characteristic of 'any' Lie algebra over F vanishes. Originally, the author gave an elementary proof of this fact in the case of a semi-simple Lie algebra over a field of characteristic zero, by making continual use of the full reducibility property of the representation space of a semi-simple Lie algebra.

a (linear) subspace $Z^q(L)$ of $C^q(L)$. For $q=0$, $Z^0=C^0=F$. A cochain $f\in C^q(L)$ is a 'coboundary' if it can be expressed in the form $d_{q-1}g$ for some $g\in C^{q-1}(L)$. The coboundaries form a space $B^q(L)\subset Z^q(L)$ and $B^0(L)=(0)$. Also, $B^q(L)=(0)$ for $q>n$. The quotient space $H^q(L)=Z^q(L)/B^q(L)$ is called the 'qth cohomology group' of L.

Definition. Let $R^q=\dim H^q(L)$, that is R^q is the qth Betti number of L. Then $\chi(L)=\sum_{q=0}^{n}(-1)^qR^q$.

The purpose of this paper is to show that $\chi(L)=0$.

All one needs is the algebraic equivalent of the usual Euler relation for a complex (alternating sum of Betti numbers equals alternating sum of numbers of cells of various dimensions), namely: If K is a finite chain (or here cochain) complex; *i.e.* a finite sequence of finite vector spaces $C^0, \cdots, C^n$ and homomorphisms d_q: $C^q\to C^{q+1}$ with $d_{q+1}d_q=0$, then $\sum(-1)^q \dim C^q=\sum(-1)^qR^q$.

Now following [**3**], p. 214, we note that

$$\dim H^q = \dim Z^q - \dim B^q \tag{2}$$

and

$$\dim B^q = \dim C^{q-1} - \dim Z^{q-1}. \tag{3}$$

Putting (2) and (3) together we obtain

$$\dim H^q = \dim Z^q + \dim Z^{q-1} - \dim C^{q-1}. \tag{4}$$

For $q=0$ this reduces to $\dim H^0=\dim Z^0$, for $q=n+1$ to $0=\dim Z^n-\dim C^n$.

Taking the alternating sum, one verifies immediately the above relation. For the Lie algebra L the numbers $\dim C^k(L)$ are given by $\binom{n}{k}$, as mentioned above. It follows that

$$\sum(-1)^qR^q = \sum(-1)^q\binom{n}{q} = 0,$$

which we wished to prove.

We note that the explicit form of the coboundary operator (the fact that it involves the Lie-bracket) is completely immaterial.

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MATHEMATICAL NOTES

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A GENERALIZATION OF THE EULER-FERMAT THEOREM

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In this note the well known Euler-Fermat theorem ($a^{\phi(m)} \equiv 1 \pmod{m}$) for all a such that $(a, m) = 1$) is sharpened by showing that the congruence often holds for a modulus $k > m$. The condition $(a, m) = 1$ is replaced by an analogous condition $(a, t) = 1$, where $t \leqq k$.

THEOREM: *Let n be any integer and let $p_1, \cdots, p_r$ be the set of all primes such that $(p_i - 1) \mid n$ for all i, and let $\alpha_1, \cdots, \alpha_r$ be the maximum powers of these primes such that $\phi(p_i^{\alpha_i}) \mid n$; then $a^n \equiv 1 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}$, for all a such that $(a, p_1 \cdots p_r) = 1$.*

Proof. We have $a^{\phi(p_i^{\alpha_i})} \equiv 1 \pmod{p_i^{\alpha_i}}$ for $(a, p_i) = 1$. Now since $\phi(p_i^{\alpha_i}) \mid n$, it follows that

$$a^n \equiv 1 \pmod{p_i^{\alpha_i}} \quad \text{for} \quad (a, p_i) = 1.$$

However, since $(p_i, p_j) = 1$ for $i \neq j$, it follows at once from the elementary theory of congruences that

$$a^n \equiv 1 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}} \quad \text{for} \quad (a, p_1 \cdots p_r) = 1.$$

The case of most interest is that in which there exists an m such that $\phi(m) = n$. For example, the congruence $a^4 \equiv 1 \pmod{12}$ for $(a, 12) = 1$ (Euler's theorem) gives rise to the stronger relation

$$a^4 \equiv 1 \pmod{2^3 \cdot 3 \cdot 5 = 120} \qquad \text{for } (a, 30) = 1.$$

Similarly the congruence

$$a^{60} \equiv 1 \pmod{61} \quad \text{for} \quad (a, 61) = 1$$

gives rise to

$$a^{60} \equiv 1 \pmod{k = 61 \cdot 2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 = 3{,}407{,}203{,}800}$$

$$\text{for } \left(a, \frac{k}{60}\right) = 1.$$

A SPECIAL DETERMINANT

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Put

$$
(1)\qquad \Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots n \\ 1 & 2^2 & 3^2 & \cdots n^2 \\ 1 & 2^4 & 3^4 & \cdots n^4 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2^{2n-2} & 3^{2n-2} & \cdots n^{2n-2} \end{vmatrix}.
$$

Shapiro [1] has proved that if $2n-1=p$, where p is a prime, then Δ_n is not divisible by p. This result can be obtained very rapidly as follows. Subtract the $(n-1)$-th column of Δ_n from the n-th column. Since $(n-1)^{2s}\equiv n^{2s} \pmod{p}$ it follows that

$$
\Delta_n \equiv (-1)^{n-1}D_n \qquad \pmod{p},
$$

where

$$
D_n = \left| r^{2s} \right|, \qquad (r,\ s = 1, \cdots, n-1).
$$

Since

$$
D_n = ((n-1)!)^2 \prod_{1\leqq r<s\leqq n-1} (r^2 - s^2) \not\equiv 0 \qquad \pmod{p},
$$

the assertion follows at once.

We may observe that when $2n+1=p$, then again Δ_n is not divisible by the prime p. To see this we add the first, second, $\cdots$, $(n-1)$-th column of Δ_n to the n-th column. Since

$$
\sum_{r=1}^{p} r = \tfrac{1}{2}n(n+1) \not\equiv 0 \qquad \pmod{p},
$$

while

$$
\sum_{r=1}^{n} r^{2s} \equiv \frac{1}{2}\sum_{r=1}^{p-1} r^{2s} \equiv 0 \pmod{p} \qquad (1 \leqq s \leqq n-1),
$$

it follows as in the previous case that $\Delta_n \not\equiv 0 \pmod{p}$.

We now show that*

$$
(2)\qquad \Delta_n = \frac{1!3!5!\cdots(2n-1)!}{(2n-1)\cdot(n-1)!}
$$

for arbitrary $n \geqq 1$. Indeed expanding Δ_n with respect to the elements of the first row we get

* This result was obtained independently but slightly later by Dr. Morris Newman of the National Bureau of Standards.

$$\Delta_n = \sum_{r=1}^{n} (-1)^{r-1} r D_r, \tag{3}$$

where

$$D_r = \left| i^{2j} \right| \qquad (i = 1, \cdots, n;\ i \neq r;\ j = 1, \cdots, n-1).$$

Then by a familiar formula

$$\begin{aligned} D_r &= \frac{(n!)^2}{r^2} \prod_{1<i<j<n,\, i\neq r,\, j\neq r} (j^2 - i^2) \\ &= \frac{(n!)^2}{r^2} \frac{V_n}{(r-1)!(n-r)!} \frac{1}{(r+1)\cdots(2r-1)(2r+1)\cdots(r+n)} \qquad (4) \\ &= \frac{2(n!)^2 V_n}{(n-r)!(n+r)!} \end{aligned}$$

where

$$V_n = \prod_{1\leqq i<j\leqq n} (j^2 - i^2) = \frac{1!3!5!\cdots(2n-1)!}{n!}. \tag{5}$$

Substituting from (4) in (3) we get

$$\begin{aligned} \Delta_n &= 2V_n \frac{(n!)^2}{(2n)!} \sum_{r=1}^{n} (-1)^{r-1} r \binom{2n}{n-r} \\ &= 2V_n \frac{(n!)^2}{(2n)!} \sum_{r=0}^{n-1} (-1)^{n-r-1} (n-r) \binom{2n}{r} \\ &= 2V_n \frac{(n!)^2}{(2n)!} \left\{ n \binom{2n-1}{n-1} - 2n \binom{2n-2}{n-2} \right\} \\ &= \frac{nV_n}{2n-1}. \end{aligned}$$

Using (5) it is clear that we get (2).

If p is a prime such that $n<p\leqq 2n+1$, it follows at once from (2) that the highest power of p dividing Δ_n is

$$\begin{cases} n - \frac{1}{2}(p-1) & (p < 2n-1), \\ 0 & (p = 2n-1,\ 2n+1). \end{cases} \tag{6}$$

The latter half of (6) evidently includes the two special cases mentioned above.

Reference

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A DIOPHANTINE MATRIX EQUATION

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M. H. Ingraham and H. C. Trimble [1] considered the matrix equation $XA = BX + C$. W. V. Parker [2] considered the matrix equation $AX = XB$. W. E. Roth [3] found a solution to the matrix equation $AX^2 + BX + XC = D$, and [4] also found a necessary and sufficient condition for a solution to matrix equations of the types $AX - YB = C$, and $AX - XB = C$. The principal purpose of this paper is to prove the following theorem.

THEOREM. *A necessary and sufficient condition that the Diophantine matrix equation* $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda)$ *have a solution,* $X(\lambda)$, $Y(\lambda)$, *of* n *by* n *matrices with elements which are analytic functions of a single complex variable* λ *in* $\mathfrak{R}$, *where the elements of the* n *by* n *matrices* $A(\lambda)$, $B(\lambda)$, *and* $C(\lambda)$ *are also analytic functions of a single complex variable* λ *in* $\mathfrak{R}$, *is that the matrices*

$$\begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix}, \quad \begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix} \tag{1}$$

be equivalent.

If a solution of the above matrix equation $A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda)$ exists, we have

$$\begin{pmatrix} I & Y(\lambda) \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix} \cdot \begin{pmatrix} I & -X(\lambda) \\ 0 & I \end{pmatrix} = \begin{pmatrix} A(\lambda) & C(\lambda) - A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) \\ 0 & B(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix}$$

where I is the identity matrix; therefore the matrices (1) are equivalent.

Let the matrices (1) be equivalent; we are to show that the matrix equation of the theorem has a solution. J. H. M. Wedderburn [5] has shown that if $A(\lambda)$ is a matrix with elements which are analytic functions of a single complex variable in a region $\mathfrak{R}$, there exist two non-singular matrices $P(\lambda)$ and $Q(\lambda)$ which have elements which are analytic in $\mathfrak{R}$ and are such that

$$P(\lambda) \cdot A(\lambda) \cdot Q(\lambda) = E_1(\lambda) \dotplus E_2(\lambda) \dotplus \cdots \dotplus E_r(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0,$$

where $E_i(\lambda) \mid E_{i+1}(\lambda)$, for $(i = 1, 2, \cdots, r)$, *i.e.*, $E_i(\lambda)$ are functions which are analytic in $\mathfrak{R}$ and such that $E_s(\lambda)$ is a factor of $E_t(\lambda)$ when $s < t$, $(s, t = 1, 2, \cdots, r)$. Choose non-singular n by n matrices $P(\lambda)$, $Q(\lambda)$, $R(\lambda)$ and $S(\lambda)$ with elements which are analytic functions of a single complex variable in $\mathfrak{R}$ such that

$$\begin{aligned} P(\lambda) \cdot A(\lambda) \cdot Q(\lambda) &= A'(\lambda) = a_1(\lambda) \dotplus a_2(\lambda) \dotplus \cdots \dotplus a_\alpha(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0, \\ R(\lambda) \cdot B(\lambda) \cdot S(\lambda) &= B'(\lambda) = b_1(\lambda) \dotplus b_2(\lambda) \dotplus \cdots \dotplus b_\beta(\lambda) \dotplus 0 \dotplus \cdots \dotplus 0, \end{aligned} \tag{1'}$$

where $a_i(\lambda)$ and $b_j(\lambda)$ are the invariant factors of $A(\lambda)$ and $B(\lambda)$, respectively,

$(i=1, 2, 3, \cdots, \alpha)$, $(j=1, 2, 3, \cdots, \beta)$. Since we have assumed the matrices (1) to be equivalent, we have

$$\begin{aligned} M(\lambda) &= \begin{pmatrix} P(\lambda) & 0 \\ 0 & R(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{pmatrix} \begin{pmatrix} Q(\lambda) & 0 \\ 0 & S(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} A'(\lambda) & 0 \\ 0 & B'(\lambda) \end{pmatrix} = \begin{pmatrix} P(\lambda)A(\lambda)Q(\lambda) & 0 \\ 0 & R(\lambda)B(\lambda)S(\lambda) \end{pmatrix}, \\ N(\lambda) &= \begin{pmatrix} P(\lambda) & 0 \\ 0 & R(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{pmatrix} \begin{pmatrix} Q(\lambda) & 0 \\ 0 & S(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} A'(\lambda) & C'(\lambda) \\ 0 & B'(\lambda) \end{pmatrix} = \begin{pmatrix} P(\lambda)A(\lambda)Q(\lambda) & P(\lambda)C(\lambda)S(\lambda) \\ 0 & R(\lambda)B(\lambda)S(\lambda) \end{pmatrix}, \end{aligned}$$

where $P(\lambda)C(\lambda)S(\lambda)=(c_{ij})$, $(i, j=1, 2, \cdots n)$. Thus the matrices $M(\lambda)$ and $N(\lambda)$ are equivalent. Associated with the matrices of (1) is the matric equation, $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$. Now, if we show that there exists a solution to a matric equation of the type

$$A'(\lambda)U(\lambda) - V(\lambda)B'(\lambda) = C'(\lambda), \tag{2}$$

associated with the pair of matrices $M(\lambda)$ and $N(\lambda)$ equivalent to (1), then we can obtain a solution to the original equation associated with the matrices of (1), where the elements of $U(\lambda)$ and $V(\lambda)$ are analytic functions of a single complex variable in $\mathfrak{R}$. Consider (2) element-wise,

$$a_i(\lambda)u_{ij}(\lambda) - v_{ij}(\lambda)b_j(\lambda) = c_{ij}(\lambda), \tag{3}$$

where $(i, j=1, 2, 3, \cdots, n)$, and $a_i(\lambda)\in A'(\lambda)$, $u_{ij}(\lambda)\in U(\lambda)$, $b_{ij}(\lambda)\in B'(\lambda)$, $v_{ij}(\lambda)\in V(\lambda)$, and $c_{ij}(\lambda)\in C'(\lambda)$. Consider elements of $C'(\lambda)$ in four cases: case (i) $c_{ij}(\lambda)$ where $(1\leqq i\leqq\alpha)$, $(1\leqq j\leqq\beta)$; cases (ii) and (iii) $c_{ij}(\lambda)$ where $(1\leqq i\leqq\alpha)$, $(\beta<j\leqq n)$ and $(\alpha<i\leqq n)$, and $(1\leqq j\leqq\beta)$; and case (iv) $c_{ij}(\lambda)$ where $(\alpha<i\leqq n)$ and $(\beta<j\leqq n)$.

In case (i), equation (3) has a solution. For any $c_{ij}(\lambda)$, the pair $a_i(\lambda)$, $b_j(\lambda)$ either have a common factor not a non-zero constant or they are relatively prime to each other. If they are relatively prime to each other, then there exist analytic functions in $\mathfrak{R}$, $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$, such that

$$a_i(\lambda)p_{ij}(\lambda) - b_j(\lambda)q_{ij}(\lambda) = 1. \tag{4}$$

By using a lemma due to Wedderburn [5], we may expand $1/a_i(\lambda)b_j(\lambda)$ in terms of its principal parts in a Mittag-Leffler series.

$$1/a_i(\lambda)b_j(\lambda) = F_{ij}(\lambda) + G_{ij}(\lambda) + \phi_{ij}(\lambda), \tag{5}$$

where $F_{ij}(\lambda)$ and $G_{ij}(\lambda)$ are the parts of this series arising from the zeros of $a_i(\lambda)$ and $b_j(\lambda)$ respectively, which lie in $\mathfrak{R}$, and $\phi_{ij}(\lambda)$ is a function which is analytic in $\mathfrak{R}$. Let

$$(6) \qquad p_{ij}(\lambda) = b_j(\lambda)[G_{ij}(\lambda) + \phi_{ij}(\lambda)]; \qquad q_{ij}(\lambda) = -a_i(\lambda)F_{ij}(\lambda).$$

Both $p_{ij}(\lambda)$ and $q_{ij}(\lambda)$ are analytic in $\mathfrak{R}$ and on multiplying both sides of (5) by $a_i(\lambda)b_j(\lambda)$, we have

$$(7) \quad 1 = a_i(\lambda)b_j(\lambda)[G_{ij}(\lambda) + \phi_{ij}(\lambda)] + a_i(\lambda)b_j(\lambda)F_{ij}(\lambda) = a_i(\lambda)p_{ij}(\lambda) - b_j(\lambda)q_{ij}(\lambda).$$

Hence by multiplying (7) by $c_{ij}(\lambda)$ we get

$$(8) \qquad c_{ij}(\lambda) = a_i(\lambda)[p_{ij}(\lambda)c_{ij}(\lambda)] - b_j(\lambda)[q_{ij}(\lambda)c_{ij}(\lambda)],$$

where $(i=1, 2, \cdots, \alpha)$, $(j=1, 2, 3, \cdots, \beta)$, and therefore there exists a solution for any $c_{ij}(\lambda)$ which is analytic in $\mathfrak{R}$. By making use of a technique due to Roth [4] of using determinantal divisors of $N(\lambda)$, it is seen in cases (ii), (iii), and (iv) above that $c_{ij}(\lambda) = z_{ij}(\lambda)[a_i(\lambda), b_j(\lambda)]$, hence equation (3) always has a solution. Thus there exist matrices $U(\lambda)$ and $V(\lambda)$ of order n with elements which are analytic functions of a single complex variable λ in $\mathfrak{R}$ and satisfy equation (2). Upon substituting (1′) in (2) we have,

$$(9) \qquad X(\lambda) = Q(\lambda)U(\lambda)S^{-1}(\lambda), \quad \text{and} \quad Y(\lambda) = P^{-1}(\lambda)V(\lambda)R(\lambda).$$

The theorem is also true when the elements of the matrices $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ belong to a principal ideal ring $\mathfrak{P}$. The necessity is proved as in the preceding theorem. To prove the sufficiency we make use of a theorem due to C. C. MacDuffee [6] which states that every matrix A of rank α with elements in a principal ideal ring $\mathfrak{P}$ is equivalent to a diagonal matrix $h_1, h_2, \cdots, h_\alpha, 0, \cdots, 0$ where $h_i \neq 0$ and $h_i \mid h_{i+1}$. The h_i are invariant up to a unit factor in the principal ideal ring $\mathfrak{P}$, so that the diagonal form of A is unique if the h_i belong to a set of non-associates in $\mathfrak{P}$. Thus there exist unimodular matrices P, Q, R, and S with elements in $\mathfrak{P}$ such that

$$PAQ = \text{diagonal matrix } [a_1, a_2, a_3, \cdots a_\alpha, 0, \cdots 0] = A',$$
$$RBS = \text{diagonal matrix } [b_1, b_2, b_3, \cdots, b_\beta, 0, \cdots 0] = B',$$

where A is of rank α and B is of rank β. In a principal ideal ring $\mathfrak{P}$ every element neither zero nor a unit can be factored uniquely (except for unit factors) into a product of primes according to MacDuffee [6]. Thus by making use of the same type of argument as that given by W. E. Roth in [4] we obtain a solution (9) to the matrix equation up to a unit factor in $\mathfrak{P}$. However, if the elements of A' and B' belong to a set of non-associates, then X and Y will be unique in $\mathfrak{P}$ for a given P, Q, R and S which reduce A and B to diagonal form.

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A PAIR OF MATRICES WITH PROPERTY P

HANS SCHNEIDER, Queen's University, Belfast

A set $A_1, \cdots, A_s$ of $n\times n$ matrices with coefficients in an algebraically closed field is said to have property P if there exists an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}$, $i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$ for which the characteristic roots of any polynomial $p(A_1, \cdots, A_s)$ are $p(\alpha_i^{(1)}, \cdots, \alpha_i^{(s)})$, $i=1, \cdots, n$. In 1936 McCoy [3] proved that the set $A_1, \cdots, A_s$ has property P if and only if every matrix of the form $(A_iA_j-A_jA_i)R$, where R is a polynomial in the A_i, is nilpotent (for an elementary proof see Drazin, Dungey and Gruenberg [2]). More recently two very special cases of this theorem have been proved separately. Thus in 1950 Parker [5] showed that if $AB=B^2=0$, then the characteristic roots of $A+B$ are the same as those of A. (The characteristic roots of B are all zero.) In 1953 Perfect [6], completing a theorem of A. Brauer [1], showed that if $(C-\lambda I)v=0$, and B is a matrix of rank 1 all of whose columns are multiples of the column vector v, then the characteristic roots of $C+B$ are obtained from those of C by replacing one λ by $\lambda+$trace B. (One characteristic root of B is trace B, the rest are zero, and $AB=0$ if $A=(C-\lambda I)$.) We shall give a very simple proof of a special case of McCoy's theorem which includes the two results quoted above.

In order to make our theorem applicable to $n\times n$ matrices with coefficients in a field which is not necessarily algebraically closed we shall state the result in terms of the characteristic polynomial $|xI-A|$ of a square matrix A.

LEMMA. *Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k. If $(\sum_{i=1}^{j} A_i)A_{j+1}=0$, for $j=1, \cdots, s-1$, then the characteristic polynomial of $\sum_{j=1}^{s} A_j$ is $(\prod_{j=1}^{s} |xI-A_j|)/x^{(s-1)n}$.*

Proof: Since $A_1A_2=0$,

$$\begin{aligned} |xI - A_1|\,|xI - A_2| &= |x^2 - x(A_1 + A_2) + A_1A_2| \\ &= |x(xI - (A_1 + A_2))| = x^n|xI - (A_1 + A_2)|. \end{aligned}$$

The result follows if $s=2$. The lemma is now obtained by induction on s.

THEOREM. *Let A and B be $n\times n$ matrices with coefficients in k. If $AB=0$, then the characteristic polynomial of the polynomial $p(A, B)$ without constant term is*

$$|xI - p(A, 0)| \cdot |xI - p(0, B)|/x^n.$$

Proof. Since $AB=0$, we have $p(A, B)=A_1+A_2+A_3$, where $A_1=p(A, 0)$,

$A_3=p(0, B)$ and A_2 is of form $Bq(A, B)A$. Clearly $A_2^2=0$, and so the characteristic polynomial of A_2 is x^n. As $A_1A_2=(A_1+A_2)A_3=0$, it follows by the lemma that

$$\begin{aligned} |xI - p(A, B)| &= |xI - A_1|\,|xI - A_2|\,|xI - A_3|/x^{2n} \\ &= |xI - p(A, 0)|\,|xI - p(0, B)|/x^n. \end{aligned}$$

COROLLARY. *If k is algebraically closed, then the pair of matrices A, B has property P.*

Proof. If α_i, β_i, $i=1, \cdots, n$, are the characteristic roots of A and B respectively then $x^n|xI-(A+B)|=|xI-A|\,|xI-B|=\prod_{i=1}^n (x-\alpha_i)(x-\beta_i)$. We may easily show from this that there is an ordering α_i, β_i, $i=1, \cdots, n$, for which $\alpha_i\beta_i=0$, $i=1, \cdots, n$. It is enough to prove that with this ordering the characteristic roots of $p(A, B)$ are $p(\alpha_i, \beta_i)$, $i=1, \cdots, n$, for any polynomial $p(A, B)$ without constant term.

For such a polynomial, $p(\alpha_i, \beta_i)=p(\alpha_i, 0)+p(0, \beta_i)$ and $p(\alpha_i, 0)p(0, \beta_i)=0$, $i=1, \cdots, n$.

By the theorem,

$$\begin{aligned} x^n|xI - p(A, B)| &= |xI - p(A, 0)|\,|xI - p(0, B)| \\ &= \prod_{i=1}^{n} (x - p(\alpha_i, 0))(x - p(0, \beta_i)) \\ &= x^n \prod_{i=1}^{n} (x - p(\alpha_i, \beta_i)). \end{aligned}$$

The corollary is proved.

We remark that if $AB=0$ then $((AB-BA)R)^2=0$, where R is any polynomial in A and B. Thus we have indeed proved a special case of McCoy's theorem. A slight extension of the arguments used in the theorem and corollary leads to the following results: Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k such that $A_iA_j=0$ if $i<j$. If $p(A_1, \cdots, A_s)$ is a polynomial without constant term, then the characteristic polynomial of $p(A_1, \cdots, A_s)$ is

$$\left(\prod_{j=1}^{s} |xI - p_j(A_j)|\right)/x^{(s-1)n},$$

where $p_1(A_1)=p(A_1, 0, \cdots, 0)$, *etc.* If in addition, k is algebraically closed, then the set $A_1, \cdots, A_s$ has property P. It may also be shown that if $A_1, \cdots, A_s$ satisfy the conditions of the lemma and k is algebraically closed, then the characteristic roots of $\sum_{j=1}^s A_j$ are $\sum_{j=1}^s \alpha_i^{(j)}$, $i=1, \cdots, n$, for an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}$, $i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$. This is a property weaker than property L (*cf.* Motzkin and Taussky [4]).

I am indebted to a referee for materially simplifying the proof of the lemma.

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CLASSROOM NOTES

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AREA OF A TRIANGLE AND THE DETERMINANT

G. M. PETERSEN, University of Oklahoma

It is easy to demonstrate to a class that the equation of a straight line through (x_2, y_2) and (x_3, y_3) is given by

$$\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The altitude through (x_1, y_1) of a triangle with vertices (x_1, y_1) (x_2, y_2) and (x_3, y_3) is found in the usual way from the normal form of the above:

$$\frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Another step shows the area of the triangle to be

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

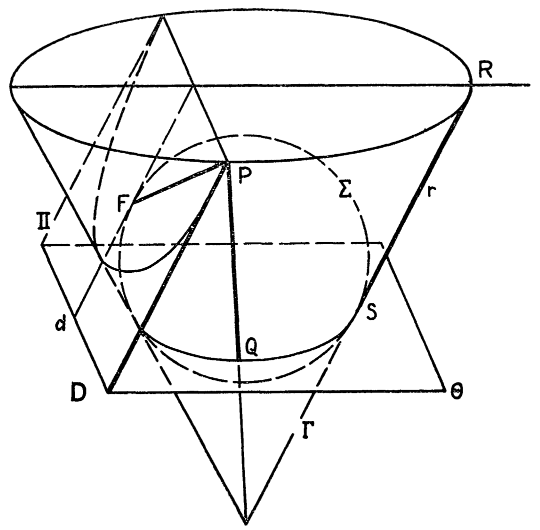
This exposition reveals the *modus operandi* more clearly than the usual proof of showing a collection of terms to be identical with a determinant.

ON DEFINING CONIC SECTIONS

G. B. Huff, The University of Georgia

The conventional American text in analytic geometry begins the study of conic sections with a paragraph describing these curves as sections of a right circular cone by suitably chosen planes. It is customary to remark that this definition of a conic section is not adapted to the methods and objectives of the course, and each one of the parabola, the ellipse, and the hyperbola is then given its own definition as a locus in a later paragraph. Ordinarily there is no attempt to give even an intuitive proof of the equivalence of the two definitions.

Apparently it is not well known that this may be done quickly and simply in the case of the ellipse and the hyperbola. In *Anschauliche Geometrie* (Hilbert, Cohn-Vossen; Dover; 1944; pp. 7, 8), a figure is given from which a simple



intuitive argument may be made that the two definitions for the ellipse (or the hyperbola) are equivalent. This note is written to call these figures to the attention of analytic geometry teachers and to give the construction of a third figure which leads to a demonstration of the equivalence of the two definitions of the parabola.

Let the parabola be defined as the intersection of a right circular cone Γ and a plane Π parallel to a ruling r of Γ and perpendicular to the plane on r and the axis of Γ. Construct a sphere Σ internally tangent to Γ and tangent to the plane Π. (The existence and unicity of Σ is made clear by thinking of fitting a small balloon in Γ and inflating it until it meets Π.) Let Θ be the plane determined by the intersection of Σ and Γ. Let F be the common point of Π and Σ,

and let d be the common line of Π and Θ. I claim that F is the focus and d is the directrix of the parabola defined by Γ and Π.

Let P be any point of the parabola and let Φ be the plane through P and perpendicular to the axis of the cone Γ. Let D be the foot of the perpendicular from P to d; let Q be the intersection of the ruling on P with the plane Θ, and let R, S be the points on r cut out by Φ and Θ. A simple argument shows that RS is parallel to PD.

Since PF and PQ are tangents to the sphere, then $PF=PQ$.

Since PQ and RS are segments of rulings of Γ cut out by the parallel planes Θ and Φ, then $PQ=RS$.

Since PD is parallel to RS, these segments are cut out on parallel lines by parallel planes and $RS=PD$. It follows that $PF=PD$, as claimed.

It is easy to generalize the figure and the argument to definition of a conic of eccentricity e in terms of its directrix and focus. (Zwikker, *Advanced Plane Geometry*, Amsterdam, 1950, p. 108.)

A DIOPHANTINE EQUATION CHARACTERIZING THE LAW OF COSINES

I. A. BARNETT, University of Cincinnati

The purpose of this remark is to show that the most general rational solution of the cubic equation

$$x^2 + y^2 + z^2 + 2xyz = 1 \tag{1}$$

is given by

$$x = \frac{b^2 + c^2 - a^2}{2bc}, \qquad y = \frac{a^2 + c^2 - b^2}{2ac}, \qquad z = \frac{a^2 + b^2 - c^2}{2ab}. \tag{2}$$

In the first place it is seen at once that equation (1) may be written in the form

$$\begin{vmatrix} y & x & -1 \\ -1 & z & y \\ z & -1 & x \end{vmatrix} = 0.$$

Hence there exist rational numbers a, b, c such that

$$\begin{aligned} bx + ay &= c \\ cy + bz &= a \\ cx + az &= b. \end{aligned}$$

From these equations we obtain the solution (2). Conversely, the expressions in (2) satisfy (1).

Equation (1) has an immediate geometrical interpretation. For, if a semicircle of radius 1 is divided into three parts subtending angles of magnitude 2α, 2β, 2γ at the center, with the corresponding opposite chords of lengths

$2x, 2y, 2z$ respectively, we have $\alpha+\beta=90°-\gamma$ so that $\cos(\alpha+\beta)=\sin\gamma$ and hence, $\sqrt{(1-x^2)(1-y^2)}=xy+z$. When this is freed of radicals we obtain (1).

This interpretation suggests a generalization of the law of cosines. Dividing the semicircle of radius 1 into n angles $2\alpha_1, 2\alpha_2, \cdots, 2\alpha_n$, and designating the opposite chords by $2x_1, 2x_2, \cdots, 2x_n$, we have immediately $\cos(\alpha_1+\alpha_2 \cdots +\alpha_n)=0$. The resulting equation when freed of radicals will be symmetric in $x_1, x_2, \cdots, x_n$.

However, even for $n=4$ the equation will be of the sixteenth degree in x_1, x_2, x_3 and x_4. In fact if $\alpha+\beta+\gamma+\delta=90°$ we see that $\cos(\alpha+\beta)=\sin(\gamma+\delta)$.

Hence

$$\sqrt{(1-x^2)(1-y^2)}-u\sqrt{1-z^2}=xy+z\sqrt{1-u^2},$$

where we have called the lengths of the opposite chords $2x, 2y, 2z, 2u$. When this equation is freed of radicals we obtain

$$\begin{aligned}[(1-x^2-y^2-z^2-u^2)^2-4(x^2y^2z^2+x^2y^2u^2+x^2z^2u^2+y^2z^2u^2-2x^2y^2z^2u^2)]^2 \\ =64x^2y^2z^2u^2(1-x^2)(1-y^2)(1-z^2)(1-u^2).\end{aligned}$$

In terms of the elementary symmetric functions of x^2, y^2, z^2, u^2,

$$r=\sum x^2, \qquad s=\sum x^2y^2, \qquad t=\sum x^2y^2z^2, \qquad w=x^2y^2z^2u^2$$

the preceding equation may be written

$$[(1-r)^2-4(t-2w)]^2=64w(1-r+s-t+w).$$

It is not clear how one would go about solving such a diophantine equation.

ON THE $\lim_{\theta\to 0}\cos\theta$

M. J. PASCUAL, Siena College

In most texts on calculus, the proof of the fundamental limit equation $\lim_{\theta\to 0}\sin\theta/\theta=1$ involves the assumption that the limit equation $\lim_{\theta\to 0}\cos\theta=1$ is evident. The apparent reason for this supposedly obvious conclusion must have its basis in the fact that $\cos\theta$ is continuous at $\theta=0$. But the notion of limit is a more primitive one than that of continuity. Hence to avoid the possible confusion which may easily arise in the mind of a beginner in calculus, the teacher should at least assert that the $\lim_{\theta\to 0}\cos\theta$ could be proved to be 1 by direct appeal to the definition of limit. Indeed, we could give the following proof, which would simultaneously introduce the student to the analytical method of proof and also prepare an example for the lesson on continuity, in which the instructor shows that $\cos\theta$ is actually continuous at $\theta=0$. Perhaps it could be left to the students to show it is continuous at any value of θ.

Since this proof would be given at an early stage of a calculus course, prior to which inverse trigonometric functions may not have been covered, it will be desirous to avoid such functions, so we shall define $\cos\theta$ by the conventional

method employing rectangular coordinates. This will also prove helpful in establishing the continuity at $\theta=0$, as the old triangle definition will not suffice, there being no triangle.

Placing the vertex of θ at the origin of the rectangular system, with its initial side along the positive x-axis, and measuring θ as usual counterclockwise, we define $\cos\theta$ by choosing an arbitrary point other than (0, 0) on the terminal side of θ having coordinates (x, y) and let $\cos\theta=x/\sqrt{x^2+y^2}$. Since $y\to 0$ as $\theta\to 0$, we wish to establish the equation $\lim_{y\to 0} x/\sqrt{x^2+y^2}=1$. We shall restrict θ to the first or fourth quadrants and for convenience we may choose $x=1$.

To fulfill the definition of limit, for any $\epsilon>0$, we must produce a δ_ϵ such that

$$\left|\frac{1}{\sqrt{1+y^2}}-1\right|<\epsilon \quad \text{for} \quad |y|<\delta_\epsilon.$$

We analyze as follows: for any $y\neq 0$

$$0<\frac{1}{\sqrt{1+y^2}}<1 \quad \text{so that} \quad \left|\frac{1}{\sqrt{1+y^2}}-1\right|=1-\frac{1}{\sqrt{1+y^2}}\cdot$$

Hence we wish to have

$$1-\frac{1}{\sqrt{1+y^2}}<\epsilon$$

and for $0<\epsilon<1$ (for $\epsilon\geqq 1$ this inequality obviously holds)

$$1-\epsilon<\frac{1}{\sqrt{1+y^2}}$$

$$1+y^2<(1-\epsilon)^{-2}$$

$$|y|<\sqrt{(1-\epsilon)^{-2}-1}$$

giving the desired δ_ϵ as

$$\sqrt{(1-\epsilon)^{-2}-1}.$$

A DE-CENTRALIZED CENTROID

J. P. BALLANTINE, University of Washington

My students are always interested to hear about a region in Quadrant I whose centroid is on the x-axis. This is, of course, slightly impossible. Moreover, this particular region is wide at one end, and tapers out to a sharp point, with the centroid at the end of the point. This is getting more impossible. Finally, the height of impossibility is reached with the statement that the sharp point reaches out infinitely far.

The region is that included between $y=1/x$ and $y=0$, from $x=1$ to $x=\infty$.

THE PERIOD OF POLAR CURVES

R. LARIVIERE, University of Illinois, Chicago

Many calculus teachers find students in their classes who are not aware that the period of the polar curves of many trigonometric functions is not the same as that of the corresponding curve in rectangular coordinates.

An interesting way of enlightening these students is to teach them generalized polar coordinates (if they have not already learned them) and to point out that the curve cannot become periodic until the points indicated by increasing values of θ have the same coordinates as those of points already on the curve.

Thus, the generalized coordinates of a point (ρ, θ) on the curve $\rho=f(\theta)$ are $[(-1)^r\rho, (\theta+r\pi)]$, where r is any integer. Since we are concerned with increasing values of θ only we limit our consideration to $r>0$. When $f(\theta+r\pi)=(-1)^rf(\theta)$ for all values of θ the curve repeats, and the minimum $r\pi$ at which this occurs is the polar period R.

Examples.

Curve	$f(\theta + r\pi)$	$(-1)^rf(\theta)$	R
$\rho = \sin 5\theta$	$\sin (5\theta + 5r\pi)$	$(-1)^r \sin 5\theta$	1π
$\rho = \cos \dfrac{3\theta}{4}$	$\cos\left(\dfrac{3\theta}{4} + \dfrac{3r\pi}{4}\right)$	$(-1)^r \cos \dfrac{3\theta}{4}$	8π
$\rho = \tan \dfrac{\theta}{3}$	$\tan\left(\dfrac{\theta}{3} + \dfrac{r\pi}{3}\right)$	$(-1)^r \tan \dfrac{\theta}{3}$	6π
$\rho = 1 + \sin\theta$	$1 + \sin(\theta + r\pi) = 1 + \sin\theta\cos r\pi$	$(-1)^r(1 + \sin\theta)$	2π

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1161. *Proposed by C. S. Ogilvy, Hamilton College*

Professor E. P. B. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving this equation using nothing higher than quadratic equations.

E 1162. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find two non-congruent similar triangles having two sides of one equal to two sides of the other.

E 1163. *Proposed by H. Gupta, Hoshiarpur, India*

Find the number of ways in which m similar objects can be distributed among n persons where there is no restriction as to the number of objects that any of them may receive.

E 1164. *Proposed by M. J. Mansfield, Purdue University*

Every rational number in the closed interval [0, 1] appears at least once in the sequence

$$0, 1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \cdots.$$

Find the nth, or general, term of this sequence.

E 1165. *Proposed by Andrew Sobczyk, Los Alamos Scientific Laboratory*

A vertex V of a closed polygon C having an odd (even) number of sides is *regular* in case a triangle formed by extending the sides incident on V and having for base a line segment containing the opposite side (vertex) to V circumscribes C. Show that every convex C has at least one regular vertex. (A convex pentagon may have non-regular vertices.)

SOLUTIONS

The Student's Predicament

E 1131 [1954, 568]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

A student solves the differential equation $(dy/dx)^2 = x^2$ and reports as follows: "General solution $(2y-2c)^2 = x^4$; p-discriminant $x=0$; c-discriminant $x=0$. Since the p-discriminant locus consists of the envelope, cuspidal, and tac loci while the c-discriminant consists of the envelope, cuspidal, and nodal loci and since the line $x=0$ is not an envelope, it ought to be a cuspidal locus. But the general solution consists of the parabolas $y=x^2/2+c$ and $y=-x^2/2+c$, and parabolas don't have cusps. How can there be a cuspidal locus if there are no cusps?" Resolve the predicament.

Editorial Note. All the contributed solutions accounted for a cuspidal locus by pointing out that the solution of the differential equation is a pencil of composite quartic curves, each factorable into a pair of parabolas tangent to one

another along the line $x=0$, whence $x=0$ is a (double) cuspidal locus. This argument seems hardly satisfactory. The theory that a locus which is part of both the p-discriminant locus and the c-discriminant locus, and whose equation does not satisfy the differential equation, is a cusp locus, assumes that the p-equation is irreducible in the real domain. This is not the case here. The locus $x=0$, which appears as a squared factor in the p-discriminant, is a tac locus; there is no cusp locus as generally understood. See W. W. Johnson, *A Treatise on Ordinary and Partial Differential Equations*, 3rd ed., art. 54, pp. 49–50. The student's predicament arose from employing theory concerning the p and c-discriminants in a case where that theory does not apply. The analytical theory concerning the p-discriminant and c-discriminant loci presents grave difficulties.

Three Tangent Circles and Two Equal Arcs

E 1132 [1954, 568]. *Proposed by Leon Bankoff, Los Angeles, California*

A common external tangent of two circles, tangent externally at C, cuts their smallest circumscribed circle in P and Q. The common internal tangent at C intersects the minor arc PQ in E, and the major arc QP in F. PC extended meets the outer circumference in K. Show that arc $QK=$arc KF.

Solution by the Proposer. The inversion (E, EC^2) transforms the outer circumference into the line PQ; the line PK becomes the circle through P, C, E. Hence K', the inverse of K, lies on the intersection of the line PQ and the circle PCE. It follows that angles CEK' and CPK' are equal, and arc $QK=$arc KF.

Also solved by Hüseyin Demir and Josef Langr (synthetically) and R. B. Plymale (analytically).

A Theorem on Polynomials

E 1133 [1954, 568]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Show that a necessary and sufficient condition that the imaginary roots of

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

occur in conjugate pairs is that the numbers $a_0, a_1, \cdots, a_n$ all lie on a common ray emanating from the origin in the complex plane.

Solution by F. D. Parker, Clarkson College of Technology. The theorem should read, "a common line through the origin" rather than "a common ray emanating from the origin."

If the coefficients lie on a common line, $\theta=\theta_1$, through the origin, then writing the coefficients in polar form produces a factor $e^{i\theta_1}$. When this factor is divided out, the coefficients are real and the imaginary roots are in conjugate pairs.

If the imaginary roots occur in conjugate pairs, then an equation with real coefficients can be formed with those roots. Any other equation with the same roots must differ by only a multiplicative constant. Any such constant will place the new coefficients on a common line through the origin.

Also solved by C. N. Campopiano, D. F. Criley, Hüseyin Demir, E. Frank, A. M. Glicksman, Bernard Greenspan, D. S. Greenstein, Virginia Hanly, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, M. J. Pascual, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, David Sachs, Paul Schaefer, Berthold Schweizer, R. H. Sprague, J. A. Tierney, and the proposer. Late solution by C. F. Pinzka.

Squares and Perfect Numbers

E 1134 [1954, 568]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that a square integer is not a perfect number.

I. *Solution by C. F. Pinzka, Princeton, N. J.* If $N=\Pi p^{2a}$, the sum of the divisors of N is

$$\Pi \frac{p^{2a+1}-1}{p-1}.$$

Since the latter is always odd, it cannot equal $2N$ as required for a perfect number.

II. *Solution by C. D. Olds, San Jose State College.* Euler proved that an odd perfect number must have the form $r^{4k+1}P^2$, where r is a prime of the form $4n+1$. An even perfect number must be of Euclid's type, that is, of the form $2^{p-1}(2^p-1)$ where 2^p-1 is a prime. Thus a square cannot be perfect.

Also solved by Leon Bankoff, R. L. Beinert, Arne Benson, W. E. Briggs, J. E. Darraugh, A. D. Freedman, Phil Freygood, Bernard Greenspan, D. S. Greenstein, Virginia Hanly, Douglas Holdridge, A. R. Hyde, Bernard Jacobson, M. S. Klamkin, Sidney Kravitz, Octave Levenspiel, D. C. B. Marsh, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, F. D. Parker, L. A. Ringenberg, Azriel Rosenfeld, Georgia Smith, D. R. Sudborough, and the proposer. Late solutions by H. W. Becker and C. W. Trigg.

Maximum Value of a Determinant

E 1135 [1954, 568]. *Proposed by Vern Hoggatt, San Jose State College*

If a third order determinant has elements 1, 2, 3, $\cdots$, 9, what is the maximum value it may have?

Solution by F. W. Saunders, Coker College. Let P be the sum of the three positive terms of the determinant and $-N$ the sum of the three negative terms. The maximum value of P is

$$9\cdot 8\cdot 7+6\cdot 5\cdot 4+3\cdot 2\cdot 1=630$$

The minimum value of N is

$$9\cdot 8\cdot 1+7\cdot 5\cdot 2+6\cdot 4\cdot 3=214.$$

The minimum combination for N consistent with P is

$$9\cdot 6\cdot 1 + 8\cdot 5\cdot 2 + 7\cdot 4\cdot 3 = 218.$$

Any change in P would result in lowering that sum by more than 4. Therefore 412 is the maximum value for the determinant, and one form for the determinant is

$$\begin{vmatrix} 9 & 4 & 2 \\ 3 & 8 & 6 \\ 5 & 1 & 7 \end{vmatrix}.$$

Also solved by Julian Braun, M. S. Klamkin, T. F. Mulcrone, J. B. Muskat, C. F. Pinzka, and T. H. Sumner.

F. Stancliff, in *Scripta Mathematica*, Dec. 1954, p. 278, has obtained, by trial and error, determinants which use each of the nine digits once and having values from 0 to 323 (with the exception of 202) and even values from 324 to 412.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4633. *Proposed by W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

An even number, n, of chairs are arranged in a circle and numbered 1 through n. An equal number of people are numbered 1 through n and line up in an arbitrary manner. Prove that they can be seated in such a way as to preserve their order and such that no person's number is the same as that of his chair. What happens if n is odd?

4634. *Proposed by Chih-yi Wang, University of Minnesota*

Let a real number k, $0<k<1$, be given. Define the elliptic integrals

$$A_s = \int_0^{\pi/2} \sin^{2s} t\sqrt{1 - k^2 \sin^2 t}\, dt, \qquad B_s = \int_0^{\pi/2} \frac{\sin^{2s} t}{\sqrt{1 - k^2 \sin^2 t}}\, dt$$

for $s=0, 1, 2, \cdots$. Prove that

$$\text{(a)}\ B_r = \frac{B_{r-1} - A_{r-1}}{k^2}, \qquad \text{(b)}\ A_r = \frac{(2rk^2 - 1)A_{r-1} + (1 - k^2)B_{r-1}}{(2r + 1)k^2},$$

for $r=1, 2, 3, \cdots$.

4635. *Proposed by Albert Babbitt, Rutgers University*

Let G be an additive Abelian group which has no element of order 2 and let f be a one to one mapping of G onto G. Then f is the inversion (*i.e.*, $f(x) = -x$) if and only if the following two conditions are satisfied:

I. The neutral element 0 is the only element of G invariant under f.
II. For every element $a \in G$, there exists an element $b = b(a)$ in G such that, for all $x \in G$,

$$f(x - a) = f^{-1}(x + b) - a.$$

4636. *Proposed by Emma Lehmer, Berkeley, California*

Show that, if n is a prime of the form $8m+5$, then

$$\sum_{\nu=0}^{n-1} \cos \frac{2\pi\nu^4}{n} = \sqrt{n}.$$

For what other values of n is this true?

4637. *Proposed by W. E. Briggs, University of Colorado*

Let $f(n)$ be the smallest number such that $u_{f(n)}$ is divisible by n, where $u_1 = 1$, $u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$. Prove (or disprove) that for any prime p greater than 2 and for all n,

$$f(p^n) = p^{n-1}f(p).$$

SOLUTIONS

Inequality

4553 [1953, 554; 1954, 720]. *Proposed by D. J. Newman, Republican Aviation Corporation, Farmingdale, N. Y.*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2} \sum_{n=1}^{\infty} \sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}.$$

Comment by R. P. Boas, Jr., Northwestern University. The more precise inequality

$$\sum_{n=1}^{\infty} a_n \leqq (4/3)^{1/2} \sum_{n=1}^{\infty} (r_n/n)^{1/2}, \qquad r_n = \sum_{k=n}^{\infty} a_k^2,$$

was proved by Stechkin [*Mat. Sbornik N.S.* 29(71), 225–232 (1951)]. His proof is as follows. First we observe that

$$\sum_{n=k}^{\infty} \frac{1}{n^2(n+1)^2} = \sum_{n=k}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 = \sum_{n=k}^{\infty}\left\{\int_n^{n+1}\frac{dx}{x^2}\right\}^2$$
$$\leqq \sum_{n=k}^{\infty}\int_n^{n+1}\frac{dx}{x^4} = \int_k^{\infty}\frac{dx}{x^4} = \frac{1}{3k^3}.$$

Using this, we have by Cauchy's inequality

$$\sum_{n=1}^{\infty} a_n = 2\sum_{n=1}^{\infty}\frac{a_n}{n(n+1)}\sum_{k=1}^{n} k = 2\sum_{k=1}^{\infty} k \sum_{n=k}^{\infty}\frac{a_n}{n(n+1)}$$
$$\leqq 2\sum_{k=1}^{\infty} k\left\{\sum_{n=k}^{\infty} a_n^2\right\}^{1/2}\left\{\sum_{n=k}^{\infty}\frac{1}{n^2(n+1)^2}\right\}^{1/2}$$
$$\leqq \frac{2}{\sqrt{3}}\sum_{k=1}^{\infty}(r_k/k)^{1/2}.$$

The constant $(4/3)^{1/2} = 1.1547$ is not the best possible, and the best constant does not seem to be known. By taking $a_1 = x$, $a_2 = 1$, $a_3 = a_4 = \cdots = 0$, one can show that the best constant cannot be less than 1.12.

A Consequence of Fermat's Equation

4571 [1954, 52]. *Proposed by Robert Kissling, University of California, Berkeley*

Suppose $a^p + b^p = c^p$ in relatively prime integers with p a prime greater than 3. If q is any prime dividing $a^2 + ab + b^2$ but not $a + b - c$, show that $(q-1)/6p$ is an integer.

Solution by C. R. Phelps, Rutgers University. We have $a \not\equiv 0 \pmod{q}$, for otherwise q is a common factor of a and b, from $a^2 + ab + b^2 \equiv 0$. Thus $a^{-1} \pmod{q}$ exists, and the hypotheses give

$$1 + B^p \equiv C^p \text{ and } 1 + B + B^2 \equiv 0 \pmod{q},$$

where $B = ba^{-1}$ and $C = ca^{-1}$. From the latter congruence we get $B^3 \equiv 1$, with $B \not\equiv 1$. (Otherwise $a \equiv b$ and $3a^2 \equiv 0$ with $a \not\equiv 0$, so that $q = 3$; but if $q = 3$ and $a \equiv b \pmod{3}$ then either $a \equiv b \equiv 1$ and $c \equiv 2$ or $a \equiv b \equiv 2$ and $c \equiv 1$, and in either case $a + b - c \equiv 0$ contrary to the hypothesis.) Thus 3 is a divisor of $q - 1$, from $B^{q-1} \equiv 1$.

Suppose that $(p, q-1) = 1$. Then there exist x, y such that $px + y(q-1) = 1$ with x odd, and since 3 divides $q - 1$, $p \equiv x \pmod{3}$. If $p \equiv 1 \pmod{3}$, then $C^p \equiv 1 + B^p \equiv 1 + B \equiv -B^2 \pmod{q}$, since $B^3 \equiv 1$; also $C \equiv C^{px} \cdot C^{y(q-1)} \equiv C^{px} \equiv -B^{2x} \equiv -B^2 \equiv 1 + B \pmod{q}$, contrary to the hypothesis $1 + B \not\equiv C$. Similarly, if $p \equiv 2 \pmod{3}$, then $C^p \equiv 1 + B^p \equiv 1 + B^2 \equiv -B$ and $C \equiv C^{px} \equiv (-B)^x \equiv -B^2 \equiv 1 + B$,

again contrary to the hypothesis. Thus $(p, q-1)\neq 1$, so that p divides $q-1$.

Evidently $q-1$ is even, and therefore $(q-1)/6p$ is an integer.

Note. The condition $a^p+b^p=c^p$ can be replaced by the more general condition $a^p+b^p\equiv c^p \pmod{q}$, without change in the proof. In this less esoteric form examples can be constructed, e.g.: $a=4$, $b=7$, $c=13$, $p=5$, $q=31$.

A Class of Irrational Numbers

4572 [1954, 263]. *Proposed by Paul Erdös, University of Notre Dame*

Let b_k be a sequence of non-negative integers such that

$$\overline{\lim}\ \frac{1}{n}\sum_{k=1}^{n} b_k < \infty.$$

Let $f(n)\to\infty$ denote the number of b_k's with $1\leqq k\leqq n$ for which $b_k>0$. If $\underline{\lim}\ f(n)/n=0$, show that

$$c = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

is irrational.

Solution by G. G. Lorentz, Wayne University. Let

$$c_k = b_k + b_{k+1}/2 + b_{k+2}/2^2 + \cdots.$$

If $(b_1+\cdots+b_k)/k\leqq M$, we have

$$\begin{aligned}\frac{c_1+\cdots+c_k}{k} &\leqq \frac{b_1+\cdots+b_k}{k} + \frac{k+1}{2k}\cdot\frac{b_1+\cdots+b_{k+1}}{k+1}\\ &\quad + \frac{k+2}{2^2k}\cdot\frac{b_1+\cdots+b_{k+2}}{k+2} + \cdots \qquad (1)\\ &\leqq \frac{M}{k}\left\{k+\frac{k+1}{2}+\frac{k+2}{2^2}+\cdots\right\} = 2M+\frac{2M}{k}\leqq 4M.\end{aligned}$$

We prove that

$$\underline{\lim}\ c_k = 0. \qquad (2)$$

Assume $c_k\geqq q_0>0$ for all large k. Then, since $2c_{k-1}=2b_{k-1}+c_k$, we have $c_k=2c_{k-1}$ for all k with $b_{k-1}=0$. Let s be a fixed positive integer. We see that $c_k\geqq 2^s q_0$ if b_k is preceded by s vanishing terms b_i. The number of k's between 1 and n which do not have this last property does not exceed $sf(n)$ and is therefore less than $n/2$ for an infinity of values of n. For such n, $c_k\geqq 2^s q_0$ is true for at least $n/2$ values of k, $1\leqq k\leqq n$, so that

$$\frac{c_1+\cdots+c_n}{n}\geqq 2^{s-1}q_0,$$

which contradicts (1) if s is sufficiently large, and proves (2).

Now let $c=p/q$ be rational. Then for some integer a, $c=p/q=a/2^{k-1}+c_k/2^k$, so that

$$p2^{k-1} - qa = \text{integer} = qc_k/2,$$

which is a contradiction.

Also solved by Harry Kesten, P. J. Owens and Howard Robbins.

An Improper Integral

4573 [1954, 126]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Show that, if $p=\log(\sqrt{2}+1)$,

$$\int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\, dx = \frac{\pi}{4} \log 2.$$

I. *Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota.* The substitution $\coth x = y$ yields

$$n_1 = \int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\, dx = \frac{1}{2} \int_{\sqrt{2}}^{\infty} \frac{\log \dfrac{y+1}{y-1}}{(y^2-1)\sqrt{y^2-2}}\, dy.$$

Now consider the integral

$$n(\alpha) = \frac{1}{2} \int_{\sqrt{2}}^{\infty} \frac{\log \dfrac{\alpha y+1}{\alpha y-1}}{(y^2-1)\sqrt{y^2-2}}\, dy.$$

Differentiation with respect to α yields

$$n'(\alpha) = \frac{-1}{\alpha^2} \int_{\sqrt{2}}^{\infty} \frac{y\,dy}{(y^2-1)(y^2-\alpha^{-2})\sqrt{y^2-2}}.$$

This integral is easily evaluated by familiar methods and we obtain

$$n'(\alpha) = \frac{\pi}{2}\left[\frac{\alpha}{(\alpha^2-1)\sqrt{2\alpha^2-1}} - \frac{1}{\alpha^2-1}\right].$$

Hence

$$n(\alpha) = \frac{\pi}{4} \log \frac{(\alpha+1)(\sqrt{2\alpha^2-1}-1)}{(\alpha-1)(\sqrt{2\alpha^2-1}+1)}$$

in which the constant of integration vanishes since $n(\alpha)\to 0$ as $\alpha\to\infty$. On noting that $n(\alpha)\to n_1$ as $\alpha\to 1$, we obtain $n_1=(\pi/4)\log 2$ as required.

II. *Solution by the Proposer.* It is not difficult to show that

$$\log\left(\sqrt{1+\sin\theta}+\sqrt{\sin\theta}\right)=\sin\theta+\frac{1}{2}\,\frac{\sin 3\theta}{3}+\frac{1\cdot 3}{2\cdot 4}\,\frac{\sin 5\theta}{5}+\cdots$$

for $0\leqq\theta\leqq\pi$. Integrating from 0 to $\pi/2$ we obtain

$$\int_0^{\pi/2}\log\left(\sqrt{1+\sin\theta}+\sqrt{\sin\theta}\right)d\theta=1+\frac{1}{2}\,\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5^2}+\cdots\doteq\frac{\pi}{2}\log 2,$$

from the result of problem 4483 [1953, 424].

Letting $\sqrt{1+\sin\theta}+\sqrt{\sin\theta}=e^x$ we have the required result.

Also solved by Leonard Carlitz, P. L. Chessin, and Chih-yi Wang.

Coefficients in the Taylor Expansion for a Rational Function

4574 [1954, 126]. *Proposed by H. S. Shapiro, New York University*

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be the power series of a rational function, and let the a_n be rational integers. If $a_n=0$ for $n=(k!+k)$, $k=1, 2, \cdots$, then f is a polynomial.

Solution by John B. Kelly, Michigan State College. According to a recent theorem of C. Lech (Arkiv für Matematik, vol. 2, 1953, pp. 417–421) the indices n for which $a_n=0$, belong, for n sufficiently large, to a finite set of residue classes modulo some integer, r, and every sufficiently large element n of any of these classes is such that $a_n=0$. But for any modulus r, $k!+k$ will lie in each residue class, modulo r, infinitely often, because when $k\geqq r$, $k!+k\equiv k \pmod r$. Hence, for n sufficiently large $a_n=0$; $f(z)$ is a polynomial. The condition that the coefficients be integers is superfluous.

Also solved by Leonard Carlitz.

A Diophantine System

4575 [1954, 126]. *Proposed by the late Joseph Rosenbaum, Hartford, Connecticut*

Find the complete solution in positive integers of the Diophantine system

$$2uv-xy=16,\qquad xv-uy=12.$$

Solution by C. R. Phelps, Rutgers University. Solving the given equations for x and u, we get

$$x=\frac{24v+16y}{2v^2-y^2}\qquad u=\frac{16v+12y}{2v^2-y^2}\cdot$$

Evidently $2v^2-y^2$ is positive. We find also

$$2(12)^2 - (16)^2 = 32 = (x^2 - 2u^2)(2v^2 - y^2),$$

whence $2v^2-y^2$ is a divisor of 32. Let $2v^2-y^2=2^s$, $0 \leqq s \leqq 5$.

(A) If s is odd, say $2k+1$, from $2v^2-y^2=2^{2k+1}$ by successive applications of divisibility by 2 we get

$$v = 2^k v_0, \qquad y = 2^{k+1} y_0, \qquad v_0^2 - 2y_0^2 = 1, \qquad v_0 \text{ odd.}$$

(B) If s is even, say $2k$, we similarly get

$$v = 2^k v_0, \qquad y = 2^k y_0, \qquad y_0^2 - 2v_0^2 = -1, \qquad y_0 \text{ odd.}$$

In either case all solutions v_0, y_0 are obtained from the solutions of the Pell equation $p^2-2q^2=\pm 1$, which are given by

$$2p_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n, \qquad 2\sqrt{2}q_n = (1+\sqrt{2})^n - (1-\sqrt{2})^n.$$

In case (A), v_0 and y_0 are given by p_n and q_n, respectively, for even values of n; and in case (B), v_0 and y_0 are given by q_n and p_n, respectively, for odd values of n. Since

$$(p_n + q_n\sqrt{2})(1+\sqrt{2})^2 = p_{n+2} + q_{n+2}\sqrt{2},$$

we have the recurrence formulas

$$p_{n+2} = 3p_n + 4q_n, \qquad q_{n+2} = 2p_n + 3q_n.$$

We note also that p_i and q_i are both positive if and only if $i \geqq 0$.

The complete solutions are then put together as follows:

$$(A)\ v = 2^k p_n, \quad y = 2^{k+1} q_n, \quad x = 2^{2-k} p_{n+2}, \quad u = 2^{2-k} q_{n+2}, \qquad n \text{ even}$$

$$(B)\ v = 2^k q_n, \quad y = 2^k p_n, \quad x = 2^{3-k} q_{n+2}, \quad u = 2^{2-k} p_{n+2}, \qquad n \text{ odd;}$$

where in both cases $k=0, 1, 2$.

The solutions in integers all less than 100 are:

n	k	x	u	v	y
1	0	40	28	1	1
	1	20	14	2	2
	2	10	7	4	4
2	0	68	48	3	4

n	k	x	u	v	y
2	1	34	24	6	8
	2	17	12	12	16
3	2	58	41	20	28
4	2	99	70	68	96

Also solved by H. D. Block, Monte Dernham, C. V. Gregg, Louisa S. Grinstein, Stephen Lichtenbaum and Joseph D'Atri, T. M. Little, D. C. B. Marsh, R. S. Underwood, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

COLLEGE TEXT BOOKS

Editorial Note. It has been felt for some time that as complete as possible a list of text books in the important fields of mathematics, taught in most colleges, should be made available. With this in mind, the MONTHLY has endeavored to collect such information, and list all the recent text books covering the fields of Intermediate Algebra, College Algebra, Trigonometry, Analytic Geometry, Unified Freshman Mathematics, Analytic Geometry and Calculus, Calculus, and Differential Equations, together with the author, publisher and the date published. Also the number of pages and price are included. References following the price are to critical reviews which have appeared in this MONTHLY. In some instances short paragraphs are included which describe unusual contents, features, or methods of presentation. These are not intended as critical reviews and the presence or absence of a paragraph has no relation to the quality of the book.

We have made every effort to prepare a complete list, and hope that this has been accomplished. We fear, however, that there may be omissions or errors, and apologize in advance for these. Moreover, we do not wish to recommend any particular books, but hope that the material will be of sufficient help to warrant the entire listing.

Except in a few cases, the books have been published since 1945. They have been classified by subject first, then by date of publication, and within any given year, alphabetically by author.

INTERMEDIATE ALGEBRA

Except where noted, these books assume one year of high school algebra.

Intermediate Algebra. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1947. 337 pages, $3.00. Review: vol. 55, p. 376 (1948).

This volume consists of the first twelve chapters of the authors' survey text plus four additional chapters.

Intermediate Algebra for Colleges. By E. B. Miller (Illinois College). The Ronald Press Company, 1947. 361 pages, $3.00. Review: vol. 55, p. 177 (1948).

Intermediate Algebra. By R. S. Underwood (Texas Technological College), T. R. Nelson (Texas A. and M. College), and S. Selby (University of Akron). The Macmillan Company, 1947. 283 pages, $3.00. Review: vol. 55, p. 517 (1948).

Intermediate Algebra for Colleges. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1948. 276 pages, $3.50. Review: vol. 56, p. 194 (1949).

Intermediate Algebra for Colleges. By P. R. Rider (Washington University). The Macmillan Company, 1949. 242 pages, \$3.00. Review: vol. 57, p. 494 (1950).

Intermediate College Algebra. By E. M. J. Pease. Prentice-Hall, Inc., 1950. 420 pages, \$3.50. Review: vol. 59, p. 192 (1952).

Elements of Algebra. By L. C. Peck (Ohio University). McGraw-Hill Book Company, 1950. 230 pages, \$3.25. Review: vol. 59, p. 192 (1952).

This text is written primarily for college students who have had no algebra in high school, and thus presupposes no mathematics background other than ordinary arithmetic.

Intermediate Algebra. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 295 pages, \$3.25.

This book, a second edition, consists of the first fifteen chapters of Brink's College Algebra, Second Edition.

Algebra for Commerce and Liberal Arts. By A. K. Bettinger (Creighton University) and W. A. Dwyer. Pitman Publishing Company, 1951. 225 pages, \$3.00. Review: vol. 59, p. 192 (1952).

Intermediate Algebra. By P. K. Rees (Louisiana State University) and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1951. 328 pages, \$3.50. Review: vol. 59, p. 192 (1952).

Intermediate Algebra for Colleges. By J. B. Rosenbach and E. A. Whitman (Carnegie Institute of Technology). Ginn and Company, 1951. 241 pages, \$3.00. Review: vol. 61, p. 274 (1954).

This book, designed for those students who have had a minimum of algebra, covers those topics usually completed in two years of high school algebra through progressions, but written at a college maturity level.

Beginning Algebra for College Students. By L. L. Lowenstein (Kent State University). John Wiley and Sons, Inc., 1953. 279 pages, \$3.50.

Geared for an introductory or terminal course, this introduces algebra through the use of arithmetic.

Intermediate Algebra for College Students (Revised edition). By T. S. Peterson, (Portland State Extension Center). Harper and Brothers, 1953. 369 pages, \$3.00.

Elements of Algebra. By V. B. Caris (Ohio State University). Ginn and Company, 1953. 307 pages, \$3.30. Review: vol. 61, p. 272 (1954).

Intermediate Algebra (Revised edition). By L. J. Adams (Santa Monica City College). Henry Holt and Company, 1954. 366 pages, \$3.40.

Algebra for College Students. By W. M. Whyburn (University of North Carolina) and P. H. Daus (University of California at Los Angeles). Prentice-Hall, Inc., 1955. 255 pages, $4.25.

COLLEGE ALGEBRA

College Algebra. By F. M. Morgan (Clark School). American Book Company, 1943. 374 pages, $3.00. Review: vol. 51, p. 41 (1944).

Brief College Algebra. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1947. 292 pages, $3.25.

College Algebra. By E. R. Heineman (Texas Technological College). The Macmillan Company, 1947. 359 pages, $3.50. Review: vol. 55, p. 437 (1948).

College Algebra. By F. S. Nowlan (University of British Columbia). McGraw-Hill Book Company, 1947. 371 pages, $4.00. Review: vol. 55, p. 374 (1948).

College Algebra. By T. S. Peterson (University of Oregon). Harper and Brothers, 1947. 334 pages, $3.00. Review: vol. 56, p. 49 (1949).

College Algebra. By Moses Richardson (Brooklyn College). Prentice-Hall, Inc., 1947. 472 pages, $3.95. Review: vol. 55, p. 107 (1948).

College Algebra. By Gordon Fuller (Texas Technological College). D. Van Nostrand Company, Inc., 1948. 255 pages, $3.25. Review: vol. 56, p. 281 (1949).

This text is for freshman college students who are to take a course in algebra composed of material of high school algebra and some chapters of a more advanced nature, usually classed as college algebra.

College Algebra. By L. M. Reagan (University of Wichita), E. R. Ott (Rutgers University), and D. T. Sigley (Johns Hopkins University). Rinehart and Company, 1948. 447 pages, $4.00.

This book departs from conventional college algebra texts in the order of topics and in the use of the inductive approach in developing new concepts.

College Algebra. By H. A. Simmons (Northwestern University). The Macmillan Company, 1948. 619 pages, $4.00. Review: vol. 56, p. 193 (1949).

College Algebra. By E. A. Cameron and E. T. Browne (both of University of North Carolina). Henry Holt and Company, 1949. 406 pages, $3.80. Review: vol. 57, p. 50 (1950).

College Algebra (Third edition). By J. B. Rosenbach and E. A. Whitman (both of Carnegie Institute of Technology). Ginn and Company, 1949. 565 pages, $3.50. Review: First edition, vol. 41, p. 258 (1934); Revised edition, vol. 56, p. 647 (1949).

The flexible organization of the third edition makes it adjustable to the

needs and abilities of different groups of students and to courses of various lengths and objectives.

College Algebra. By H. A. Bender (University of Rhode Island). Pitman Publishing Company, 1950. 452 pages, $3.75.

College Algebra. By E. B. Miller (Illinois College) and R. M. Thrall (University of Michigan). The Ronald Press, 1950. 493 pages, $4.25. Review: vol. 58, p. 200 (1951).

A textbook for first-year college students who propose to make a career of mathematics or of some science where a thorough knowledge of mathematics is indispensable.

Essentials of College Algebra. By J. B. Rosenbach and E. A. Whitman (Carnegie Institute of Technology). Ginn and Company, 1950. 352 pages, $3.00. Review: vol. 59, p. 192 (1952).

This textbook is based on, but shorter than, the authors' *College Algebra.*

College Algebra. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 495 pages, $4.00.

This book, a second edition, is designed for college students who are in need of a review of high-school higher algebra.

Algebra: College Course. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1951. 378 pages, $3.50.

This book is intended for college students who have had adequate instruction in high-school algebra and do not require a review of the subject.

College Algebra. By H. K. Fulmer and Walter Reynolds (Georgia Institute of Technology). Ginn and Company, 1951. 218 pages, $3.00. Review: vol. 59, p. 192 (1952)

This short (quarter or semester) course in algebra condenses, into as brief a text as possible, a review of the essentials of elementary algebra together with an adequate treatment of the theory of equations.

College Algebra. By H. L. Rietz, A. R. Crathorne, and J. W. Peters (University of Illinois). Henry Holt and Company, 1951. 387 pages, $3.50. Review: vol. 59, p. 192 (1952).

This contains new material on functional notation, graphs, and elementary algebra.

College Algebra. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1952. 344 pages, $3.50. Review: vol. 61, p. 273 (1954).

College Algebra. By R. H. Bardell and Abraham Spitzbart (University of Wisconsin, Milwaukee Extension). Addison-Wesley Publishing Company, Inc., 1953. 208 pages, $3.50. Review: vol. 61, p. 270 (1954).

College Algebra. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1953. 502 pages, $5.00. Review: vol. 61, p. 271 (1954).

College Algebra (Fourth edition). By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1953. 420 pages, $3.50.

Algebra for College Students. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1953. 394 pages, $3.75. Review: vol. 61, p. 274 (1954).

This text is identical with the author's *College Algebra* with the following exceptions: the first three chapters of *College Algebra* have been replaced by five chapters, covering this same elementary material at a more leisurely pace.

College Algebra. By H. G. Apostle (Grinnell College). Henry Holt and Company, 1954. 422 pages, $4.50. Review: vol. 62, p. 192 (1955).

Algebra for College Students. By J. R. Britton and L. C. Snively (University of Colorado). Rinehart and Company, 1954. 537 pages, $4.25. Review: vol. 62, p. 192 (1955).

College Algebra. By M. W. Keller (Purdue University). Houghton Mifflin Company, 1954. 471 pages, $4.90.

College Algebra (Third edition). By P. K. Rees (Louisiana State University) and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1954. 414 pages, $4.25. Review: vol. 62, p. 192 (1955).

Understanding College Algebra. By Samuel Selby (University of Akron), E. R. Smith (Iowa State College) and Murray Kleiman (City College of New York). The Dryden Press, 1954. 505 pages, $3.50. Review: vol. 62, p. 192 (1955).

College Algebra. By P. R. Rider (Washington University). The Macmillan Company, 1955 (in press).
This is a revised edition of the author's *College Algebra*, 1940.

TRIGONOMETRY

Plane and Spherical Trigonometry. By H. A. Simmons (Northwestern University). John Wiley and Sons, Inc., 1945. 387 pages, $3.75 with tables, $3.00 without tables. Review: vol. 53, p. 95 (1946).

A revision of Simmons and Gore's *Plane Trigonometry.* Gives applications to surveying, navigation and nautical astronomy, with appendixes on complex numbers and the slide rule.

Essentials of Plane and Spherical Trigonometry (Revised edition). By Clifford Bell (University of California at Los Angeles) and T. Y. Thomas (Indiana

University). Henry Holt and Company, 1946. 246 pages, $3.15 with tables, $2.60 without tables.

Trigonometry. By H. K. Hughes and G. T. Miller (both at Purdue University). John Wiley and Sons, Inc., 1946. 175 pages, $3.25 with tables, $2.75 without tables.

Plane Trigonometry. By W. K. Morrill (Johns Hopkins University). Rinehart and Company, 1946. 245 pages. $2.50.

Plane Trigonometry. By E. B. Mode (Boston University). Prentice-Hall, Inc., 1947. 214 pages, $2.95. Review: vol. 55, p. 176 (1948).

Plane and Spherical Trigonometry. By J. Shibli (Pennsylvania State College). Ginn and Company, 1949. 262 pages, $3.60. Review: vol. 58, p. 645 (1951).

This book contains practical applications including problems in surveying, engineering, physics, navigation, aviation, and astronomy.

Plane Trigonometry. By J. J. Corliss (University of Illinois) and W. V. Berglund (University of Illinois). Houghton Mifflin Company, 1950. 388 pages, $3.25. Review: vol. 58, p. 645 (1951).

Plane Trigonometry. By H. M. Dadourian (Trinity College). Addison-Wesley Publishing Company, Inc., 1950. 194 pages, $3.50.

This is a textbook for a brief course in trigonometry intended particularly for science and engineering students. Of special interest is the author's application of trigonometry to other topics, including an entire chapter devoted to applications to physics.

Plane Trigonometry. By Gordon Fuller (Alabama Polytechnic Institute). McGraw-Hill Book Company, 1950. 202 pages, $3.25 with tables, $2.75 without tables. Review: vol. 58, p. 645 (1951).

Plane Trigonometry. By E. R. Heineman (Texas Technological College). McGraw-Hill Book Company, 1950. 252 pages, $3.25 with tables, $2.75 without tables. Review: vol. 58, p. 645 (1951).

This is an alternate edition of the author's *Plane Trigonometry*, published in 1942, containing a new set of problems and exercises.

Plane and Spherical Trigonometry (Revised edition). By J. A. Northcott (Columbia University). Rinehart and Company, 1950. 234 pages, $3.50 with tables, $2.50 without tables.

Plane and Spherical Trigonometry (Fifth edition). By C. I. Palmer, C. W. Leigh, and Spofford Kimball (University of Maine). McGraw-Hill Book Company, 1950. 164 pages, $4.50 with tables, $3.25 without tables.

Plane and Spherical Trigonometry. By Moses Richardson (Brooklyn College). The Macmillan Company, 1950. 343 pages, $3.75 with tables, $3.40 without tables. Review: vol. 60, p. 720 (1953).

This text presents a full treatment of plane and spherical trigonometry adaptable to long or short courses with varying emphasis.

Plane and Spherical Trigonometry (Third edition). By H. L. Rietz, J. F. Reilly, and Roscoe Woods (University of Iowa). The Macmillan Company, 1950. 205 pages, $3.00 with tables, $2.75 without tables.

Essentials of Plane Trigonometry. By J. B. Rosenbach, E. A. Whitman and David Moskovitz (all of the Carnegie Institute of Technology). Ginn and Company, 1950. 183 pages, $3.00 with tables, $2.85 without tables. Review: vol. 58, p. 645 (1951).

Trigonometry. By C. T. Holmes (Bowdoin College). McGraw-Hill Book Company, 1951. 153 pages, $3.25 with tables, $2.75 without tables.

This text connects trigonometry more closely with other college mathematics. This is accomplished by introducing some very elementary concepts from analytic geometry and using them consistently in proofs throughout the book.

Plane and Spherical Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland (all of the United States Naval Academy). McGraw-Hill Book Company, 1951. 290 pages, $4.00 with tables, $3.25 without tables.

Plane Trigonometry. By L. M. Kells, W. F. Kern, and J. R. Bland (all of the United States Naval Academy). McGraw-Hill Book Company, 1951. 220 pages, $3.75 with tables, $3.25 without tables. Review: vol. 60, p. 720 (1953).

This textbook consists of those sections of the authors' *Plane and Spherical Trigonometry* which are restricted to plane trigonometry.

Plane and Spherical Trigonometry. By F. M. Morgan (Clark School). American Book Company, 1951. 324 pages, $3.25 with tables.

Brief Trigonometry. By E. A. Cameron (University of North Carolina). Henry Holt and Company, 1952. 153 pages, $2.45.

The material is presented so that it can be covered in fewer then thirty assignments.

Trigonometry, Plane and Spherical. By L. L. Smail (Lehigh University). McGraw-Hill Book Company, 1952. 324 pages, $4.00 with tables, $3.25 without tables. Review: vol. 60, p. 720 (1953).

Plane Trigonometry. By F. W. Sparks (Texas Technological College) and P. K. Rees (Louisiana State University). Prentice-Hall, Inc., 1952. 199 pages, $3.25.

Plane Trigonometry. By D. H. Ballou (Middlebury College), and F. H. Steen (Allegheny College). Ginn and Company, 1953. 160 pages, $3.25 with tables.

Plane and Spherical Trigonometry. By D. H. Ballou (Middlebury College), and F. H. Steen (Allegheny College). Ginn and Company, 1953. 218 pages, $3.50 with tables, $3.25 without tables.

This textbook includes the authors' *Plane Trigonometry* as well as additional material and is constructed for a maximum of self-study and self-drill.

Trigonometry. By J. F. Randolph (University of Rochester). The Macmillan Company, 1953. 220 pages, $3.00. Review: vol. 61, p. 656 (1954).

The body of this text is devoted to purely trigonometric concepts and their applications, but pertinent principles of analytic geometry and logarithms are included in appendices. A bridge between high school and college work is provided by another appendix containing a review of elementary algebra.

Plane Trigonometry. By P. R. Rider (Washington University). The Macmillan Company, 1953. 180 pages. $3.00. Review: vol. 61, p. 656 (1954).

This book consists of the plane trigonometry sections of the author's *First-Year Mathematics for Colleges*.

Plane Trigonometry. By A. W. Weeks and H. G. Funkhouser (Phillips Exeter Academy). D. Van Nostrand Company, Inc., 1953. 193 pages, $2.68. Review: vol. 61, p. 656 (1954).

Trigonometry. By W. L. Hart (University of Minnesota). D. C. Heath and Company, 1954. 368 pages, $3.75.

Trigonometry. By E. P. Vance (Oberlin College). Addison-Wesley Publishing Company, 1954. 158 pages, $3.00. Review: to appear in June–July (1955).

This textbook emphasizes analytic trigonometry rather than triangle solution, although both are included. The rectangular coordinate system is introduced immediately and used throughout the book.

Trigonometry. By Roy Dubisch (Fresno State College). The Ronald Press Company, 1955. 396 pages, $5.00. Review: to appear in June–July (1955).

This textbook defines the circular functions as functions of a real variable, by use of the unit circle. It presents a comprehensive treatment.

COLLEGE ALGEBRA AND TRIGONOMETRY

College Algebra and Trigonometry. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1945. 324 pages, $4.00. Review: vol. 52, p. 450 (1945).

This textbook for a basic integrated course is an attempt to give an integrated treatment of the two subjects on the college level and to lay a suitable

foundation for the study of more advanced mathematics. Throughout, algebra and trigonometry support and complement each other, and the techniques of both are applied continually.

College Algebra and Plane Trigonometry. By J. H. Zant, (Oklahoma Agricultural and Mechanical College). Ginn and Company, 1953. 387 pages, $4.00.

Throughout this book, which gives the student an integrated foundation in algebra and plane trigonometry, principles and skills from both fields are applied continually. The natural relations between them are brought out but the material is not combined when the integration would be artificial.

College Algebra and Plane Trigonometry. By R. H. Bardell and Abraham Spitzbart (both of University of Wisconsin, Milwaukee Extension). Addison-Wesley Publishing Company, Inc., March 1955.

This textbook which integrates Algebra and Plane Trigonometry, has avoided artificial integration for integration's sake. It is designed for either a one or a two semester course. The algebra portion is similar to that of the authors' *College Algebra.*

Unified Algebra and Trigonometry. By E. P. Vance (Oberlin College). Addison-Wesley Publishing Company, Inc., March, 1955.

This is a completely integrated textbook which emphasizes analytic trigonometry rather than triangle solution and has many illustrative examples from analytic geometry. It is designed primarily for students who plan to continue in mathematics or science and may be used for a one or two semester course.

ANALYTIC GEOMETRY

New Analytic Geometry. By P. F. Smith, A. S. Gale (University of Rochester), and J. H. Neelley (Carnegie Institute of Technology). Ginn and Company, 1945. 326 pages, $3.20.

Analytic Geometry. By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1945. 306 pages, $3.75. Review: vol. 53, p. 214 (1946).

Analytic Geometry. By F. D. Murnaghan (Johns Hopkins University). Prentice-Hall, Inc., 1946. 402 pages, $3.75. Review: vol. 53, p. 530 (1946).

Analytic Geometry. By F. S. Nowlan (University of British Columbia). McGraw-Hill Book Company, 1946. 355 pages, $3.75. Review: vol. 53, p. 529 (1946).

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton (Northwestern University). D. C. Heath and Company, 1947. 269 pages, $3.25. Review: vol. 56, p. 48 (1949).

Brief Analytic Geometry. By T. E. Mason and C. T. Hazard (Purdue University). Ginn and Company, 1947. 228 pages, $3.00.

This textbook is a revision of the authors' longer *Analytic Geometry*.

Analytic Geometry. By D. S. Nathan (College of the City of New York) and Olaf Helmer (Douglas Aircraft Co.). Prentice-Hall, Inc., 1947. 402 pages, $3.75. Review: vol. 55, p. 595 (1948).

Analytic Geometry. By P. R. Rider (Washington University). The Macmillan Company, 1947. 383 pages, $3.50. Review: vol. 56, p. 487 (1949).

Analytic Geometry. By R. S. Underwood (Texas Technological College) and F. W. Sparks (Texas Technological College). Houghton Mifflin Company, 1947. 255 pages, $3.25. Review: vol. 55, p. 597 (1948).

Analytic Geometry (Fourth edition). By C. E. Love (University of Michigan). The Macmillan Company, 1948. 306 pages, $3.50.

Analytic Geometry. By P. K. Rees (Louisiana State University) and E. D. Mouzon (Southern Methodist University). The Dryden Press, 1948. 305 pages, $2.75. Review: vol. 56, p. 488 (1949).

Analytic Geometry. By Roscoe Woods (University of Iowa). The Macmillan Company, 1948. 322 pages, $3.50.

A second edition, this book is designed for either a short course (first ten chapters) or a longer course. Its special feature is the representation of the quadric surfaces by means of photographs and wire models.

Analytic Geometry. By J. J. Corliss, I. K. Feinstein, and H. S. Levin (University of Illinois). Harper and Brothers, 1949. 370 pages, $3.25.

This text considers both plane and solid analytic geometry.

Analytic Geometry. By A. L. Nelson, K. W. Folley, and W. M. Borgman (Wayne University). The Ronald Press Company, 1949. 215 pages, $3.25. Review: vol. 57, p. 426 (1950).

Analytic Geometry. By L. M. Kells and H. C. Stotz (both of U. S. Naval Academy). Prentice-Hall, Inc., 1949. 288 pages, $3.50. Review: vol. 58, p. 201 (1951).

Analytic Geometry. By Robin Robinson (Dartmouth College). McGraw-Hill Book Company, 1949. 147 pages, $3.25. Review: vol. 57, p. 124 (1950).

Analytic Geometry (Revised edition). By C. H. Sisam (Colorado College). Henry Holt and Company, 1949. 304 pages, $3.40.

Analytic Geometry (Third edition). By W. A. Wilson and J. I. Tracey (Yale University). D. C. Heath and Company, 1949. 328 pages, $3.50.

Analytic Geometry. By R. D. Douglass and S. D. Zeldin (Massachusetts Institute of Technology). McGraw-Hill Book Company, 1950. 212 pages. $3.75.

Elements of Analytic Geometry. By W. L. Hart, (University of Minnesota). D. C. Heath and Company, 1950. 229 pages, $3.25. Review: vol. 58, p. 275 (1951).

Elements of Analytic Geometry. By C. E. Love (University of Michigan). The Macmillan Company, 1950. 218 pages, $3.00.

This text, similar to the author's *Analytic Geometry*, has been written in a similar fashion but designed for a briefer course.

Analytic Geometry. By J. W. Cell (North Carolina State College). John Wiley and Sons, Inc., 1951. 326 pages, $3.75.

This book contains numerous illustrations and problems in science and engineering.

Brief Course in Analytics. By M. A. Hill, Jr., and L. J. Burton (both of the University of North Carolina). Henry Holt and Company, 1951. 224 pages, $2.75.

Analytic Geometry. By Gordon Fuller (Texas Technological College). Addison-Wesley Publishing Company, Inc., 1954. 216 pages, $3.85. Review: vol. 62, p. 196 (1955).

An important feature is the introduction of vectors and their applications.

Analytic Geometry (Second edition). By E. S. Smith, Meyer Salkover, and H. K. Justice (University of Cincinnati). John Wiley and Sons, Inc., 1954. 306 pages, $4.00. Review: vol. 62, p. 196 (1955).

Analytic Geometry (Third edition). By F. H. Steen (Allegheny College) and D. H. Ballou (Middlebury College). Ginn and Company, 1955.

UNIFIED FRESHMAN TEXTS

An Introduction to Mathematical Analysis (Revised edition). By F. L. Griffin (Reed College). Houghton Mifflin Company, 1936. 546 pages, $4.50. Review: vol. 44, p. 172 (1937).

This is a thoroughly integrated treatment of calculus, analytic geometry, algebra, and trigonometry.

Introductory College Mathematics. By W. E. Milne (Oregon State College) and D. R. Davis (Montclair State Teachers College). Ginn and Company, 1941. 438 pages, $4.50.

This textbook correlates the essential topics of college algebra, trigonometry, analytic geometry, and some principles of the calculus. The early introduction of the simpler elements of the calculus is one of the methods of the correlation. An elementary treatment of finite differences and curve fitting is included.

Fundamentals of Mathematics. By Moses Richardson (Brooklyn College). The Macmillan Company, 1941. 525 pages, $5.00. Review: vol. 48, p. 472 (1941).

This textbook covers, in addition to the traditional subject matter, many topics which emphasize the broad scope of mathematical ideas. It begins with a logical treatment of the number system and presents other topics in this same manner. In addition, there are discussions of some of the simpler important problems of pure mathematics, many of them of the non-traditional type.

College Mathematics. By C. H. Sisam (Colorado College). Henry Holt and Company, 1946. 469 pages, $4.50. Review: vol. 54, p. 118 (1947).

This is a general introduction of unified college algebra, trigonometry, and analytic geometry.

Elementary Concepts of Mathematics. By B. W. Jones (University of Colorado). The Macmillan Company, 1947. 294 pages, $4.75. Review: vol. 55, p. 515 (1948).

This text is designed for students who have had a minimum of training in mathematics, who do not expect to specialize in mathematics or science, but who want a firmer grounding in what useful mathematics they have had, and such additional training as they may find interesting and useful in later life.

First Year College Mathematics with Applications. By P. H. Daus (University of California at Los Angeles) and W. M. Whyburn (University of North Carolina). The Macmillan Company, 1949. 495 pages, $5.00. Review: vol. 57, p. 124 (1950).

This book provides an ample background for the study of the calculus, and simultaneously, integrates the subjects of college algebra, analytic geometry, and analytic trigonometry. Applications taken from science and engineering illustrate each of the principles discussed.

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur, completely rewritten by C. V. Newsom (Associate Commissioner of Education, The State of New York). Rinehart and Company, 1949. 559 pages, $5.25. Review: vol. 57, p. 428 (1950).

Algebra, trigonometry, and analytical geometry are presented together as a tandem course and there is no repetition or overlapping of material.

First Year Mathematics for Colleges. By P. R. Rider (Washington University). The Macmillan Company, 1949. 714 pages, $5.00. Review: vol. 58, p. 53 (1951).

The topics are logically arranged and grouped about the function concept, yet the individual chapters are sufficiently independent for the teacher to adapt the book to his own sequence and needs.

Introduction to Mathematics. By H. R. Cooley, David Gans, Morris Kline and H. E. Wahlert (all of New York University). Houghton Mifflin Company, 1949. 710 pages, $5.00.

This book is intended for liberal arts students and presents a great variety of topics with a cultural rather than a technical approach.

Living Mathematics. By R. S. Underwood and F. W. Sparks (Texas Technological College). McGraw-Hill Book Company, 1949. 363 pages, $5.50.

This textbook is an elementary introduction up to but not including calculus. Part I contains the conventional material for a one-semester course in College Algebra. Part II provides a terminating course, as well as much interesting reference material.

Basic Mathematical Analysis. By H. G. Ayre (Western Illinois State College). McGraw-Hill Book Company, 1950. 584 pages, $6.00.

This textbook brings together coherently the essentials of algebra, trigonometry, analytic geometry and a minimum of the easier concepts of the calculus.

A First Course in Mathematics. By Pompey Mainardi, Carl Konove and Edward Baker (Newark College of Engineering). D. Van Nostrand Company, Inc., 1950. 380 pages, $5.85.

This third edition is designed for freshman students of engineering and the physical sciences with emphasis on the technique of applying mathematics to physical situations. It includes the usual material including three dimensional geometry with a chapter on derivatives and integrals.

Primer of College Mathematics. By J. F. Randolph (University of Rochester). The Macmillan Company, 1950. 545 pages, $5.00. Review: vol. 58, p. 350 (1951).

This text is a unification of college algebra, trigonometry, and analytic geometry with an introduction to calculus. Sufficient material is included to permit considerable selection even in a three-semester course. Even though the subjects are integrated, the book may be used in a fairly traditional presentation of the three elementary courses.

Elements of Mathematical Analysis. By S. E. Urner (Los Angeles City College) and W. B. Orange. Ginn and Company, 1950. 562 pages, $4.50. Review: vol. 58, p. 577 (1951).

In this textbook the simpler elements of the calculus are introduced at an early stage. Algebraic techniques are included, trigonometric material is complete with the main emphasis on the analytic side. Supplementary material includes discussions of such fundamental topics as limits, continuity, existence of a derivative, and infinity.

College Mathematics (Second edition). By W. W. Elliott (Duke University) and E. R. C. Miles (Johns Hopkins University). Prentice-Hall, Inc., 1951. 472 pages, $4.95.

This text is a non-integrated treatment of algebra, trigonometry, analytic geometry, and the elements of calculus.

General College Mathematics. By W. L. Ayres, C. G. Fry, and H. F. S. Jonah (Purdue University). McGraw-Hill Book Company, 1952. 280 pages, $3.75. Review: vol. 60, p. 486 (1953).

This textbook is a relatively simple book for freshman mathematics designed as a terminal course for scientific students. Important operations in arithmetic are reviewed, leading the student through the elements of algebra, trigonometry, and the elementary concepts of analytic geometry. The book concludes with a chapter on simple logic.

Basic Skills in Mathematics. By H. V. Price and L. A. Knowler (both of the State University of Iowa). Ginn and Company, 1952. 249 pages, $3.25.

This textbook is designed to provide basic training in mathematics to meet the demands of various programs and is planned particularly for students whose background is inadequate, and whose interests lie in other fields. The material covered is almost identical to that of the Commission on Post-war Plans of the National Council of Teachers of Mathematics.

Elementary Analysis. By K. O. May (Carleton College). John Wiley and Sons, Inc., 1952. 625 pages, $5.00. Review: vol. 59, p. 708 (1952).

Although the subject matter is traditional, and includes an introduction to calculus, the approach is quite modern with much time spent on basic ideas. Carefully drawn explanations of involved mathematical concepts are given and the book presents more material, more problems, and more functional information than is usually found in books of this type.

Fundamentals of College Mathematics. By R. E. Johnson (Smith College), N. H. McCoy (Smith College) and Anne F. O'Neill (Wheaton College). Rinehart and Company, 1953. 479 pages, $6.00. Review: vol. 61, p. 58 (1954).

This basic textbook for integrated courses in trigonometry, analytic geometry, and calculus, is designed primarily for liberal arts freshmen with at least three units of entrance mathematics. Emphasis is on the understanding of basic ideas rather than on the development of problem-solving techniques.

A First Year of College Mathematics. By R. W. Brink (University of Minnesota). Appleton-Century-Crofts, Inc., 1954. 725 pages, $5.00.

This standard textbook has been reexamined and rewritten and presents a

complete and integrated course in the material of college algebra, trigonometry, and analytic geometry.

Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree (both of the University of Oklahoma). Henry Holt and Company, 1954. 620 pages, $5.90. Review: vol. 62, p. 193 (1955).

This textbook for college freshmen includes college algebra, trigonometry, and analytic geometry (which are carefully integrated) as well as an introduction to differential and integral calculus and to mathematical statistics.

Introductory College Mathematics. By C. G. Jaeger (Pomona College) and H. M. Bacon (Stanford University). Harper and Brothers, 1954. 377 pages, $4.75.

This textbook is designed for use on the elementary level, and provides a unified treatment of basic topics in algebra, trigonometry, and analytic geometry together with an introduction to the calculus. The unifying element throughout the book is the function concept. It is intended for use in a full year course, and presupposes one year each of high school algebra and geometry.

Introductory College Mathematics. By Adele Leonhardy (Stephens College). John Wiley and Sons, Inc., 1954. 459 pages, $4.90.

The material in this text is organized around the nature of mathematics and its role in modern civilization and is designed to meet the requirements of a broad group of readers: those interested in a terminal work, those whose main field of interest is in education, humanities, or the social sciences, and those who wish a broader treatment than is ordinarily offered in introductory works.

An Introduction to College Mathematics. By C. V. Newsom (Associate Commissioner of Education, State of New York) and Howard Eves (University of Maine). Prentice-Hall, Inc., 1954. 416 pages, $5.75. Review: vol. 61, p. 724 (1954).

This is a revised edition by Eves of Newsom's textbook which is designed for a terminal course in first year college mathematics. The revision has introduced certain historical material and incorporated a chapter which introduces the idea of calculus.

Principles of Mathematics. By C. B. Allendoerfer (University of Washington), C. O. Oakley (Haverford College). McGraw-Hill Book Company, 1955. 450 pages, in press.

Basic Mathematics for General Education (Second edition). By H. C. Trimble and L. C. Peck (both of Iowa State Teachers College) and F. C. Bolser. Prentice-Hall, Inc., 1955. 336 pages, in press.

This book is an elementary introduction to mathematics for the general student. It emphasizes ideas rather than techniques.

ANALYTIC GEOMETRY AND CALCULUS

Analytic Geometry and Calculus. By F. S. Woods and F. H. Bailey. Ginn and Company, 1938. 524 pages, $5.00.

This book opens with the analytic geometry of two dimensions. The geometry of three dimensions is treated later when it is required for the study of functions of two variables. The calculus is introduced early through the discussion of slope and area and from this point on is intermingled with analytic geometry.

Analytic Geometry and Calculus. By H. B. Phillips (The Massachusetts Institute of Technology). John Wiley and Sons, Inc., 1946. 504 pages, $5.50. Review: vol. 54, p. 56 (1947).

This textbook treats analytic geometry and calculus in the form and order in which they are required for courses in science and engineering. It covers the essentials but does not burden the student with a mass of detail.

Analytic Geometry and Calculus. By J. F. Randolph (University of Rochester) and Mark Kac (Cornell University). The Macmillan Company, 1946. 642 pages, $5.25. Review: vol. 54, p. 293 (1947).

In this textbook analytic geometry and calculus are treated together in such a way that each complements the other. Strong emphasis is placed on the functional notation. Integration is introduced early and in the first chapters a review is given of some of the fundamental algebraic notions, such as inequalities and absolute values.

Introduction to Analytic Geometry and the Calculus. By H. M. Dadourian (Trinity College). The Ronald Press Company, 1949. 246 pages, $3.50. Review: vol. 57, p. 354 (1950).

Avoiding the formal manner of presentation, this textbook is aimed at presenting the elements of analytic geometry and the calculus in order to make these subjects intelligible to the beginner. New concepts are introduced through an appeal to the student's experience. Clear and concise directions are given in solving problems.

Analytic Geometry and Calculus: A Unified Treatment. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1949. 658 pages, $5.50.

This textbook is a correlated study of analytic geometry and calculus designed for engineering or science students who desire a unified treatment entailing the early introduction of differential and integral calculus.

Analytical Geometry and Calculus. By H. J. Gay. Edited by R. K. Morley (Worcester Polytechnic Institute). McGraw-Hill Book Company, 1950. 524 pages, $5.00. Review: vol. 58, p. 577 (1951).

This textbook is intended to cover the elements of analytical geometry and calculus in an order of topics that will introduce the basic ideas of calculus fairly early, but not until the student has had enough work with coordinates to make the graphs of the common curves familiar, so that they may be used readily for illustrative purposes in calculus.

Calculus and Analytic Geometry. By C. T. Holmes (Bowdoin College). McGraw-Hill Book Company, 1950. 416 pages, $5.00. Review: vol. 58, p. 433 (1951).

This textbook is planned for a combination course in which the concepts and techniques of the calculus are the main objectives. The analytic geometry material is closely interwoven with calculus throughout the first half of the book, and the concept of integration is introduced early. The text is suitable for either arts and science students or engineering students.

Analytic Geometry and Calculus. By L. M. Kells (United States Naval Academy). Prentice-Hall, Inc., 1950. 608 pages, $5.50. Review: vol. 58, p. 577 (1951).

This textbook unifies the analytic geometry and calculus and covers all the ordinary material.

Analytic Geometry and Calculus. By W. R. Longley, P. F. Smith and W. A. Wilson (all of Yale University). Ginn and Company, 1951. 598 pages, $5.00.

This textbook combines analytic geometry and calculus including one chapter on differential equations, so that a student may be introduced to the calculus before completing analytic geometry.

Calculus and Analytic Geometry. By G. B. Thomas, Jr. (Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1953. 731 pages, $7.50. Review: vol. 61, p. 435 (1954).

In this textbook on analytic geometry and calculus, designed primarily for students of science and engineering, integration is introduced early, with many applications to geometry and mechanics. In addition to standard topics, this book contains discussions of determinants and hyperbolic functions, introduction to vector analysis, infinite series (including Taylor and Fourier Series) and the theory of complex variables, including the Cauchy-Riemann differential equations.

Analytic Geometry and Calculus. By L. L. Smail (Lehigh University). Appleton-Century-Crofts, Inc., 1953. 714 pages, $5.50. Review: vol. 61, p. 435 (1954).

After the basic ideas of analytic geometry appear, the fundamental concepts and processes of calculus are introduced. After considerable discussion on differentiation with simple applications, the indefinite integral and the definite integral with simple applications, the conic sections are introduced. More complicated functions follow, and all ordinary topics such as solid analytic geometry,

partial derivatives and multiple integrals and series are considered. The last chapter deals with elementary differential equations.

Introductory Calculus with Analytic Geometry. By E. G. Begle (Yale University). Henry Holt and Company, 1954. 301 pages, $4.50. Review: vol. 62, p. 54 (1955).

This textbook, in a completely rigorous fashion, covers the differentiation and integration of algebraic, logarithmic, exponential, and trigonometric functions, and includes the elements of plane analytic geometry through conics. It includes much more theory than most calculus textbooks contain.

CALCULUS

Elements of Calculus. By W. A. Granville (Gettysburg College), P. F. Smith, and W. R. Longley (both of Yale University). Ginn and Company, 1946. 549 pages, $4.75. Review: vol. 53, p. 332 (1946).

This extremely well-known calculus is arranged so that the applications of the differential and integral calculus can be presented early in the course.

Differential and Integral Calculus (Second edition). By R. R. Middlemiss (Washington University). McGraw-Hill Book Company, 1946. 497 pages, $4.75.

Calculus. By F. H. Miller (The Cooper Union School of Engineering). John Wiley and Sons, Inc., 1946. 416 pages, $4.50. Review: vol. 54, p. 176 (1947).

Calculus (Revised edition). By A. L. Nelson, K. W. Folley and W. M. Borgman (all of Wayne University). D. C. Heath and Company, 1946. 386 pages, $4.00.

Unified Calculus. By E. S. Smith, Meyer Salkover and H. K. Justice (all of the University of Cincinnati). John Wiley and Sons, Inc., 1947. 507 pages, $4.75. Review: vol. 56, p. 562 (1949).

This textbook gives particular attention to the subject as a whole with great emphasis on the practical applications of calculus to both physics and mechanics. Although it is designed primarily for engineering students, it may be used by liberal arts students as well.

Differential and Integral Calculus. By F. M. Morgan (Director of Clark School, Hanover, N. H.). American Book Company, 1948. 400 pages, $3.25.

This textbook is designed for an elementary course and is characterized by its brevity and non-technical approach. The simpler parts of integration are introduced early so as to assist students who are taking science courses simultaneously with the calculus.

Calculus. By L. M. Kells (U. S. Naval Academy). Prentice-Hall, Inc., 1949. 508 pages, $4.75.

Calculus. By L. L. Smail (Lehigh University). Appleton-Century-Crofts, Inc., 1949. 592 pages, $5.00. Review: vol. 57, p. 427 (1950).

The aim of the textbook is to give due emphasis to the meaning of fundamental concepts, as well as to teach the technique of differentiation and integration and the methods of applying this technique. Much attention has been given to the careful formulation of fundamental definitions and theorems. Integration has been introduced early.

Elements of Calculus. By T. S. Peterson (University of Oregon). Harper and Brothers, 1950. 369 pages, $4.50.

Fundamentals of the Calculus. By D. E. Richmond (Williams College). McGraw-Hill Book Company, 1950. 233 pages, $3.75. Review: vol. 58, p. 431 (1951).

This brief textbook, designed for freshman courses, was written to provide sufficient knowledge of the calculus to furnish an adequate background for courses in physics, and to enable the liberal arts student who elects one year of college mathematics to acquire some feeling for mathematical thinking by presenting the material with as much attention to logical clarity as possible. Great emphasis is placed throughout upon fundamental principles, with problems especially designed to contribute to understanding.

Calculus. By Tomlinson Fort (University of Georgia). D. C. Heath and Company, 1951. 576 pages, $5.00. Review: vol. 60, p. 134 (1953).

This textbook, which covers the usual material, differs from others in that it introduces the idea of limit through the use of infinite series. This novel method is introduced in the first chapter, and is used throughout the book.

Calculus. By J. V. McKelvey (Iowa State College). The Macmillan Company, 1951. 405 pages, $4.50. Review: vol. 60, p. 134 (1953).

This textbook, which presents the differential and integral calculus with its geometrical, physical and mechanical applications, emphasizes the rate of change principle in the study of the derivative and stresses the summation definition for the definite integral.

Elements of the Differential and Integral Calculus. By W. A. Granville (Pennsylvania College). Ginn and Company, 1952. 463 pages, $4.00. Review: vol. 18, p. 170 (1911).

This is the 1911 edition of the "original Granville," long the classic college text in calculus, reissued with a foreword by J. W. Lasley, Jr.

Calculus. By J. F. Randolph (University of Rochester). The Macmillan Company, 1952. 483 pages, $6.00. Review: vol. 60, p. 134 (1953).

This is a flexible, well-proportioned treatment of the theory and applications of calculus, and the text maintains a balance between formal problem material

and general concepts. In addition to the standard material, it presents many proofs and several more advanced topics which include Law of Mass Action, Line Integrals, Graphical Integration, and Green's Theorem.

Calculus. By A. H. Sprague (Amherst College). The Ronald Press Company, 1952. 576 pages, \$6.50. Review: vol. 60, p. 134 (1953).

This textbook constitutes a logically complete course in the calculus, with the addition of a chapter on polar coordinates and an extensive chapter on solid analytic geometry. The treatment is rigorous, but analytic proofs are accompanied by detailed explanations. Intuitive geometric illustrations pave the way for such proofs.

Differential and Integral Calculus. By Philip Franklin (Massachusetts Institute of Technology). McGraw-Hill Book Company, 1953. 638 pages, \$6.00. Review: vol. 62, p. 56 (1955).

This is a complete basic textbook for the full-year course for engineers and science majors. It is so designed that a sequence of material appears for those needing a review of trigonometry and analytic geometry as well as a sequence for those not needing this review.

Calculus. By G. B. Thomas, Jr. (Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1953. 614 pages, \$6.50. Review: vol. 61, p. 575 (1954).

This textbook for a first course in calculus is designed for students of science and engineering. Integration is introduced at any early stage, and after this every differentiation formula is related to the corresponding integration formula. Thus, students taking a concurrent physics course are able to make early use of this valuable tool.

Calculus. By T. L. Wade (Florida State University). Ginn and Company, 1953. 506 pages, \$4.75.

This is a textbook which presents calculus for students with the usual background in trigonometry and analytic geometry. Particular attention is given to the recall of prerequisite mathematics courses as the need arises. The flexibility of arrangement enables the course to be taught with varying degrees of formalism.

Calculus. By C. R. Wylie, Jr. (University of Utah). McGraw-Hill Book Company, 1953. 565 pages, \$6.00. Review: vol. 60, p. 722 (1953).

This textbook covers the usual material and is so organized that, by making a judicious choice from among the many worked examples, an instructor can equally well meet the needs and interests of a class of mathematics majors or a class of engineering students.

First Course in Calculus. By H. R. Cooley (Washington Square College, New York University). John Wiley and Sons, Inc., 1954. 643 pages, $6.00. Review: vol. 62, p. 52 (1955).

In this textbook, the emphasis is less on formal proofs than on a clear understanding of the basic concepts. Manipulations are treated thoroughly but only as an efficient means of carrying out the reasoning.

Differential and Integral Calculus. By C. E. Love and E. D. Rainville (both of University of Michigan). The Macmillan Company, 1954. 526 pages. $5.75. Review: to appear in June–July (1955).

This textbook is a complete revision by Rainville of Love's well-known textbook. In addition to the previous material, there is a short appendix containing a rigorous presentation of limits.

Calculus—An Introduction to Analysis and a Tool for the Sciences. By G. M. Merriman (University of Cincinnati). Henry Holt and Company, 1954. 617 pages, $6.50. Review: to appear in June–July (1955).

This textbook includes a careful explanation of the principles and hypotheses of calculus, and proceeds gradually from an intuitive to a rigorous discussion of theorems, hypotheses, and their applications before offering proof.

Calculus (Third edition). By G. F. Sherwood and A. E. Taylor (both at University of California at Los Angeles). Prentice-Hall, Inc., 1954. 624 pages, $5.75. Review: to appear in June–July (1955).

Although many sections have been completely revised, this edition follows the same pattern as the earlier ones. Integration is introduced early and the book is suitable for either arts and science students or engineers.

Differential and Integral Calculus (Second edition). By H. M. Bacon (Stanford University). McGraw-Hill Book Company, 1955. 559 pages, $6.00.

DIFFERENTIAL EQUATIONS

Differential Equations. By R. P. Agnew (Cornell University). McGraw-Hill Book Company, 1942. 341 pages, $4.50. Review: vol. 50, p. 323 (1943).

This textbook may be used, not only for classroom work for good students, but as a reference book from which all teachers may profit.

A Short Course in Elementary Differential Equations. By E. D. Rainville (University of Michigan). The Macmillan Company, 1949. 210 pages, $3.00. Review: vol. 57, p. 694 (1950).

Designed to furnish an introduction to differential equations, this textbook places great emphasis on the development and execution of methods. The omission of infinite series methods has made possible a thorough treatment of topics essential to a first course.

Differential Equations. By H. W. Reddick (New York University). John Wiley and Sons, Inc., 1949. 288 pages, $3.75.

This textbook treats methods of solving ordinary differential equations and problems in applied mathematics involving ordinary differential equations. It includes a chapter on the linear equation of second order, additional material in hyperbolic functions, systems of curves, and vibratory motion.

Elements of Ordinary Differential Equations. By Michael Golomb and M. E. Shanks (both of Purdue University). McGraw-Hill Book Company, 1950. 352 pages, $4.50. Review: vol. 59, p. 50 (1952).

This text is planned for the standard course as it is offered to engineering students, although it also fits many courses for non-engineering students. The subject is presented in such a way as to stimulate the student's imagination and at the time to inculcate correct mathematical thinking. Many techniques not usually found are included.

Differential Equations. By J. E. Powell and C. P. Wells (both of Michigan State College). Ginn and Company, 1950. 205 pages, $3.00. Review: vol. 59, p. 50 (1952).

This textbook is designed for a brief first course in methods of solving elementary differential equations for students majoring in engineering, science, and mathematics. Series solutions are given more than the usual amount of attention.

Differential Equations (Third edition). By H. B. Phillips (The Massachusetts Institute of Technology). John Wiley and Sons, Inc., 1951. 149 pages, $3.00. Review: vol. 59, p. 50 (1952).

An Introduction to the Theory of Differential Equations. By Walter Leighton (Carnegie Institute of Technology). McGraw-Hill Book Company, 1952. 174 pages, $3.75. Review: vol. 60, p. 201 (1953).

An unusually rigorous presentation, designed for pure mathematicians and college students who are majoring in mathematics, this text covers the usual material given in a first course but with exceptional attention to the underlying theory. Emphasis is upon the fundamental existence theorem which is used as a central idea.

Differential Equations. By A. L. Nelson, K. W. Folley, and Max Coral (all of Wayne University). D. C. Heath and Company, 1952. 309 pages, $4.00.

Elementary Differential Equations. By E. D. Rainville (University of Michigan). The Macmillan Company, 1952. 392 pages, $5.00. Review: vol. 60, p. 58 (1953).

This complete introduction to elementary differential equations contains all the material in Rainville's *Short Course*, and in addition, includes such material

as power series methods, and an extensive introduction to Fourier Series and the solution of boundary value problems in partial differential equations.

Differential Equations. By R. C. Yates (United States Military Academy). McGraw-Hill Book Company, 1952. 212 pages, \$3.75. Review: vol. 60, p. 202 (1953).

In addition to the usual material, this text includes numerical solutions and solutions by series, an account of Fourier Series, and certain simple boundary value problems.

Differential Equations (Third edition). By Max Morris and O. E. Brown (Case Institute of Technology). Prentice-Hall, Inc., 1952. 361 pages, \$4.50.

Elementary Differential Equations. By L. M. Kells (United States Naval Academy). McGraw-Hill Book Company, 1953. 266 pages, \$4.00.

Combining theory and applications, this revision deals with the most important types of differential equations. After each type, the author shows interesting and important applications of this type to geometry, electricity, mechanics, physics, and science generally.

Differential Equations with Applications. By Herman Betz, P. B. Burcham, and G. M. Ewing (all of the University of Missouri). Harper and Brothers, 1954. 310 pages, \$4.50. Review: vol. 62, p. 130 (1955).

This textbook, in addition to the standard material, includes applications to the physical, biological and engineering sciences not merely as illustrations, but as an integral part of the text. It also includes a thorough treatment of classical operational methods and an introduction to modern operational methods and to non-linear problems.

A First Course in Ordinary Differential Equations. By R. E. Langer (University of Wisconsin). John Wiley and Sons, Inc., 1954. 249 pages, \$4.50. Review: vol. 61, p. 437 (1954).

This book is a complete presentation which treats the subject from the standpoint of its mathematical interest and its utility as a technological tool. Emphasis, throughout, is on equations of the first and second order. The book uses an approach which rests solidly upon reason, playing down the necessity for memorization or special tricks.

Differential Equations in Engineering Problems. By M. G. Salvadori and R. J. Schwarz (Columbia University). Prentice-Hall, Inc., 1954. 455 pages, \$6.50.

Differential Equations. By Lester Ford (Illinois Institute of Technology). McGraw-Hill Book Company, 1955. 263 pages, \$4.75.

This new edition will appear this spring.

Elementary Differential Equations. By W. T. Martin and Eric Reissner (both of

Massachusetts Institute of Technology). Addison-Wesley Publishing Company, Inc., 1955.

This textbook, to be published this fall, stresses two important separate aspects of the study of differential equations: (1) the formulation of problems in science and engineering as problems of differential equations, and (2) the systematic study of methods of solutions for differential equations arising in applications. An added feature of the book is a chapter on numerical analysis.

Differential Equations. By F. H. Steen (Allegheny College). Ginn and Company, 1955.

This new text, to be published in 1955, will cover all the traditional material. It will be especially designed for a maximum of self-study.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

UNIVERSITY OF OKLAHOMA MATHEMATICS LETTER

The *University of Oklahoma Mathematics Letter* is a four-page publication of interest to high school and beginning college students and teachers. Copies will be sent without charge to persons requesting them and enclosing a stamped, self addressed envelope plus a $3'' \times 5''$ card giving their name and school address. Send all requests to Professor R. V. Andree, Department of Mathematics, The University of Oklahoma, Norman, Oklahoma. You owe it to your students to send for this interesting free publication.

SUMMER INSTITUTES FOR MATHEMATICS TEACHERS

A Summer Institute for Mathematics Teachers will be held in air-conditioned classrooms at the University of Oklahoma, Norman, Oklahoma, from June 6 to June 17, 1955. Teachers may attend either week (one hour credit) or both weeks (two hours credit). The fee is $15 for one week or $25 for two weeks—either credit or non-credit. Official certificates of attendance will be issued. Leaders in the teaching of elementary, junior high school, high school, and beginning college mathematics will be on hand to assist teachers with individual problems related to both classroom procedures and subject matter. Material for the interest and

enrichment of standard courses will be provided. For further information write to F. L. Hayden, Short Courses and Conferences, the University of Oklahoma, Norman, Oklahoma.

The seventh Institute for Teachers and Professors of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Middlebury College, Middlebury, Vermont, August 18–25, 1955. Professor J. G. Bowker is General Chairman. For further information write: Miss Harriett Howard, Publicity Chairman, The Ethel Walker School, Simsbury, Connecticut.

SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY

The first part of the Third Berkeley Symposium on Mathematical Statistics and Probability was held at the Statistical Laboratory of the University of California, Berkeley, during the last week of December, 1954. Planned as a part of the 121st Annual Meeting of the AAAS, it emphasized applications of statistics and probability. Thus, there were sessions given especially to problems of astronomy, biology, statistical mechanics, industrial research, psychology, and problems of health. One special session was assigned to philosophy of science. Also, there were two sessions given to the theory of statistics.

The second part of the Third Symposium will be held during the forthcoming summer. The second part of the Symposium is planned as a prolonged seminar, extending over July and August, 1955. The central subject of studies will be stochastic processes. The guest speakers include: T. W. Anderson, Columbia University; M. S. Bartlett, University of Manchester; J. Berkson, Mayo Clinic; A. J. L. Blanc-Lapierre, University of Algiers; D. Blackwell, Howard University; J. L. Doob, University of Illinois; W. Feller, Princeton University; R. Fortet, University of Paris; A. Girshick, Stanford University; J. M. Hammersley, Oxford University; W. Hoeffding, University of North Carolina; P. Levy, Paris École Polytechnique; H. Robbins, Columbia University; C. M. Stein, Stanford University.

Because of this event, the summer session program at the Statistical Laboratory will be limited solely to the usual undergraduate courses. The meetings of the Symposium will be open to graduate students and to visiting scholars. Details of the Symposium program may be had by writing to the Statistical Laboratory, University of California, Berkeley 4, California. It is hoped that the *Proceedings of the Third Berkeley Symposium* will be published early in 1956.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1955.

Boston University. May 31 to July 9: Professor Johanson, vector analysis; Professor Scheid, methods of applied mathematics. July 11 to August 20: Dr. Ross, differential equations; Professor Brown, infinite series.

The Catholic University of America. June 27 to August 6: Professor Moller,

higher algebra; Professor Wiegmann, introduction to matrix theory; Professor Taam, advanced calculus I; Professor Rice, advanced calculus II; Professor Finan, basic concepts; Professor Ramler, analytic projective geometry, college geometry, differential equations.

Columbia University, Teachers College. July 5 to August 12: Professor Fehr, teaching arithmetic in the elementary school, applications of geometry; Professors Fehr and Rosskopf, research and departmental seminar in teaching mathematics; Professor Rosskopf, teaching of geometry, foundations of geometry for teachers; Mr. Rourke, business arithmetic in mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Kays, materials and models in mathematics education; Mr. Sobel, supervision and teaching of mathematics in the junior high school.

DePaul University. June 24 to August 4: Professor DeCicco, functions of real variables, principles of geometry; Professor Caton, solid analytic geometry. June 10 to August 3 (evenings): Professor Caton, functions of a complex variable.

George Washington University. June 20 to August 15: Professor Taylor, introduction to boundary value problems, tensor analysis.

Indiana University. June 15 to August 12: Professor Wolfe, theory and application of statistics, non-euclidean geometry; Visiting Professor Polley (Wabash College), history of mathematics, topics for mathematics teachers; Professor MacKenzie, differential equations, algebra and elementary number theory.

Massachusetts Institute of Technology. June 20 to July 1: Professor Hildebrand, numerical analysis.

Michigan State College. June 21 to July 29: Professor Arnold, analysis of variance; Professor J. B. Kelly, theory of equations; Mr. Laurent, correlation analysis; Professor Nordhaus, complex variable I; Professor Olkin, mathematical analysis for the social sciences, factor analysis; Professor Powell, complex variable III; Professor Stewart, college geometry, theory of matrices and groups. June 21 to August 19: Professor Herzog, advanced calculus I, real variable; Professor L. M. Kelly, foundations of mathematics, introductory topology; Mr. Laurent, statistical methods in engineering; Professor Stelson, differential equations, mathematics of engineering; Professor Wells, vector analysis, numerical methods in partial differential equations.

Northwestern University. June 18 to August 13: differential equations, non-euclidean geometry, vector analysis and differential geometry, probability, introduction to the theory of numbers, the continuum, introduction to the theory of groups, engineering mathematics III.

Syracuse University. July 5 to August 13: Professor Baum, introduction to probability; Professor Hemmingsen, intermediate course in algebra; Professor Gelbart, fundamentals of analysis; Professor Exner, heuristic methods in elementary geometry, analysis of elementary mathematics. August 15 to September 16: Dr. Leger, higher mathematics for engineers and scientists I.

University of Buffalo. July 5 to August 12: Professor Gehman, foundations of mathematics; Professor Montague, history of mathematics; Professor Schneckenburger, theory of sets.

University of California. June 20 to July 30: Visiting Professor Steenrod, introduction to algebraic topology.

University of Chicago. June 27 to September 2: A special program will be offered in abstract analysis, including seminars and the following courses: Professor Lashof, introduction to topological algebra; Professor Kaplansky, rings of operators; Visiting Professor Mackey, representation of locally compact groups; Professor Segal, mathematics foundations of quantum mechanics; Professor Halmos, ergodic theory. Other advanced courses are: Professor Chern, differential geometry; Professor Halmos, introduction to modern algebra; Professor Lashof, linear algebra; Professor MacLane, multilinear algebra; Professor Chern, topology and differential geometry of one and two dimensions; Professor Graves, theory of functions of a complex variable.

University of Colorado. June 17 to July 22 and July 25 to August 26: Visiting Professor Hirsch, fundamental ideas in mathematics; Professor Thron, modern algebra; Professor Britton, vector analysis, operational calculus; Professor Hutchinson, functions of a complex variable, partial differential equations.

University of Florida. June 17 to August 13. Graduate Courses in Mathematics: Professor Moore, introduction to mathematical thought; Professor Hadlock, advanced topics in calculus; Professor Rohde, vector analysis; Professor Kokomoor, synthetic projective geometry; Professor Hutcherson, foundations of geometry, differential geometry; Professor Pirenian, Fourier series; Professor Smith, functions of a complex variable; Professor Gager, history of elementary mathematics; Professor South, calculus of variations.

Graduate Courses in Statistics: Professor Marshall, research methods; Professor Harshbarger, design of experiments; Professor Nicholson, theory of probability and theory of sampling; Professor Duncan, advanced statistics; Professors Burrows, Smith and Anderson, special topics; Professor South, advanced topics in calculus; Professor Rulon, educational statistics; Professor Meyer, recent advances in statistics.

The second cooperative Summer Session in Statistics, sponsored by the University of Florida, North Carolina State College, Virginia Polytechnic Institute and the Southern Regional Education Board will be held at the University of Florida from June 20 to July 29. (See this MONTHLY, vol. 61, 1954, p. 65). Inquiries should be addressed to Professor H. A. Meyer, Statistical Laboratory, University of Florida, Gainesville, Florida.

University of Illinois. June 20 to August 13: Professor Day, theory of fields; Professor Ketchum, functions of real variables; Professor Landin, elementary geometry from a modern viewpoint.

University of Michigan. June 20 to August 12: Professor Bott, calculus of variations in the large, functions of a complex variable with applications; Professor Copeland, probability, theory of games; Professor Craig, mathematical

statistics, estimation and significance tests; Professor Hay, advanced mechanics, operational mathematics; Professor Jones, history of algebra, applications of mathematics for teachers; Professor Kaplan, nonlinear differential equations; Professor Leisenring, non-euclidean geometry; Professor LeVeque, advanced calculus; Professor Lyndon, higher algebra; Professor McLaughlin, matrix theory; Professor Nesbitt, mathematics of life insurance, finite differences; Professor Rainich, relativity, vector analysis; Professor Rainville, intermediate differential equations; Professor Rothe, topics in mathematical physics, Fourier series; Professor Samelson, functions of real variables, differential geometry; Professor Young, foundations of mathematics, unified topology.

University of Minnesota, Institute of Technology. June 13 to July 16: Professor Munro, intermediate calculus; Professor Turrittin, variational problems in engineering; Professor Wilcox, vector analysis. July 18 to August 20: Professor Thompson, advanced calculus.

University of Nebraska. June 16 to August 5: Professor Basoco, differential equations; Professor Camp, theory of equations; Professor Jackson, Laplace transforms.

University of North Carolina. June 9 to July 16: Professor Hoyle, elementary algebra from an advanced viewpoint; Professor Winsor, introduction to higher geometry; Professor Garner, history of mathematics; Professor Lasley, analytic geometry from a higher standpoint; Professor Linker, differential equations; Professor Brauer, some recent results in algebra. July 18 to August 24: Professor Hill, elementary mathematical statistics; Professor Cameron, fundamental concepts; Professor Mackie, theory of equations; Professor MacNerny, topics in analysis; Professor Jones, foundations of geometry.

University of Oklahoma. June 10 to August 5: Professor Pan, differential equations; Professor Springer, solid analytical geometry; Professor Bernhart, vector analysis; Professor LaFon, ordinary and partial differential equations; Professor Brixey, theory of groups; Professor Goffman, topics in infinite series.

University of South Carolina. June 10 to August 13: Professor Hedberg, theory of equations, theory of numbers; Professor Novak, introduction to mathematical statistics, college geometry; Professor Williams, advanced calculus, vector analysis.

University of Washington. June 20 to August 19: Professor Kokoris, linear algebra; Professor Jerbert, differential equations; Professor Ballantine, advanced calculus and vector analysis; Professors McFarlan and Cramlet, topics in applied analysis; Professors Dekker and Kingston, advanced euclidean geometry; Professor Beaumont, foundations of mathematics.

University of Wyoming. June 13 to July 15: Professor Varineau, theory of equations, fundamental concepts of mathematics, higher algebra; Professor Schwid, ordinary differential equations; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability. July 18 to August 19: Professor Barr, advanced calculus, seminar in geometry; Pro-

fessor S. R. Smith, partial differential equations; Professor Steen, college geometry.

West Virginia University. June 6 to July 15: Professor Peters, introduction to algebraic theories, special topics (for teachers); Professor Stewart, higher plane curves. July 18 to August 26: Professor Cunningham, theory of numbers; Professor Posey, group theory, special topics; Professor Vest, advanced differential equations.

PERSONAL ITEMS

University of New Brunswick announces: Dr. J. E. L. Peck, formerly a senior lecturer at the University of Natal, South Africa, has been appointed to an assistant professorship; Associate Professor L. P. Edwards has been promoted to a professorship.

Mr. P. M. Anselone, formerly a mathematician with the General Electric Company, Hanford Works, Richland, Washington, has been appointed Research Associate at the Radiation Laboratory, Johns Hopkins University, Baltimore, Maryland.

Miss Mary N. Applegate, previously a student at the University of Oklahoma, has a position as an engineer at General Electric Company, Schenectady, New York.

Mr. J. D. Armstrong, formerly a junior development engineer for Goodyear Aircraft Corporation, Akron, Ohio, has been appointed to a position as a physics instructor at Bolles School, Jacksonville, Florida.

Dr. E. L. Crow, recently head of the Statistics Branch, United States Naval Ordnance Test Station, China Lake, California, has a position as a statistician in the Office of the Director, Boulder Laboratories, National Bureau of Standards, Boulder, Colorado.

Mr. D. J. Davis, previously a physicist with the Rocket Development Group, Redstone Arsenal, Huntsville, Alabama, has accepted a position as project engineer with the R. C. A. Service Company, Missile Test Project, Patrick Air Force Base, Florida.

Assistant Professor Mamie M. Davis of Francis T. Nicholls Junior College of Louisiana State University has been appointed to an associate professorship at Wesleyan College.

Mr. August Deckert, formerly a mathematician at White Sands Proving Grounds, Las Cruces, New Mexico, has a position as a research engineer with Boeing Airplane Company, Seattle, Washington.

Mr. W. C. Dixon, previously a research associate at the Computation Laboratory, Wayne University, is now a member of the Computer Systems Division, Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. J. R. Fleming, recently a graduate student at the University of Washington, has a position at the Radiation Laboratory, University of California, Livermore.

Mr. J. B. Freier of St. Peter's College has been appointed to an assistant professorship at the Rensselaer Polytechnic Institute.

Mr. B. A. Fusaro has been appointed to an instructorship at Middlebury College.

Dr. L. V. Good, formerly a graduate student at the State College of Washington, is now Director of Anchorage Community College, Alaska.

Professor Emeritus G. H. Graves of Purdue University is a visiting lecturer at Valparaiso University.

Mr. J. B. Hansen, previously a student at Harvard University, has a position as a mathematician with the Computer Control Company, Naval Air Missile Test Center, Point Mugu, California.

Assistant Professor Katharine E. Hazard of New Jersey College for Women, Rutgers University, has been promoted to an associate professorship.

Mr. Peter Henrici of American University has been promoted to an assistant professorship.

Mr. J. G. Horne, Jr., has been appointed to an instructorship at Tulane University.

Reverend Brother Calixtus James, recently at LaSalle Military Academy, Oakdale, New York, is now at Bishop Bradley High School, Manchester, New Hampshire.

Assistant Professor J. A. Jenkins of Johns Hopkins University has been appointed to an associate professorship at the University of Notre Dame.

Mr. J. S. Klein, formerly a research assistant at the University of Michigan, has been appointed to an instructorship at Oberlin College.

Mr. A. H. Kruse, previously a graduate student at the University of Chicago, has been appointed to a research instructorship at the University of Kansas.

Mr. L. H. Lange of Valparaiso University has been promoted to an assistant professorship.

Mr. R. E. Lewkowicz, previously in military service, has a position as an associate mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Dr. Mark Lotkin, formerly chief of Machines Section, Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, is now Manager, Computer Mathematics Unit, R. C. A. Service Company, Patrick Air Force Base, Florida.

Mr. L. C. Marshall, previously a mathematician for White Sands Proving Ground, Las Cruces, New Mexico, is a computing analyst for North American Aviation, Downey, California.

Mr. R. M. Meisel of New York University has accepted a position as Applied Mathematical Aerodynamicist with Republic Aviation Corporation, Farmingdale, New York.

Mr. R. A. Moreland, Jr., formerly a teaching fellow at Texas Technological

College, has a position as a mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Professor Zeev Nehari of Washington University has been appointed to a professorship at Carnegie Institute of Technology.

Associate Professor C. L. Perry of the United States Naval Postgraduate School, Monterey, California, has been promoted to a professorship.

Dr. J. H. Powell, previously a graduate research assistant at Michigan State College, has been appointed to an instructorship at the University of Detroit.

Mr. John J. Quinn, formerly a student at St. John's University, New York, is now a graduate assistant in the Department of Physics, University of Maryland.

Assistant Professor Anatol Rapoport of the University of Chicago is a fellow at the Center for Advanced Study in the Behavioral Sciences, Stanford, California.

Mr. E. A. Rasor, recently an actuarial mathematician for the Social Security Administration, Washington, D. C., has accepted a position as an actuary for the Retirement Division of the United States Civil Service Commission, Washington, D. C.

Dr. B. A. Rattray, formerly a lecturer at the University of New Brunswick, has been appointed to an assistant professorship at McGill University.

Dr. E. K. Ritter, Director of Computation and Ballistics, United States Naval Proving Ground, Dahlgren, Virginia, has accepted an appointment as Professor and Director of the Rich Electronic Computer Center, Engineering Experiment Station, Georgia Institute of Technology.

Miss Mildred G. Schlapkohl, previously a student at Northwestern University, is now an assistant mathematician at Argonne National Laboratory, Lemont, Illinois.

Dr. C. V. L. Smith, recently head of Computer Branch, Office of Naval Research, Washington, D. C., is now a scientific liaison officer at the Office of Naval Research (London Branch), New York City.

Mr. H. F. Smith, formerly an instructor at Elgin High School and Community College, Illinois, is now an applied science representative with the I. B. M. Corporation, Chicago, Illinois.

Mr. E. A. Stavinoha, previously a mathematics teacher at Brownsville High School, Texas, has accepted a position as a computing analyst for Douglas Aircraft Company, Tulsa, Oklahoma.

Mr. Isay Stemp is Assistant Director of Research for J. A. Deknatel & Son, Inc., Queens Village, Long Island, New York.

Mr. A. D. Stewart of the University of Wisconsin has been appointed to an assistant professorship at the Prairie View Agricultural and Mechanical College, Texas.

Miss Roberta A. Wilcox, formerly a student at the University of Rhode Is-

land, is now a research trainee in the Department of Biostatistics, School of Hygiene and Public Health, Johns Hopkins University.

Mr. R. F. Willis, previously a mathematician with the Operations Research Group of Arthur D. Little, Inc., Cambridge, Massachusetts, has a position as a mathematician with the Weapons Systems Laboratory, Aberdeen Proving Ground, Maryland.

Mr. W. H. Winnis, formerly a junior actuary with the Union Labor Life Insurance Company, New York City, has accepted a position as an actuarial assistant for the Insurance Department, State of Maryland, Baltimore, Maryland.

Dr. R. A. Zemlin, previously a National Science Foundation fellow at Ohio State University, is now a senior mathematician at Remington Rand Corporation, St. Paul, Minnesota.

Professor Irvin S. Cohen of the Massachusetts Institute of Technology died on February 14, 1955.

Professor Emeritus Edward Kasner of Columbia University died on January 7, 1955. He was a charter member of the Association.

Professor D. W. Pugsley, head of the Department of Mathematics of Berea College, died on December 15, 1954. He was a member of the Association for 27 years.

Professor Emeritus C. N. Reynolds of West Virginia University died on December 15, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1954

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1954. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

At the end of the year, the total funds of the Association included balances held for the Committee on Visiting Lecturers and the Committee on the Undergraduate Program, most of which will be expended during 1955.

During 1954, the Association paid its share of the expenses of the Combined Membership Lists for 1952 and 1953. Since a large part of the cost of the 1952 List was a capital expense, the Board of Governors voted to appropriate $2,000 from the Dunkel Fund for this item.

The Current Fund suffered an actual deficit of $5,134 during 1954, as compared with a surplus of $393 in 1953. Most of this deficit is due to the charges for the Combined Membership List. It is the opinion of the Finance Committee that either the publication of the List should be discontinued or that it should be financed entirely by one of the special funds of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1954

M & T Trust Co., Buffalo	$11,124.99	Current Fund	$11,124.99
Securities	83,283.00	Carus Fund	14,833.38
		Chace Fund	12,979.19
		Houck Fund	8,792.72
		Chauvenet Fund	1,115.58
		Dunkel Fund	16,373.09
		General Fund	28,189.04
	$94,407.99		$94,407.99

II. CURRENT FUND

Balance, January 1, 1954	$11,124.99	MONTHLY	
Dues	21,199.59	Publication	$17,699.29
Initiation fees	696.00	Reprints	572.53
Subscriptions	5,110.34	Editor's Office	1,758.39
Sale of back numbers	1,665.73	Secretary-Treasurer's Office	
Advertisements	3,577.50	Clerical help	7,982.84
Contributions	3.00	Postage and printing	1,721.27
Sale of exchange periodicals	150.80	Office expenses	518.65
Interest on General Fund	1,171.49	Bank fees	215.82
Income from Hardy Fund	120.00	Auditing fee	50.00
Charges against Funds	233.46	Board of Governors	1,766.96
		Meetings	822.16
		Committees	1,000.00
		Subventions	505.17
		1952 Combined Membership List	1,290.51
		1953 Combined Membership List	3,158.88
		Transfer to General Fund	5,494.81
		Balance, December 31, 1954	$ 495.62

	III. CARUS FUND	IV. CHACE FUND	V. HOUCK FUND
Balance, January 1, 1954	$14,833.38	$12,979.19	$ 8,792.72
Sale of publications	1,582.19	731.63	20.80
Interest	595.79	522.15	354.80
Increase in value of securities	2,881.08	2,525.00	1,715.70
National Research Council grant	—	1,000.00	—
Less: charges and bank fees	245.40	178.87	53.99
Fire Insurance	21.00	—	—
Printing	—	2,781.63	—
Balance, December 31, 1954	$19,626.04	$14,797.47	$10,830.03

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1954	$ 1,115.58	$16,373.09	$29,189.04
Interest	43.51	659.38	—
Increase in value of securities	210.42	3,188.62	5,665.06
Dunkel Estate payment	—	240.80	—
Transfer from Current Fund	—	—	5,494.81
Less: charges and bank fees	6.37	96.48	—
Mathematics Student Journal	—	500.00	—
1952 Combined Membership List	—	2,000.00	—
Balance, December 31, 1954	$ 1,363.14	$17,865.41	$40,348.91

	IX. Visiting Lecturers Fund	X. Fund for Committee on Undergraduate Program
National Science Foundation grant	$15,000.00	—
Social Science Research Council grant	—	$2,500.00
Transfer from Current Fund	500.00	500.00
Contributions toward travel expenses	505.00	—
Less: salaries and travel expenses	5,393.36	2,432.26
Committee expenses	149.36	58.58
Balance, December 31, 1954	$10,462.28	$ 509.16

XI. Total Funds of the Association on December 31, 1954

Current Fund	$ 495.62	M & T Trust Co., Buffalo	$ 11,467.06
Carus Fund	19,626.04	Securities	104,831.00
Chace Fund	14,797.47		
Houck Fund	10,830.03		
Chauvenet Fund	1,363.14		
Dunkel Fund	17,865.41		
General Fund	40,348.91		
Visiting Lecturers Fund	10,462.28		
Fund for Committee on Und. Program	509.16		
	$116,298.06		$116,298.06

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 97 persons have been elected to membership by the Board of Governors on applications duly certified.

D. S. ADORNO, B.A. (Texas) Senior Mathematician, Haller, Raymond & Brown, State College, Pa.

N. W. ALBRIGHT, Student, California Institute of Technology.

D. R. ARNOLD, Student, Bethany College.

F. A. BARAGAR, Student, University of Manitoba.

C. W. BARNES, A.B. (High Point) Part-time Instr. and Grad. Student, University of North Carolina.

C. B. BAYTOP, M.S. (Howard) Instr., Hampton Institute.

A. P. BERENS, Student, University of Dayton.

LIPMAN BERS, Dr.Rer.Nat. (Prague) Professor, New York University.

S. E. BOHN, M. A. (Nebraska) Asst. Professor Concordia College.

J. R. BROWN, B.A. (Oregon S.C.) Grad. Student, Oregon State College.

C. E. BURGESS, Ph.D. (Texas) Asst. Professor, University of Utah.

R. R. CHRISTIAN, Ph.D. (Yale) Instr., University of British Columbia.

DR. J. B. COMROE, M.D. (Chicago) Physician, Berkeley, Calif.

B. R. DAVIS, B.S. (Geneva) Grad. Asst., University of Pittsburgh.

J. F. DAVIS, B.A. (Pennsylvania S.U.) Quality Control Engr., Firestone Tire and Rubber Co., Akron, Ohio.

E. R. DEAL, M.S. (Kansas S.C.) Instr., University of Michigan.

E. C. DELAND, M.A. (U.C.L.A.) Grad. Student, University of California at Los Angeles.

ANDREW DELANEY, B.A. (Oberlin) Vice President and Actuary, American General Life Insurance Co., Houston, Texas.

MRS. MARTHA M. DE VALLE, M.S. (Mexico) Research Worker, Institute of Geophysics, University of Mexico.

J. O. DISTAD, B.S. (Montana S.C.) Grad. Student, Montana State College.

T. H. DYER, B.S. (U.S. Naval Academy) Captain, United States Navy, Washington, D. C.

COL. E. H. EDDY, M.S. in A.E. (Calif. I.T.) Instr., General Motors Institute, Flint, Mich.

K. H. ERDODY, M.S. (Michigan) Instr., Wayne University; Northern High School, Detroit, Mich.

CATHERINE M. FITZPATRICK, Mathematician, Rock Island Arsenal, Rock Island, Ill.

J. E. FLANAGAN, Ph.D. (Illinois) Applied Science Representative, I.B.M., Pittsburgh, Pa.

PAUL FONG, Student, University of California, Berkeley.

OLNEY FORTIER, Danielson, Conn.

Y. T. FUNG, Student, University of California, Berkeley.

B. A. FUSARO, M.A. (Columbia) Instr., Middlebury College.

M. L. GLASSER, B.A. (Chicago) Grad. Student, University of Chicago.

WILLIAM GOOGE, Student, Georgetown University.

R. P. GOSSELIN, Ph.D. (Chicago) Professor, Youngstown College.

R. G. GREEN, Student, Abilene Christian College.

J. G. HARVEY II, Student, Baylor University.

COL. ARCHIE HIGDON, Ph.D. (Iowa S.C.) Head, Department of Mathematics, United States Air Force Academy, Lowry Air Force Base, Denver, Colo.

G. W. HILLER, Student, Mississippi Southern College.

M. L. HODGINS, Student, Sacramento State College.

C. A. HOFFMAN, M.S. (Rutgers) Instr., University of Florida.

J. T. HUMPHREY, M.S. (S.U. of Iowa) Asso. Professor, Grambling College.

R. Q. JENNETT, M.S. (Purdue) Aerophysics Engr., Consolidated Vultee Aircraft Corp., Ft. Worth, Texas.

EDGAR KARST, M.S. (Breslau) Independence, Missouri.

A. G. KONHEIM, Student, Polytechnic Institute of Brooklyn.

W. A. KRZEMINSKI, B.S. in Ed. (Bucknell) Instr., Wallington High School, N. J.

C. M. LARSEN, A.M. (Cornell) Instr., San Jose State College.

T. L. LIBBY, Student, Baylor University.

J. H. LINDSAY, JR., Student, University of Toronto.

R. D. LOW, M.S. (Iowa S.C.) Private, United States Army.

E. S. LOWRY, Student, University of Toronto.

MRS. RUTH J. MACKICHAN, M.A. (Michigan) Instr., University of North Dakota.

MARTHA J. MAY, Student, University of Kentucky.

R. M. MEISEL, M.S. (N.Y.U.) Aerodynamicist, Republic Aviation Corp., Farmingdale, N. Y.

KATHRINE C. MIRES, M.A. (Arkansas) Asst. Professor, Northwestern State College, Alva, Okla.

J. E. MULLINS, Student, Siena College.

HERBERT NADLER, M.A. (Columbia) Corporal, United States Army.

J. L. NEFF, B.S. in Ed. (Otterbein) Instr., University of Dayton.

W. C. OLSON, M.S. (Delaware) Mathematician, Aberdeen Proving Ground.

E. T. PARKER, B.A. (Northwestern) Assistant, Ohio State University.

C. M. PEARCY, JR., B.A. (Texas A. & M.) Teaching Fellow, A. & M. College of Texas.

JACQUELINE L. PENEZ, Ph.D. (Minnesota) Instr., Barnard College, Columbia University.

J. B. PHARR, Student, Davidson College.

LT. A. H. PUCCI, B.S. (New Hampshire) Mathematician, Wright-Patterson Air Force Base, Ohio.

GUSTAVE RABSON, Ph.D. (Michigan) Asst. Professor, Antioch College.

E. A. RACE, M.A. (Maine) Asso. Professor, Norwich University.

DAVID ROSEN, Ph.D. (Pennsylvania) Asst. Professor, Swarthmore College.

MORRIS ROSEN, Student, Hofstra College.

W. C. ROSS, JR., Ph.D. (S.U. of Iowa) Instr., Knox College.

ANNE RYCHLICKI, Student, Alliance College.

P. P. SAWOROTNOW, Ph.D. (Harvard) Instr., Catholic University of America.

D. H. SCHMIEDER, A.B. (Kentucky) Grad. Asst., University of Kentucky.

M. M. SEGAL, M.S. in A.E. (N.Y.U.) Special Lecturer, Newark College of Engineering.

M. A. SELIM, M.S., M.A. (Texas) Research Asso., Petroleum Research Co., Denver, Colo.

DANIEL SHANKS, Ph.D. (Maryland) Acting Chief, Applied Mathematics Division, Naval Ordnance Laboratory.

A. B. SHEAFFER, M.A. (Sacramento S.C.) Major, United States Air Force.

W. C. SIKES, M.A. (Colorado) Asst. Professor, Abilene Christian College.

SISTER JULIA, M.S. (Catholic) Instr., College of Mount St. Joseph.

SISTER M. CELINE, Ph.D. (Michigan) Professor, Mercyhurst College.

SISTER MARY LORETTA, Student, Mercy College.

SISTER REGINA MARY, A.B. (Mount St. Joseph) Instr., College of Mount St. Joseph.

SISTER RUTHMARY, B.S. (La Crosse S.C.) Head, Mathematics Department, Viterbo College.

J. R. SLAGLE, Student, St. John's College, Brooklyn, N. Y.

H. R. SMITH, M.S. (South Carolina) Head, Mathematics Department, Baltimore Polytechnic Institute.

W. G. SPOHN, JR., M.A. (California) Instr., University of Delaware.

A. D. STEWART, M.S. (Howard) Asst. Professor, Prairie View A. & M. College.

W. C. SWIFT, B.S. (Kentucky) Grad. Student, University of Kentucky.

L. R. TAPPAN, M.A. (Arkansas) Instr., University of Arkansas.

W. P. UDINSKI, Ph.D. (Illinois) Mathematician, U.S. Air Force, Lackland Air Force Base, Texas.

MILDRED J. WESEEN, M.A. (California) Head, Mathematics Department, Analy Union High School, Sebastopol, Calif.

NANCY J. WHITE, B.S. (Millersville S.T.C.) Teacher, Edward Hand Junior High School, Lancaster, Pa.

R. J. WISNER, Ph.D. (Washington) Asst. Professor, Haverford College.

H. A. WOLF, B.S. (Illinois) Grad. Student, DePaul University.

F. W. WOLOCK, M.S. (Catholic) Instr., Iona College.

GREGORY WULCZYN, M.A. (Pennsylvania) Instr., Bucknell University.

KOICHI YAMAMOTO, M.S. (Tokyo) Asst. Professor, Kyushu University, Fukuoka, Japan

C. T. YANG, Ph.D. (Tulane) Member, Institute for Advanced Study.

B. H. YOUELL, JR., B.S. (W. Va. I. T.) Grad. Asst., West Virginia University.

L. N. ZACCARO, M.S. (Connecticut) Asst. Instr., Syracuse University.

ROMUALD ZALUBAS, M.A. (Vytautas) Instr., Georgetown University.

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Princeton University, Princeton, New Jersey, on November 27, 1954. Professor Alexander Tartler, Chairman of the Section, presided. There were present one hundred fifteen individuals including seventy-eight members of the Association:

J. H. Barrett, Donald C. Benson, J. S. Biggerstaff, B. H. Bissinger, Leila Dragonette Bram, H. W. Brinkmann, R. C. Carson, E. H. Cutler, L. J. Deck, F. L. Dennis, G. F. Feeman, N. J. Fine, R. F. Gabriel, E. D. Glenney, G. B. Glover, S. I. Goldberg, H. H. Goldstine, D. S. Greenstein, V. H. Haag, Theodore Hailperin, Katherine E. Hazard, S. A. Hoffman, J. R. Holzinger, C. C. Hsuing, R. F. Jackson, R. C. James, C. E. Kerr, R. W. Klopfenstein, T. L. Koehler, R. J. Kohlmeyer, J. B. Kruskal, Jr., M. D. Kruskal, Harold W. Kuhn, R. A. C. Lane, V. V. Latshaw, W. S. Lawton, Solomon Lefschetz, Marguerite Lehr, W. H. Leser, A. W. Mall, Helen M. Marston, B. H. McCandless, William McKay, S. S. McNeary, Josephine M. Mitchell, E. R. Mullins, Jr., C. A. Nelson, C. O. Oakley, Marghrita L. O'Neil, J. C. Oxtoby, R. S. Pieters, C. F. Pinzka, G. E. Raynor, Edgar Reich, Dorothy J. Rhea, B. E. Rhoades, David Rosen, F. E. Ross, C. W. Saalfrank, I. J. Schoenberg, Pincus Schub, Jerome C. Smith, Ernst Snapper, Sister Mary Stephanie, E. P. Starke, J. D. Swift, Alexander Tartler, C. M. Terry, D. L. Thomsen, Jr., Bryant Tuckerman, R. M. Walter, G. C. Webber, D. W. Western, Anna Pell Wheeler, M. E. White, Albert Wilansky, F. H. M. Williams, Marie A. Wurster.

Officers of the Section for the year 1954–55 are: Chairman, Professor T. L. Koehler, Muhlenberg College; Secretary, Professor G. C. Webber, University of Delaware; Program Committee: Chairman, Professor C. W. Saalfrank, Lafayette College; Professor N. J. Fine, University of Pennsylvania; Professor H. S. Grant, Rutgers University.

The following papers were presented:

1. *Coordinates of algebraic varieties*, by Professor Ernst Snapper, University of Southern California and Princeton University.

Let $\{V_g^m\} = A$ be a set of pure, algebraic varieties of dimension m and degree g, imbedded in an n-dimensional projective space S^n over an arbitrary groundfield k. The irreducible components of each V_g^m all have dimension m, and are counted with preassigned multiplicities. Such pure varieties, whose components are furnished with multiplicities, are called cycles of dimension m and degree g. We say that A is an *algebraic system* of cycles if, in some projective or multiply projective space, there exists an algebraic variety P and an algebraic correspondence T from P to S^n, satisfying the following conditions: (1) T associates to each point of P exactly one cycle of A, (2) T associates to

different points of P, different cycles of A, (3) every cycle of A is the image under T of a point of P.

If both (P, T) and (P', T') satisfy the above conditions, there exists a birational transformation R from P onto P' where both R and R^{-1} are regular and where $T'R=T$; namely, if $p \in P$, $R(p)$ is defined by $T'(R(p))=T(p)$. Hence, for all purposes of algebraic geometry, (P, T) and (P', T') are equivalent. These conventions free us from the unnecessary and very undesirable restriction of always having to construct Chow-varieties in a preassigned way. When (P, T) satisfies the above conditions P is called a Chow-variety of A, a point $p \in P$ is called a Chow-point of the cycle of A which it represents, and the coordinates of P are called the coordinates of that cycle. The dimension of P is termed the dimension of A and, in the same way, all of the algebra-geometric terminology is carried over from P to A.

Several examples were given. In each example, P and T were constructed precisely. As an application it was proved that, on a locally normal variety, a linear system is always contained in a complete linear system.

2. *Mathematics through the television lens*, by Professor F. G. Fender of Rutgers University, Professor Marguerite Lehr of Bryn Mawr College, and Professor R. F. Jackson, University of Delaware.

Professor Fender reported on a thirteen-week series of talks entitled "This is Mathematics" which was presented over television by Rutgers University. The series was aimed at making clearer to the general public the story of the modern mathematician and his work. Topics of the individual talks were Counting, Number Systems, Adding and Multiplying, Subtracting and Dividing, Number Theory, Machines, Greatest and Least, The Shape of Things, Uncertainty, Symbols, The Possible and Impossible, Audience-suggested Program.

Professor Lehr described a fifteen-week series of half-hour talks which was presented over Philadelphia's Channel 6, accompanied by a semester syllabus of abstracts and references for each topic. See this MONTHLY, vol. 62, 1955, pp. 15–21.

Professor R. F. Jackson presented an outline of the general philosophy and specific plans for a television series on "Thinking Machines—from Fingers to Flip-Flops."

3. *Area and volume*, by Professor A. S. Besicovitch, Trinity College, Cambridge University, England, and the Institute for Advanced Study, introduced by the Secretary.

A topological transform of a square in a plane or of a cylinder in 3-space is called a topological square or topological cylinder. The transforms of sides, vertices and bases are called sides, vertices and bases of the topological figure. The S-distance between two points of a topological figure is defined as the minimum length of curves joining them and entirely belonging to the topological figure. It is shown: (1) The area of a topological square is greater than or equal to ah, when a and h are the distances between pairs of opposite sides; (2) The volume of a topological cylinder whose height is h and for which the area of any side section is greater than a need not be greater than or equal to ah.

4. *Some remarks on numerical stability*, by Dr. H. H. Goldstine, Electronic Computer Project, Institute for Advanced Study.

The concept of numerical stability plays a fundamental role in modern day numerical analysis. Since machines do not multiply or divide with exactitude, it is of importance to estimate how these errors accumulate in the course of a calculation. Examples were given showing how round-off errors could accumulate in such a fashion that no significance could be attached to the final results.

Following the program the Electronic Computer at the Institute was demonstrated.

G. C. WEBBER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May 7, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY, Georgetown College, Georgetown, April 30, 1955.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Morgan State College, Baltimore, Maryland, April 16, 1955.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN

MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.

MISSOURI, University of Kansas City, April 22, 1955.

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METHODS OF MATRIX INVERSION

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Introduction. The physical scientist has long been plagued with large linear systems of equations, hopeless differential equations or systems of such equations, or simply enigmatic integral equations. Each of these problems in turn often lends itself to matrix analysis and the approach usually involves matrix inversion. However, it is only with the advent of high speed computation that the approach has entered the realm of practicality.

The purpose of this paper is to present some of the existing methods of matrix inversion in a fashion requiring a minimal mathematical background. ("Minimal mathematical background" implies knowing the definition of a matrix, how to add and multiply matrices, how to apply matrix notation to systems of linear equations, and how to evaluate a determinant. Most of this material is discussed in [27].)

In each of the following discussions, matrix A will be assumed non-singular and will be represented by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{bmatrix}$$

The main diagonal of A will be that part of the matrix consisting of the elements a_{ii}, $i=1, 2, \cdots, n$; and the trace $[A]$ will be the sum of these elements. The unit matrix is one such that $a_{ii}=1$; $i=1, 2, \cdots, n$; and $a_{ij}=0$; $i \neq j$; $i, j=1, 2, 3, \cdots, n$. A is called symmetric if and only if $a_{ij}=a_{ji}$ for all i and j.

Method I—Elimination method. Multiply row 1 by $1/a_{11}$. Then multiply row 1 by $-a_{i1}$ and add to row i for $i=2, 3, 4, \cdots, n$. This sequence of operations has the effect of creating a 1 in the first position of the main diagonal and zeros everywhere else in the first column of the matrix. Apply the same technique to the second row, thus creating a 1 in the second position on the main diagonal and zeros everywhere else in the second column. Apply the same technique to the third row, again resulting in a column of $(n-1)$ zeros and a 1 in the main diagonal position. Keep up this procedure until the given matrix is reduced to the unit matrix. By performing the very same sequence of operations on the unit

matrix as were performed on A, using the very same numbers necessary when operating on A, the unit matrix is transformed into A^{-1}.
Example 1: Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then we write down:

$$\left[\begin{array}{rrr|rrr} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

We multiply row 1 by $\frac{1}{2}$ and secure:

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

By multiplying row 1 by -2 and adding to row 2 and multiplying row 1 by 1 and adding to row 3 we have:

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

Multiply row 2 by $\frac{1}{5}$. Hence:

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

The sequence of operations: multiply row 2 by 1 and add to row 1; multiply row 2 by 0 and add to row 3; multiply row 3 by 1; multiply row 3 by $-\frac{8}{5}$ and add to row 1; multiply row 3 by $\frac{2}{5}$ and add to row 2, yields:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right].$$

Hence the inverse of

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

The only problem which can arise, assuming no loss of significance to make the matrix at any stage singular, is that one of the a_{ii} becomes 0. In this case, one need only interchange two rows so that the zero on the main diagonal is removed from its choice position. In order to get A^{-1} then, one need only take the final result and interchange in it the same numbered columns as those of the rows which were interchanged.

A technique like this is called an elimination technique. Application of it to a large scale matrix should be accompanied by an examination of [**30**].

Method II. Consider the matrix B such that:

$$B = \left[\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right],$$

where A is the matrix to be inverted, I is the identity matrix, and 0 the matrix composed of all zero elements. Consider the following operations:

a) multiply any row or any column by a non-zero constant.
b) multiply any row (column) by a non-zero constant and add to another row (column).
c) interchange any two rows or any two columns.

Using these operations, we reduce

$$B \rightarrow \left[\begin{array}{c|c} P & I \\ \hline 0 & Q \end{array}\right].$$

Then $A^{-1} = QP$.

Example 2:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then

$$B = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 1 & 0 & 0 & 2 & -2 & 4 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 5 & 0 \\ 1 & 0 & 0 & 2 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 5 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Hence

$$P = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^{-1} = QP = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & \frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Let us note first that method I is a special case of method II but that method I is worthy of special discussion, for although Andree [2] claims method II is the faster of the two, method I is more readily coded for high speed machines.

Method III. Given matrix A, consider matrix B such that $AB = I$. Setting the product AB equal to I term by term yields n^2 equations in n^2 unknowns which one tries to solve. These actually form n distinct sets of linear systems.

Example 3: Given

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}, \qquad \text{let } B = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}.$$

Then:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 2x_1-2x_2+4x_3 & 2y_1-2y_2+4y_3 & 2z_1-2z_2+4z_3 \\ 2x_1+3x_2+2x_3 & 2y_1+3y_2+2y_3 & 2z_1+3z_2+2z_3 \\ -x_1+\ x_2-\ x_3 & -y_1+\ y_2-\ y_3 & -z_1+\ z_2\ -z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence:

$$\begin{array}{lll} 2x_1 - 2x_2 + 4x_3 = 1 & 2y_1 - 2y_2 + 4y_3 = 0 & 2z_1 - 2z_2 + 4z_3 = 0 \\ 2x_1 + 3x_2 + 2x_3 = 0 & 2y_1 + 3y_2 + 2y_3 = 1 & 2z_1 + 3z_2 + 2z_3 = 0 \\ -x_1 + x_2 - x_3 = 0 & -y_1 + y_2 - y_3 = 0 & -z_1 + z_2 - z_3 = 1, \end{array}$$

the solutions of which are

$$\begin{array}{lll} x_1 = -\frac{1}{2}, & x_2 = 0, & x_3 = \frac{1}{2} \\ y_1 = \frac{1}{5}, & y_2 = \frac{1}{5}, & y_3 = 0 \\ z_1 = -\frac{8}{5}, & z_2 = \frac{2}{5}, & z_3 = 1. \end{array}$$

Therefore:

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

It will be noted that the problem of matrix inversion has here been converted to one of solving linear systems of equations. Much research has been done on this latter problem and hence any technique associated with solving linear systems can be applied to matrix inversion. Besides the usual methods of: a) Cramer's rule, b) Gauss elimination method, c) Gauss-Jordan method, and d) Crout's method, more modern techniques are described in: [5], [8], [12], [16], [21], [23], [26].

Method IV. Adjoint method.

Let A_{11} = determinant formed from matrix A by deleting row 1 and column 1 and multiplying by $(-1)^{1+1}$.

Let A_{21} = determinant formed from matrix A by deleting row 2 and column 1 and multiplying by $(-1)^{2+1}$.

Let A_{37} = determinant formed from matrix A by deleting row 3 and column 7 and multiplying by $(-1)^{3+7}$.

Let A_{ij} = determinant formed from matrix A by deleting row i and column j and multiplying by $(-1)^{i+j}$.

Let C be the matrix:

$$C = \begin{bmatrix} A_{11} & A_{21} & A_{31} \cdots A_{n1} \\ A_{12} & A_{22} & A_{32} \cdots A_{n2} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ A_{1n} & A_{2n} & \cdots A_{nn} \end{bmatrix}.$$

Then $A^{-1} = (1/|A|)C$, where $|A|$ is the determinant of matrix A.

Example 4. Let:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then:

$$A_{11} = (-1)^2 \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5 \qquad A_{21} = (-1)^3 \begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix} = 2$$

$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} = 0 \qquad A_{22} = (-1)^4 \begin{vmatrix} 2 & 4 \\ -1 & -1 \end{vmatrix} = 2$$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5 \qquad A_{23} = (-1)^5 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$$

$$A_{31} = (-1)^4 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} = -16$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4$$

$$A_{33} = (-1)^6 \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} = 10.$$

Also:

$$|A| = \begin{vmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix} = 10;$$

therefore

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & \frac{1}{10} \end{bmatrix}.$$

Method V—Method of Partitions. Given matrix A, let the sequence of matrices $S_1, S_2, \cdots, S_N$, be defined by:

$$S_1 = [a_{11}],\ S_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left[\begin{array}{c|c} S_1 & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right],\ S_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \left[\begin{array}{c|c} S_2 & \begin{matrix} a_{13} \\ a_{23} \end{matrix} \\ \hline a_{31}a_{32} & a_{33} \end{array}\right], \cdots$$

$$S_N = A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1N} \\ a_{21} & a_{22} \cdots a_{2N} \\ \cdot & \\ \cdot & \\ \cdot & \\ a_{N1} & a_{N2} \cdots a_{NN} \end{bmatrix} = \left[\begin{array}{c|c} S_{N-1} & \begin{matrix} a_{1N} \\ a_{2N} \\ \cdot \\ \cdot \\ \cdot \end{matrix} \\ \hline a_{N1}a_{N2} \cdots & a_{NN} \end{array}\right].$$

In general then, we shall want to partition a matrix into 4 sub-matrices which we write as: (See [11], pp. 112–115)

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

For example, for S_3, we have:

$$\alpha_{11} = S_2, \qquad \alpha_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \qquad \alpha_{21} = [a_{31}\ a_{32}], \qquad \alpha_{22} = [a_{33}].$$

If we denote the inverse of A by B and let:

$$B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix},$$

then we have:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \cdot \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = I.$$

Multiplying out and setting the resulting matrix equal to unity yields four matrix equations. Solving these matrix equations for $\beta_{11}, \beta_{12}, \beta_{21}$, and β_{22}, we have:

$$\begin{aligned} \beta_{22} &= D \\ \beta_{21} &= -D(\alpha_{21}\alpha_{11}^{-1}) \\ \beta_{12} &= -(\alpha_{11}^{-1}\alpha_{12})D \\ \beta_{11} &= \alpha_{11}^{-1} + (\alpha_{11}^{-1}\alpha_{12})D\alpha_{21}\alpha_{11}^{-1}, \end{aligned}$$

where $D = (\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12})^{-1}$.

The technique of inversion will be to apply the above formulas in a recursive fashion by first finding the inverse of S_1, then of S_2, then of $S_3, \cdots$, finally of S_N and this will be A^{-1}. One interchanges two rows to avoid trouble when a submatrix is singular and the process cannot be continued directly. This necessitates only changing the same two columns in the result to find A^{-1}.

Example 5:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then $S_1 = [2]$, $S_1^{-1} = [\frac{1}{2}]$. Hence:

$$S_2 = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} S_1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}.$$

Using the formulas for β_{ij}, it follows that: $\beta_{11} = \frac{3}{10}$, $\beta_{12} = \frac{1}{5}$, $\beta_{21} = -\frac{1}{5}$, $\beta_{22} = \frac{1}{5}$. Hence

$$S_2^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

Now:

$$S_3 = \left[\begin{array}{c|c} S_2 & \begin{matrix} a_{13} \\ a_{23} \end{matrix} \\ \hline a_{31}a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 2 & -2 & 4 \\ 2 & 3 & 2 \\ \hline -1 & 1 & -1 \end{array}\right].$$

Then:

$$\alpha_{11} = S_2 = \begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}, \qquad \alpha_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

$$\alpha_{21} = [a_{31} \ a_{32}] = [-1, \ 1], \qquad \alpha_{22} = [a_{33}] = [-1].$$

Note that $S_2^{-1} = \alpha_{11}^{-1}$ is known from the previous calculation. Again using the formulas for the β_{ij}, it follows that:

$$\beta_{11} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} \\ 0 & \frac{1}{5} \end{bmatrix}, \qquad \beta_{12} = \begin{bmatrix} -\frac{8}{5} \\ \frac{2}{5} \end{bmatrix}, \qquad \beta_{21} = [\tfrac{1}{2} \ 0], \qquad \beta_{22} = [1].$$

Therefore:

$$A^{-1} = S_3^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Concerning experiments with this method, one should consult [17].

Method VI. Given matrix A, consider the matrix: $B = A - \lambda I$, where I is the unit matrix and λ is a constant. If we set the determinant of B equal to zero, *i.e.*, $|B| = 0$, we see that this is nothing more than a polynomial equation in λ of degree n. The values $\lambda_1, \lambda_2, \cdots, \lambda_n$ which satisfy this equation are called the eigenvalues of the matrix A and the equation itself is called the characteristic equation of the matrix A. For example, if:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -2 & 4 \\ 2 & 3-\lambda & 2 \\ -1 & 1 & -1-\lambda \end{bmatrix},$$

the characteristic equation is:

$$\lambda^3 - 4\lambda^2 + 7\lambda - 10 = 0.$$

A fundamental theorem of matrix theory says that every matrix satisfies its characteristic equation. By the example above, we explain this to mean:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^3 - 4\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^2 + 7\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

$$- 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and this may be readily verified by performing the indicated operations.

A matrix may satisfy many different equations besides its characteristic equation. It can be shown that there exists an equation of least degree which the matrix satisfies and this degree may be less than that of the characteristic equation. This equation of least degree is unique to within a multiplicative constant and is appropriately called the minimal equation associated with the matrix. If $C(x) = 0$ is the characteristic equation associated with a matrix and $\mathrm{M}(x) = 0$ is the minimal equation which the matrix satisfies, then it can be shown that $\mathrm{M}(x)$ is a factor of $C(x)$.

Now, suppose some equation has been found which the matrix satisfies, the characteristic equation usually sufficing. Then if the equation is:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \qquad a_n \neq 0,$$

and if A is the matrix:

$$a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI = 0.$$

Multiplying through by A^{-1}, we have:

$$a_0A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I + a_nA^{-1} = 0,$$

or:

$$A^{-1} = \frac{1}{a_n}\left\{- a_0A^{n-1} - a_1A^{n-2} - \cdots - a_{n-1}I\right\}.$$

Example 6:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

It has been shown that A has the characteristic equation:

$$\lambda^3 - 4\lambda^2 + 7\lambda - 10 = 0.$$

Then:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^3 - 4\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^2 + 7\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

$$- 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and:

$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}^2 - 4\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 10A^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

or:

$$\begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix} - 10A^{-1} = 0,$$

and finally,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

The case where $a_n=0$ offers no exorbitant trouble, for then the matrix has no inverse.

Method VII—Frame's method (See [10].) Given matrix A, let $A_0=I$. Using:

$$(1)\quad c_k = \frac{1}{k}\text{ trace } [AA_{k-1}]$$

$$(2)\quad A_k = AA_{k-1} - c_k I,$$

we have:

$$A^{-1} = \frac{A_{n-1}}{c_n}, \qquad c_n \neq 0.$$

Example 7. Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

Then:

$$c_1 = \tfrac{1}{1}\text{ trace } [AA_0] = 4, \qquad A_1 = AA_{k-1} - 4I = AA_0 - 4I = \begin{bmatrix} -2 & -2 & 4 \\ 2 & -1 & 2 \\ -1 & 1 & -5 \end{bmatrix},$$

$$c_2 = \tfrac{1}{2}\text{ trace } [AA_1] = -7, \quad A_2 = AA_1 + 7I = \begin{bmatrix} -5 & 2 & -16 \\ 0 & 2 & 4 \\ 5 & 0 & 10 \end{bmatrix},$$

$$c_3 = \tfrac{1}{3}\text{ trace } [AA_2] = \tfrac{1}{3}(-10 + 0 + 20 + 4 + 6 + 0 + 16 + 4 - 10) = 10.$$

Finally:

$$A^{-1} = \frac{A_2}{c_3} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Method VIII. Let a_k, $k=0, 1, 2, \cdots, n-1$, be n constants. Consider the special matrix:

$$B = \begin{bmatrix} a_0 & a_{n-1} & a_{n-2} & a_{n-3} \cdots a_2 & a_1 \\ a_1 & a_0 & a_{n-1} & a_{n-2} \cdots a_3 & a_2 \\ a_2 & a_1 & a_0 & a_{n-1} \cdots a_4 & a_3 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} \cdots a_1 & a_0 \end{bmatrix}.$$

Such special matrices are called circulant matrices. To invert these we use the procedure (see [13]):

1) Compute:

$$w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\cdot$$

2) Calculate the eigenvalues of the given matrix by:

$$e_s(B) = \sum_{r=0}^{n-1} a_r w^{rs}, \qquad s = 0, 1, \cdots, n-1.$$

3) Calculate the inverses of these eigenvalues and denote these by e_s^{-1}.

4) Calculate the numbers:

$$b_s = \frac{1}{n}\sum_{r=0}^{n-1}(e_r^{-1})w^{-rs}, \qquad s = 0, 1, \cdots, n-1,$$

5)

$$B^{-1} = \begin{bmatrix} b_0 & b_{n-1} & \cdots & b_2 & b_1 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ b_{n-2} & b_{n-3} & & b_0 & b_{n-1} \\ b_{n-1} & b_{n-2} & & b_1 & b_0 \end{bmatrix}.$$

Example 8. Let

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix},$$

which is a circulant matrix where $a_0 = 1$, $a_1 = 3$, $a_2 = 2$.

1) $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

2) $e_0(A) = 6$

$$e_1(A) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 1 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$$

$$e_2(A) = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 1 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i$$

3) $e_0^{-1} = \frac{1}{6}$

$$e_1^{-1} = -\frac{1}{2} - \frac{\sqrt{3}}{6} i$$

$$e_2^{-1} = -\frac{1}{2} + \frac{\sqrt{3}}{6} i$$

4) $$b_0 = -\frac{5}{18}$$

$$b_1 = \frac{1}{3}(e_0^{-1} + e_1^{-1}w^{-1} + e_2^{-1}w^{-2}) = \frac{1}{18}$$

$$b_2 = \frac{1}{3}(e_0^{-1} + e_1^{-1}w^{-2} + e_2^{-1}w^{-4}) = \frac{7}{18}$$

5)
$$B^{-1} = \begin{bmatrix} -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \\ \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \end{bmatrix}.$$

The method is a good one in that it works readily on matrices with complex components. If B is real and symmetrical, [**13**] gives simpler formulas for the computation of the eigenvalues.

Method IX—Newton's formula. In general, given a number W, suppose one wishes to calculate $1/W$. Let $x = 1/W$. Then by Newton's formula, setting $f(x) = W - 1/x$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0,$$

$$= x_n - \frac{W - \frac{1}{x_n}}{\frac{1}{x_n^2}}$$

$$= x_n(2 - Wx_n).$$

If we let A be a matrix whose inverse is desired, and we let X_0 be a reasonable approximation to A^{-1}, then the above formula, written in terms of matrices is:

$$X_{n+1} = X_n(2I - AX_n).$$

The following example is selected from [**27**]:

Example 9. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}.$$

Then: choose

$$X_0 = \begin{bmatrix} -3.0 & 4.1 & -0.9 \\ 4.1 & -5.1 & 1.9 \\ -0.9 & 1.9 & -1.1 \end{bmatrix}.$$

Using the recursion formula above, it follows that:

$$X_1 = \begin{bmatrix} -4.26 & 4.14 & -0.86 \\ 4.14 & -5.06 & 1.94 \\ -0.86 & 1.94 & -1.06 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} -3.9976 & 3.9864 & -1.0138 \\ 3.9864 & -4.9896 & 2.0104 \\ -1.0136 & 2.0104 & -0.9896 \end{bmatrix}.$$

The exact solution is:

$$A^{-1} = \begin{bmatrix} -4 & 4 & -1 \\ 4 & -5 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Concerning convergence of iteration processes applied to matrices, one should consult [22] and [17].

Method X. On certain matrices, the following method may be applied: Let $A=(I+B)$. Then, $A^{-1}=(I+B)^{-1}=I-B+B^2-B^3+\cdots$

Example 10: Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & -1 & 4 \end{bmatrix}.$$

Then:

$$A^{-1} = \left\{ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \right\}^{-1}$$

$$= \tfrac{1}{4} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix} \right\}^{-1}$$

$$= \frac{1}{4}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix}^2 - \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix}^3 + \cdots \right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{16} & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdots \right\}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{16} & \frac{1}{4} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & 0 \\ \frac{1}{64} & \frac{1}{16} & \frac{1}{4} \end{bmatrix}.$$

XI. Other methods and concluding remarks.

A. A newer type method which necessitates knowledge of sampling theory and game theory, concepts beyond the scope of an elementary paper, is the Monte Carlo method, developed by Von Neumann and Ulam, written up by Forsythe and Liebler in [**7**], and played with by Todd in [**28**].

B. Variations of the preceding methods are often useful for special matrices. For example, consider a matrix with the following properties:

1. It is symmetric.
2. The largest element of any row lies on the main diagonal.
3. The matrix has zeros everywhere except possibly on the main diagonal and in the positions just above and just below the main diagonal.

Such matrices occur frequently in physical problems, for example, in vibration. Preceding methods can be applied but a rapid variation is given in [**24**].

C. Lastly, it should be noted that if A can be written as a product of matrices whose inverses are readily found, then A^{-1} is easily produced, for:

$$\text{if:}\quad A = BCDEF, \qquad A^{-1} = F^{-1}E^{-1}D^{-1}C^{-1}B^{-1}.$$

Hence ingenuity must be used to find matrices whose inverses are simply found and whose product is A. Cholesky's method [**27**] is such a scheme.

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SURVEY COURSES IN MATHEMATICS FOR THE LIBERAL ARTS STUDENT*

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1. Introduction. Great interest in survey courses has been developing in the past twenty years. That educators are willing to give their students brief and general introductions into the sciences reflects one or both of two trends. First, there is growing adherence to the belief that what is called by philosophers "the unity of science" is more than a fiction. Second, this willingness indicates that knowledge of the sciences, however small, somehow furnishes the individual with a capacity to make experience more meaningful and positive. The growing interest in science fiction is perhaps a popular indication of man's acceptance of this explanation.

Mathematics has played so important a role in recent scientific advances that it becomes necessary to consider what methods are most effective in furnishing the student with enough information about it and the part that it plays in our progress. Our discussion has to do with the purposes of survey courses in mathematics, with types of courses which can be designed to fulfill these purposes, and with suggestions for available textbook material to meet the needs of the courses described.

2. Needs and purposes. Among the most obvious purposes of survey courses in general is what a mathematician might call "communication closure." The abundance of scientific terminology in everyday speech makes knowledge of such terminology mandatory. Knowledge of terminology alone is not enough to provide the desired closure. An acquaintance with the ideas for which the terms stand is also necessary. Many terms and ideas have applications outside of mathematics, as well as within, and it is understanding these that is specially important. The central mathematical notion that carries scientific information is the notion of function. A study of functions and their properties can involve mathematics which is well within the grasp of industrious and imaginative college students.

A second purpose for a survey course in mathematics which is closely allied to the previous one is merely to teach the student the meaning of numerical statements as they are used in communicating scientific data. This would require little investigation into basic mathematics. It would require mainly the numerical techniques (such as those of statistics) by which scientific data may be evaluated and disseminated.

A survey course in mathematics can also serve the purpose of furnishing the student with a solid knowledge of ideas about truth and validity. The advantage of teaching logic from a mathematical point of view, rather than from the classical (or philosophical), is the familiarity which such a treatment affords with uses of

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formal language. This familiarity with formal language in turn furnishes a device through which the student may learn more about the needs and uses of theory and generalization.

The morning session of the meeting of the Association in Baltimore, December 31, 1953, discussed another important purpose of mathematics surveys, and described ably the steps being taken to fulfill the purpose. I refer now to the growing importance to social scientists of mathematical techniques in constructing models. Because the matter has already been covered in Baltimore as well as it can be, I shall not discuss it further in what follows.

The student enrolled in a science curriculum furnishes an example of a class of prospective customers for the survey course in mathematics. Such a student is usually required to take courses in techniques, but most schools fail even to offer him the opportunity to study fundamental ideas which might equip him with more thorough understanding of mathematics and the sciences, and their relation to one another. If such an opportunity is presented it is usually not until the student is doing graduate work. If the student is in the liberal arts or if he plans to teach science or mathematics before doing graduate work, it is especially important for him to be acquainted with the basic notions with which a good survey course can supply him. It is contended that little time is left in the college career of such a student for "culture" courses like these, but it is certainly true that to offer a course in basic mathematics (and basic science) would be a gallant step in the right direction, even if such a course were elected rather than required.

3. Techniques. Before discussing particular suggestions for survey course topics, it seems a good idea to list several pedagogical techniques that such courses indicate as desirable. A survey course is best devised when it presents either new material, or digests and recasts old material in a new form. To go over high school material slowly leaves most students indifferent. To do this quickly makes the course more difficult than necessary and frustrates the student. Even with novel approaches (or perhaps even because of them) it is not necessary to require the student to have any previous background in mathematics. But if this is so, then what is required is a close touch throughout the course with an intuition which has been educated by the instructor early in the course. One good way of achieving these ends is through a detailed study of both the intuitive and the formal development of the real number system. Applying such techniques will do much to get the student's interest to participate in the study and to do away with the stigma which surrounds studies in mathematics.

Other suggestions are of a more picayune nature. The student will profit by reading simple mathematics on his own and reporting on some topic not discussed in the classroom. Though many of the topics suggested below are of such a nature that they can best be presented by lecture, chances do arise for spirited

classroom discussion about issues that arise out of the student's existing misconceptions, or out of consideration of philosophical questions concerning mathematics and science. Such discussions can be stimulated. Of course meaningful exercises that are not too difficult are always instrumental in putting across a point—a technique or an idea.

4. Course contents. An intensive one-year survey of mathematics is effective in attaining both proficiency in technique, and understanding of basic ideas. There is time to discuss the ideas which underlie analysis—the real numbers, algebra, and the theory of real functions, as well as related ideas from applications of analysis to measurement in general, to statistics, and to other fields to which application is immediate. There is a chance to illustrate the ideas with examples and techniques which are more than elementary, so that a follow-up course in technical calculus can make the student proficient in the uses of analysis. Finally, through topics from mathematical logic, the student can be given an insight into axiomatics and become acquainted with the nature of formalism and generality.

There are two textbooks in particular which follow the suggested lines. [7] is a three volume paper-bound text used at the University of Chicago and written by the department there, and represents an early attempt at developing this type of course. [1] is a new, and, in my opinion, excellent book for such a course. It is available in an offset edition only at present and has been used with success at the University of Washington.*

We have been working at Penn State on a similar type of course for the past two and a half years. Our course differs from the one above in these respects: first, it is a one semester course; second, because of this, certain topics included in the previously described course are not included in ours. The most notable omission is that no attempt is made at teaching involved techniques with complicated functions. Polynomials are used for illustrations and examples throughout the course. Ideas from mathematical logic are distributed throughout the course, rather than introduced as a separate topic. The same is true of applications of analysis, which are confined largely to explanations of simple examples from physics.

What remains, then, is an intensive discussion of the real number system, and a thorough discussion of the idea of a function, or the properties of functions and their use in social and natural science. The course closes with as long a discussion of axiomatics and formalism as time allows. Two books have been used so far: [4] and [2]. Both books have been incomplete for the purposes of our course, and supplementary reading, especially of material on the real number system, has had to be given.

There are a number of acceptable alternative choice of topics for survey courses in mathematics. Certainly study of basic notions of analysis does not

* *Editorial note.* The printed edition has just been published.

exhaust all the possibilities. Let us look briefly at several other ideas.

A course that attempts to fulfill the communication closure mentioned earlier can be oriented from the point of view of numerical methods. The only techniques which have to be presented thoroughly are those of arithmetic. Other topics, such as applications, can be adequately explained through the notion of formula. The idea of function, though fruitful, need not be mentioned at all. Techniques of measurement and approximation can be given purely operationally, without the need for strict development of algebra, geometry, and calculus. A course like this leaves much to be desired but has the advantage that it can be presented to college students who have had no mathematics in high school and who are less mature thinkers than we would desire.

A course in foundations through mathematical logic has much to recommend it, being an avenue into simple abstract thinking and idea construction. This course would spend time on the predicate calculus, presumably, but would emphasize proof theory, the idea of axiomatization, and the notions which usually go under the title of metamathematics. Textbook suggestions for such a course include [6] and [5]. For the advanced student, [8] is a suggestion, but the first two books share the advantage of having a symbolism which is more immediate.

A final suggestion for those who believe that a college student in liberal arts should be able to pass a mathematics course at all costs—that he should be "exposed" to mathematics. A survey of "game" topics in mathematics can be developed that will activate the interest of the student along mathematical lines. A course like this could contain some group theory, combinatorial topology, decidable games like "nim," magic squares, and so forth. Of course, such content is very difficult to motivate, and such a course would be difficult to integrate and unify.

One factor to keep in mind in formulating a course is the interest of the student, without which many difficulties arise. Motivating vividly, educating the intuition, and keeping as close as possible to experience are effective influences in fostering this interest.

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THE POWER SERIES COEFFICIENTS OF $\zeta(s)$*

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Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s) = \sum_{1}^{\infty} n^{-s} \qquad R(s) > 1.$$

The function can be analytically continued for $R(s)>0$ by [1, Satz 440]

$$(1) \qquad g(s) = (2^{1-s} - 1)\zeta(s) = \sum_{1}^{\infty} \frac{(-1)^n}{n^s}\cdot$$

Well known is $\lim_{s\to 1} (s-1)\zeta(s)=1$, so that $\zeta(s)$ can be written as

$$(2) \qquad \zeta(s) = \frac{1}{s-1} + \sum_{0}^{\infty} A_n(s-1)^n.$$

The purpose of this paper is to investigate these coefficients.

From (1) we obtain, for k any non-negative integer,

$$(3) \qquad g^{(k)}(s) = (-1)^k \sum_{1}^{\infty} \frac{(-1)^n \log^k n}{n^s}\cdot$$

To get a power series for $g(s)$, write

$$2^{1-s} - 1 = e^{(1-s)\log 2} - 1 = \sum_{1}^{\infty} \frac{(-1)^n \log^n 2}{n!}(s-1)^n,$$

and multiply by the series in (2) to get

$$g(s) = \sum_{k=0}^{\infty}\sum_{t=1}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t}(s-1)^k, \qquad A_{-1} = 1.$$

Hence

$$(4) \qquad g^{(k)}(1) = k!\sum_{t=1}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t}.$$

From a simple geometric argument follows

$$(5) \qquad \sum_{n=1}^{x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + \gamma_k + o(1),$$

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and hence

$$\sum_{n=1}^{2x} \frac{\log^k n}{n} = \frac{(\log 2x)^{k+1}}{k+1} + \gamma_k + o(1). \tag{6}$$

Also

$$\begin{aligned} 2\sum_{n=1}^{x} \frac{(\log 2n)^k}{2n} &= \sum_{n=1}^{x} \frac{1}{n} \sum_{t=0}^{k} \binom{k}{t} (\log^{k-t} 2)(\log^t n) \\ &= \sum_{t=0}^{k} \binom{k}{t} \log^{k-t} 2 \sum_{n=1}^{x} \frac{\log^t n}{n} \\ &= \sum_{t=0}^{k} \binom{k}{t} \log^{k-t} 2 \left[\frac{\log^{t+1} x}{t+1} + \gamma_t + o(1)\right] \\ &= \frac{(\log 2x)^{k+1}}{k+1} - \frac{\log^{k+1} 2}{k+1} + \sum_{t=0}^{k} \gamma_t \binom{k}{t} \log^{k-t} 2 + o(1). \end{aligned} \tag{7}$$

Subtracting (7) from (6) gives

$$\sum_{n=1}^{2x} \frac{(-1)^{n-1} \log^k n}{n} = \frac{\log^{k+1} 2}{k+1} - \sum_{t=0}^{k-1} \gamma_t \binom{k}{t} \log^{k-t} 2 + o(1),$$

and we get, by letting $x \to \infty$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \log^k n}{n} = \frac{\log^{k+1} 2}{k+1} - \sum_{t=0}^{k} \gamma_{k-t} \binom{k}{t} \log^t 2. \tag{8}$$

From (3), (4), and (8) follows

$$k! \sum_{t=0}^{k+1} \frac{(-1)^t \log^t 2}{t!} A_{k-t} = (-1)^{k-1}\left[\frac{\log^{k+1} 2}{k+1} - \sum_{t=1}^{k} \gamma_{k-t} \binom{k}{t} \log^t 2\right],$$

and from this we easily get the

THEOREM. *If*

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n (s-1)^n,$$

then

$$A_n = \frac{(-1)^n \gamma_n}{n!},$$

where γ_n is defined in (5).

We give a second proof of the theorem. Write [**2**, p. 14]

$$\zeta(s) - \frac{s}{s-1} = \zeta(s) - \frac{1}{s-1} - 1 = h(s) = \sum_{n=0}^{\infty} B_n(s-1)^n$$
$$= s\int_1^{\infty} \frac{[x]-x}{x^{s+1}}\,dx \qquad R(s) > 0,$$

where $B_n = A_n$, unless $n=0$, where $B_0 = A_0 - 1$. Hence

$$\begin{aligned} h^{(k)}(1) &= k!B_k = (-1)^k\left[\int_1^{\infty}\frac{[x]-x}{x^2}\log^k x\,dx - k\int_1^{\infty}\frac{[x]-x}{x^2}\log^{k-1} x\,dx\right] \\ &= (-1)^k \lim_{N\to\infty}\left\{\sum_{t=1}^{N-1}\left[t\int_t^{t+1}\frac{\log^k x - k\log^{k-1} x}{x^2}\,dx\right.\right. \\ &\quad \left.\left. - \int_1^N \frac{\log^k x - k\log^{k-1}x}{x}\,dx\right\}\right. . \end{aligned} \tag{9}$$

It is easily seen that

$$\int_t^{t+1}\frac{\log^k x - k\log^{k-1}x}{x^2}\,dx = -\frac{\log^k x}{x}\Bigg|_t^{t+1} = -\frac{\log^k (t+1)}{t+1} + \frac{\log^k t}{t}.$$

Therefore

$$\begin{aligned} \sum_{t=1}^{N-1} t\int_t^{t+1}\frac{\log^k x - k\log^{k-1}x}{x^2}\,dx &= \sum_{t=1}^{N-1} t\left[\frac{\log^k t}{t} - \frac{\log^k(t+1)}{t+1}\right] \\ &= \sum_{t=1}^{N-1}\frac{\log^k t}{t}[t-(t-1)] - \frac{N-1}{N}\log^k N \\ &= \sum_{t=1}^{N-1}\frac{\log^k t}{t} - \frac{N-1}{N}\log^k N. \end{aligned}$$

$$\int_1^N \frac{\log^k x - k\log^{k-1}x}{x}\,dx = \frac{\log^{k+1}N}{k+1} - \log^k N.$$

Therefore, for $k>0$,

$$\begin{aligned} k!B_k &= (-1)^k\lim_{N\to\infty}\left[\sum_{t=1}^{N-1}\frac{\log^k t}{t} - \left(1-\frac{1}{N}\right)\log^k N - \frac{\log^{k+1}N}{k+1} + \log^k N\right] \\ &= (-1)^k\lim_{N\to\infty}\left[\sum_{t=1}^{N}\frac{\log^k t}{t} - \frac{\log^{k+1}N}{k+1}\right] = (-1)^k\gamma_k, \end{aligned}$$

where the limit is evaluated by (5). For $k=0$ the integral in (9) is similarly evaluated to give $B_0 = \gamma_0 - 1$.

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THE NUMBER OF VERTICES OF A POLYHEDRON

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1. Introduction. Consider the following (linear programming) problem: Find values of $x_1, \cdots, x_n$ which maximize (minimize) the linear form

(1) $$x_1c_1 + \cdots + x_nc_n \quad \text{subject to the conditions that}$$

(2) $$x_j \geqq 0 \; (j = 1, \cdots, n) \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \geqq b_i \; (i = 1, \cdots, m)$$

where the a_{ij}, b_j, c_j are constants.

In general, a set of linear inequalities defines a convex set in affine space; n-dimensional affine space A_n is the set of points $(c_1, c_2, \cdots, c_n)$ where $c_1, c_2, \cdots, c_n$ are real numbers. A set is convex in affine space if it contains the segment joining any two of its points. The convex set defined by (2) is the intersection of half spaces. It has vertices, edges, *etc.* Geometrically, a solution to the above problem is a point of the convex set defined by (2) which also maximizes (minimizes) (1). This point is generally a vertex of the boundary. For simplicity assume that (1) and (2) are normalized. Then this is the vertex $(x_1^0, x_2^0, \cdots, x_n^0)$ for which (1) assumes its maximum (or minimum) distance from the origin.

Briefly, the "simplex process" is one of several iterative procedures used in solving this problem. The iterations of this process consist of translating the hyperplane corresponding to (1) in a parallel direction each time evaluating its distance from the origin at the intersected vertex of the convex set defined by the inequalities. The process is repeated in such a way that the translated hyperplane in each step yields improved results leading to the optimal distance. The simplex process also involves a useful criterion which eliminates some of the vertices of the convex set as possible trial points [3].

The number of iterations involved is decided for each problem separately. Unfortunately, so far, this number is only known to be dominated by $C_{n+m,n}$ and as will be seen below this is an unsatisfactory estimate of the number of vertices of the convex sets of this problem. One would like an upper bound to the number of trial vertices knowing the number of inequalities and variables involved (the number of variables determines n, the dimension of the space). Hence the upper bound we wish to study here is of the form:

(number of vertices)$_n \leqq$ (a function of the number of inequalities)$_n$, n being the dimension of the space. Since each inequality defines at most a half-space with a hyperplane of dimension $(n-1)$ as boundary, it suffices to study this problem in general in the form: (number of vertices)$_n \leqq$ (a function of the number of $(n-1)$ dimensional hyperplanes)$_n$.

A rough upper bound which does not always take into consideration the polyhedral [1] property of the problem is available. If we denote by $F_i (i=0, 1, \cdots, n-1)$ the number of ith dimensional faces of a convex polyhedron in n-dimensional affine space then we have,

$$F_0 \leqq \frac{F_{n-1}!}{(F_{n-1} - n)!\,n!}, \tag{3}$$

which is obtained by intersecting the $(n-1)$ dimensional hyperplanes n at a time.

Digressing for a moment, let us consider the tetrahedron in n-dimensional space (the n-dimensional simplex). The number of its kth dimensional faces F_k is given by

$$F_k = \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!}, \qquad (k = 0, 1, \cdots, n). \tag{4}$$

Now it is a known fact (proof by induction) that the simplex is a minimal polyhedron, for any value of the dimension n, in that it has the least number of faces $F_i (i=0, \cdots, n-1)$ of any polyhedron of that dimension.

Remark 1: Since $F_0 = n+1$, and $F_{n-1} = n+1$, for the simplex, the latter being a minimal polyhedron, a desirable upper bound for F_0 as a function of F_{n-1} would yield equality on substituting $(n+1)$ for F_{n-1}. Now (3) satisfies this property.

The object here is to improve on (3) by a closer study thus obtaining a function $f(F_{n-1})$ with the property that

$$F_0 \leqq f(F_{n-1}) \leqq \frac{F_{n-1}!}{(F_{n-1} - n)!\,n!}. \tag{5}$$

(Note that the second equality cannot be avoided because of the above remark.)

2. Solution. The Euler-Descartes formula

$$\sum_{i=0}^{n-1} (-1)^i F_i = \begin{matrix} (0,\ n \text{ even}) \\ (2,\ n \text{ odd}) \end{matrix} \tag{6}$$

holds for polyhedra in n-dimensions that are homeomorphic images of the simplex in that dimension. We need not go into details here. It suffices to point out that the convex sets involved in our problem satisfy (6) or can be made to do so on minor changes. Thus the inequalities in (2) define a convex set, which is a polyhedron, perhaps unbounded, in n-space with a boundary of at most $m+n$ hyperplanes. Hence for our purpose $F_{n-1} = m+n$.

An easy proof of (6) for the simplex is obtained by considering the n-dimensional simplex which according to (4) has $(n+1)$ vertices. By combining these vertices two at a time, three at a time, *etc.*, the number of edges, planes, *etc.*, of the simplex is obtained. If grouped as in (6), the quantities on the right are obtained [1].

The following relations, also used here, are:

$$nF_0 \leqq 2F_1 \tag{7}$$

(*e.g.*, for $n=3$ at least 3 lines meet in a vertex with each line joining two vertices),

$$nF_{n-1} \leqq 2F_{n-2}, \tag{8}$$

$$F_k \leqq \frac{F_{n-1}!}{(F_{n-1}-n+k)!(n-k)!}. \tag{9}$$

(This relation suffers from similar defects mentioned above.)

$$F_k \geqq \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!}. \tag{10}$$

(See (4) and the statement following it.) Note that (7) and (9) yield

$$\frac{n}{2}F_0 \leqq F_1 \leqq \frac{F_{n-1}!}{(F_{n-1}-n+1)!(n-1)!}. \tag{11}$$

Now

$$\frac{2}{n}\,\frac{F_{n-1}!}{(F_{n-1}-n+1)!(n-1)!} \leqq \frac{F_{n-1}!}{(F_{n-1}-n)!n!} \tag{12}$$

since $F_{n-1} \geqq n+1$ with strict inequality implied if $F_{n-1} > n+1$. Thus the left hand side of (12) is a better upper bound for F_0 than (3).

In (6) and (7) let $n=3$. Multiply (6) by (-1) and substitute for F_1 in (6) the majorized quantity $\frac{3}{2}F_0$ and the following is obtained:

$$F_0 \leqq 2F_2 - 4 \tag{13}$$

which for $F_2 \geqq 4$ is dominated by the right side of (3) with $n=3$.

Now let $n=3$ and $F_2=10$ be assumed, then (3) yields $F_0 \leqq 10!/7!3! = 120$ while from (13) we have $F_0 \leqq 16$ (at most 16 iterations as compared to 120).

Let $n=4$ in (6), (7) and (9). We have from (9)

$$F_2 \leqq \tfrac{1}{2}F_3(F_3 - 1). \tag{14}$$

Substituting from (7) in (6), the following result is obtained

$$F_0 - 2F_0 - F_2 - F_3 \geqq 0 \quad \text{or} \quad F_0 \leqq F_2 - F_3. \tag{15}$$

Instead of F_2 we substitute the majorizing quantity from (14) and simplify. Thus (15) becomes

$$F_0 \leqq \frac{F_3}{2}(F_3 - 3), \tag{16}$$

which for $F_3 \geqq 5$ is dominated by the right side of (3) with $n=4$. Here if we assume $F_3=10$, (16) yields 35 as an upper bound whereas (3) gives 210. Note that (5) is satisfied in both cases.

Generalization.

A. n even:

We shall first state the result and then indicate how it is obtained:

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-2/2}\binom{F_{n-1}}{n-2p} - F_{n-1} - 2^n + \left[2 + \frac{n(n+3)}{2}\right]\right\}. \tag{17}$$

Since n is even (6) is given by

$$F_0 - F_1 + F_2 - F_3 + F_4 - \cdots + F_{n-2} - F_{n-1} = 0. \tag{18}$$

Using (17) we have

$$F_0 - \frac{n}{2}F_0 + F_2 - F_3 + F_4 - \cdots + F_{n-2} - F_{n-1} \geqq 0 \tag{19}$$

or

$$\frac{n-2}{2}F_0 - F_2 + F_3 - F_4 + \cdots - F_{n-2} + F_{n-1} \leqq 0. \tag{20}$$

Substitute for the F_i with positive signs (except F_0 and F_{n-1}) majorized quantities from (10) and for those with negative signs majorized quantities from (9) after multiplying the latter by (-1). The sum of the quantities obtained from the first substitution is the negative of the last two expressions to the right of (17) having made use of the expansion of $(1+1)^n-(1-1)^n$. The second substitution yields the negative of the first expression to the right of (17). By transposing $+F_{n-1}$ to the right, the second term on the right of (17) is obtained.

B. n odd:

Do the same as above with the exception that no substitution is made for F_{n-2}. The reason for this is that when transposed to the right it appears with a negative sign and from (8) it can be majorized by $\frac{1}{2}nF_{n-1}$ which is a better substitution. In the first of the two expressions below this exception is not made, but it is in the second:

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-3/2}\binom{F_{n-1}}{n-2p} + F_{n-1} - 2^n + \frac{(n-1)n}{2}\right\} \tag{21}$$

$$F_0 \leqq \frac{2}{n-2}\left\{\sum_{p=1}^{n-3/2}\binom{F_{n-1}}{n-2p} - \frac{n-2}{2}F_{n-1} - 2^n + (n-1)n\right\}. \tag{22}$$

That remark (1) is true in each case can be easily verified.

Remark 2: It is not difficult to prove that (17) and (22) are dominated by the right side of (3) for $F_{n-1} \geqq 2n$.

Remark 3: It is clear that a value of F_{n-1} as a function of n exists such that

$$\frac{2(F_{n-1}!)}{(F_{n-1} - n + 1)!n!}$$

majorizes (17) and (22) for all values of F_{n-1} greater than or equal to this value.

THEOREM 1. *There exists no upper bound on F_0 of the form $aF_{n-1}+b$, with a and b as functions of n, satisfying*

$$F_0 \leqq aF_{n-1} + b \leqq 2 \cdot \frac{F_{n-1}!}{(F_{n-1} - n + 1)!n!} \tag{23}$$

for all $F_{n-1} \geqq n+1$ and for $n=4$, or $n \geqq 7$.

Proof. Since the two extremes in (23) yield equality for the simplex, we must have:

$$n + 1 = (n + 1)a + b. \tag{24}$$

Suppose $b=0$; then $a=1$ and it is clear that $F_0 \leqq F_{n-1}$ is not true in general (*e.g.*, the hypercube has $F_0=2^n$ and $F_{n-1}=2n$). Thus

$$b = (n + 1) - (n + 1)a. \tag{25}$$

It is possible to obtain a convex n-dimensional polyhedron with $F_{n-1}=n+2$ by cutting the (n) simplex with an $(n-1)$ dimensional hyperplane. Hence from (23) and (25) the following must hold:

$$(n + 2)a + [(n + 1) - (n + 1)a] \leqq 2\frac{(n + 2)!}{3!n!} = \frac{(n + 2)(n + 1)}{3} \tag{26}$$

or

$$a \leqq \frac{(n + 1)(n - 1)}{3}. \tag{27}$$

Using the equality sign in (27) it will be seen that even with this value of a, the assumed upper bound does not hold. Thus for the hypercube one must have

$$2^n \leqq \frac{(n + 1)(n - 1)}{3} \cdot 2n + (n + 1) - \frac{(n + 1)^2(n - 1)}{3} \tag{28}$$

which is false for $n \geqq 7$.

The proof for $n=4$ follows since one of the regular polyhedra in 4 dimensions has $F_0=600$ and $F_3=120$.

At the outset it was our hope to prove that an upper bound as a linear function of F_{n-1} with coefficients as a function of n existed.

The results obtained here, though more complicated than the fundamental upper bound mentioned early in the paper, comprise a first step in producing a

finer upper bound [2]. It is possible that a generalization of the above theorem would show that finer upper bounds than those obtained here do not exist.

These improved results, as has been shown in the cases of $n=3$ and $n=4$, are valuable in estimating the number of iterations and hence the time that an electronic computor would require to solve the problem.

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NOTE ON A HEAT CONDUCTION PROBLEM

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1. Introduction. Rainville [4] using a new method solved the problem of the heat conduction in a wedge of angle β when the sides are maintained at temperature zero and the initial temperature was unity. The solution in a different form was in effect obtained by Carslaw [1] and the result has been quoted by Jaegar [3], Carslaw and Jaegar [2], and Sowerby and Cooke [5]. The purpose of this note is to give a method of obtaining the result by the use of Integral Transforms. The procedure is formal but it illustrates the great utility of the Integral Transform method of *obtaining a result*, and, as Tranter [6] pointed out, it reduces the procedure almost to a "drill." Follow the correct procedure and the result comes out automatically. The great advantage is that *no assumption of the form of the solution need be made.*

2. The problem. We have to solve the equation

$$\kappa^{-1}u_t = u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} \tag{1}$$

where subscripts denote partial differentiation, u is the temperature, κ the diffusivity and r and θ are cylindrical coordinates. The boundary conditions are:

$$\begin{aligned} &u \to 1 \quad \text{as} \quad t \to 0+ \quad \text{for} \quad 0 < r, \quad 0 < \theta < \beta, \\ &u \to 0 \quad \text{as} \quad \theta \to 0+ \quad \text{for} \quad 0 < t, \quad 0 < r, \\ &u \to 0 \quad \text{as} \quad \theta \to \beta- \quad \text{for} \quad 0 < t, \quad 0 < r. \end{aligned}$$

For convenience we shall write $u=1-w$ so that the boundary conditions are, shortly,

$$\begin{aligned} &w = 0 \quad \text{when} \quad t = 0, \\ &w = 1 \quad \text{when} \quad \theta = 0 \quad \text{or} \quad \theta = \beta. \end{aligned}$$

3. Integral Transforms. We shall use the finite Fourier sine transform defined by

$$\bar{f}(p) = \int_0^\pi f(\phi) \sin p\phi d\phi$$

where p is integral. The inverse formula is

$$f(\phi) = (2/\pi) \sum_{p=0}^{\infty} \bar{f}(p) \sin \phi p. \tag{2}$$

We shall also use the Hankel Transform defined by

$$\bar{f}(p) = \int_0^\infty r f(r) J_s(pr) dr$$

whose inverse is

$$f(r) = \int_0^\infty p\bar{f}(p) J_s(rp) dp. \tag{3}$$

In our case we shall denote the finite sine transform of w by $\bar{w}$, and the Hankel transform of $\bar{w}$ by $\bar{w}'$ following the notation of Tranter [6].

4. The solution. In equation (1) we write $\theta = \beta\phi/\pi$ and so the equation to solve is

$$\kappa^{-1} w_t = w_{rr} + (1/r) w_r + (\pi^2/r^2\beta^2) w_{\phi\phi} \tag{4}$$

with boundary conditions

$$w = 0 \quad \text{when} \quad t = 0, \tag{5}$$

$$w = 1 \quad \text{when} \quad \phi = 0 \quad \text{or} \quad \phi = \pi. \tag{6}$$

Multiply equation (4) by $\sin p\phi$ and integrate between 0 and π. Performing a double integration by parts on the last term we obtain, observing that w_ϕ must be finite and taking account of the boundary conditions (6),

$$\kappa^{-1}\bar{w}_t = \bar{w}_{rr} + (1/r)\bar{w}_r - \bar{w}\pi^2 p^2/\beta^2 r^2 + (\pi^2 p/\beta^2 r^2)\{1 - (-1)^p\}.$$

We shall see later that p must be an odd integer and so this equation reduces to

$$\kappa^{-1}\bar{w}_t = \bar{w}_{rr} + (1/r)\,\bar{w}_r - s^2\bar{w}/r^2 + 2\pi s/\beta r^2 \tag{7}$$

on writing $s = \pi p/\beta$.

We now multiply equation (7) by $rJ_s(qr)$ and integrate to obtain the Hankel transform. Two integrations by parts are again required. Omitting the details of the integration, which are given by Tranter and are straightforward, we obtain

$$\kappa^{-1}\bar{w}'_t = -q^2\bar{w}' + (2\pi s/\beta)\int_0^\infty r^{-1}J_s(qr)dr. \tag{8}$$

Now (Watson [**7a**]) the integral in this equation is equal to $1/s$. Hence

$$\kappa^{-1}\bar{w}'_t = -q^2\bar{w}' + 2\pi/\beta. \tag{9}$$

We now solve this equation in the one independent variable t, taking account of the boundary condition (5) which gives $\bar{w}'=0$ when $t=0$. We obtain

$$\bar{w}' = (2\pi/\beta q^2)(1 - e^{-q^2t\kappa}).$$

On inverting this by equation (3) we obtain

$$\bar{w} = (2\pi/\beta)\int_0^\infty q^{-1}J_s(qr)dq - (2\pi/\beta)\int_0^\infty e^{-q^2t\kappa}q^{-1}J_s(qr)dq$$

or

$$\bar{w} = (2/p) - (2\pi/\beta)I_s,$$

writing

$$I_s = \int_0^\infty e^{-q^2t\kappa}q^{-1}J_s(qr)dq.$$

Inverting again by equation (2) we obtain

$$w = \frac{4}{\pi}\sum_p \frac{\sin p\phi}{p} - \frac{4}{\beta}\sum_p I_s \sin p\phi.$$

Remembering that p is odd, say $p=2n+1$, we thus finally obtain on summing the sine series

$$w = 1 - (4/\beta)\sum_{n=0}^\infty \sin k\theta \int_0^\infty e^{-q^2t\kappa}J_k(qr)q^{-1}dq,$$

where $k=(2n+1)\pi/\beta$.

This gives

$$u = \frac{4}{\beta}\sum_{n=0}^\infty \sin k\theta \int_0^\infty \frac{e^{-q^2\kappa t}J_k(qr)}{q}dq,$$

which reduces to the form given by Rainville on expressing the integral in terms of hypergeometric functions. See Watson [**7b**]. (There are two slight errors of detail in Rainville's result. The power of $\rho^2/4h^2t$ in his expression for $G_n(\rho)$ should be $(2n+1)\pi/2\beta$ instead of $(2n+1)\pi/\beta$, and the term $2n+1$ appearing in his formula for a_n should be in the denominator instead of the numerator.)

Note 1. For even values of p the last term of equation (7) vanishes and so

there is no last term in equation (9). It is then not possible to solve this equation and satisfy the boundary condition $\bar{w}'=0$ when $t=0$. Hence p must be odd.

Note 2. Equation (8) was obtained on the assumption that $[r\bar{w}J_s'(qr)]_0^\infty$ and $[r\bar{w}_rJ_s(qr)]_0^\infty$ vanish at both limits. This assumption must be verified *a posteriori*. Alternatively since the method aims at obtaining a *result*, and is purely formal, the result must be justified directly in any case. This can easily be done.

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ON QUASI-IDEMPOTENT MATRICES*

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Introduction. Let A be a square matrix over the field of complex numbers C, and let $F(x)$ be a polynomial matrix of the same order over the ring $C[x]$ of polynomials in the indeterminate x with complex coefficients. If

$$A^r = F(r)$$

for all positive integers r, we will say that A is *quasi-idempotent* and that $F(x)$ is an *exponential polynomial matrix* associated with A. It is clear that any idempotent matrix P is a quasi-idempotent matrix associated with the polynomial matrix P. An example of a quasi-idempotent matrix A which is not idempotent is given by the A and the $F(x)$ displayed below:

$$A = \begin{pmatrix} -9 & 48 & -18 \\ -1 & 5 & -2 \\ 3 & -16 & 6 \end{pmatrix}; \qquad F(x) = \begin{pmatrix} -12x+3 & 48x & -24x+6 \\ -x & 4x+1 & -2x \\ 4x-1 & -16x & 8x-2 \end{pmatrix}.$$

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Other examples of such matrices are found in the literature [**2**, **3**, **4**], although this property does not seem to have been studied explicitly. In this paper, a miniature theory of quasi-idempotent matrices is developed by elementary methods.

An examination of the example above is illuminating in predicting what may be expected of quasi-idempotent matrices in general. It is easy to verify that $F(1/2)$ is a square root of A and it is indeed true that $F(1/n)$ is an n-th root of A for any positive integer n. Moreover, A does not have an inverse in the ordinary sense (the third column is the double of the first) and $F(0)$ is not the identity matrix, E. Nevertheless, it may be verified by direct computation that:

$$[F(0)]^2 = F(0), \qquad F(0)A = AF(0) = A, \quad \text{and} \quad F(-1)A = AF(-1) = F(0).$$

This means that A and $F(-1)$ generate a multiplicative group with identity $F(0)$ and that $F(-1)$ is the inverse of A in this group. In the sequel, if A is a matrix and B is an idempotent matrix such that $B_0A = AB_0 = A$, we will say that *B_0 is an identity for A.*

The arguments in Section 1 depend on the natural isomorphism [**1**, **5**] between the ring $(C[x])_n$ of square polynomial matrices of order n over the ring $C[x]$ of polynomials in the indeterminate x with coefficients in C, and the ring $C_n[X]$ of matric polynomials in the indeterminate X with coefficients in the ring C_n of square matrices of order n over C. In this isomorphism, if $F(x) \in (C[x])_n$ and the element in the i-th row and j-th column of $F(x)$ is $\sum_k a_{ij,k}x^k$, then the image $F^*(X)$ is defined to be $\sum_k A_kX^k$, where A_k is the matrix with $a_{ij,k}$ in the i-th row and j-th column. In the arguments below, $F(x)$ and $F^*(X)$ will be used to designate an image pair in this isomorphism. A similar notation will be used in the similar isomorphism between $(C[x, y])_n$ and $C_n[X, Y]$.

If the matric polynomial $F^*(X)$ is the image of the polynomial matrix $F(x)$ in the natural isomorphism and r is a positive integer, then $F^*(rE) = F(r)$. It follows that a matrix A is quasi-idempotent if and only if there is a matric polynomial $F^*(X)$ such that $A^r = F^*(rE)$ for all positive integers r. Such a matric polynomial will be called an exponential matric polynomial and the original problem will be studied in this form.

1. The characterization of exponential matrix polynomials. A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $[F^*(E)]^r = F^*(rE)$ for all positive integers r. A natural direct argument extends this to the following:

THEOREM 1. *A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $F^*(rE)F^*(sE) = F^*(rE+sE)$ for all pairs of positive integers r and s.*

Now let $F^*(X)$ be an exponential matric polynomial. The fact that scalar matrices commute with all matrices implies that $F^*(rE)F^*(sH)$ is equal to $[F^*(X)F^*(Y)]_{X=rE,Y=sE}$. That is, if $F^*(X)$ satisfies the condition in Theorem 1, then the matric polynomial $F^*(X)F^*(Y)$ takes on the same value as the matric

polynomial $F^*(X+Y)$ for each pair rE, sE, where r and s are any positive integers. This in turn implies that the polynomial matrix $F(x)F(y)$ takes on the same values as the polynomial matrix $F(x+y)$ for all pairs of positive integers r and s. Hence, the polynomial in the i-th row and j-th column of $F(x)F(y)$ takes on the same values as the polynomial in the i-th row and j-th column of $F(x+y)$ for all pairs of positive integers r, s. For each i and j, these polynomials are equal and $F(x)F(y)=F(x+y)$. Finally, it follows from the isomorphism that $F^*(X)F^*(Y)=F^*(X+Y)$. Since the converse implication is clear, we have:

THEOREM 2. *A matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $F^*(X)F^*(Y)=F^*(X+Y)$.*

Comparing coefficients results in:

THEOREM 3. *If $F^*(X)=B_0+B_1X+B_2X^2+\cdots+B_nX^n$, $B_n\neq 0$, then $F^*(X)$ is an exponential matric polynomial if and only if*

$$B_iB_j = B_{i+j}\binom{i+j}{i}, \qquad \textit{where } B_t = 0 \textit{ for } t > n.$$

2. The characterization of quasi-idempotent matrices. If $F^*(X)$ and $G^*(X)$ are matric polynomials and $F^*(rE)=G^*(rE)$ for all positive integers r, then $F^*(X)=G^*(X)$. Thus if A is quasi-idempotent, there is a unique exponential matric polynomial associated with A. Let A be quasi-idempotent and let

$$F^*(X) = B_0 + B_1X + B_2X^2 + \cdots + B_nX^n, \qquad B_n\neq 0,$$

be the exponential matric polynomial associated with A. Since the matrices B_i satisfy the condition in Theorem 3, B_0 is an identity for $A=F^*(E)$. The same condition (Theorem 3) also implies that $B_iB_j=B_jB_i$ and hence that the right side of

$$(A - B_0)^r = (B_1 + B_2 + \cdots + B_n)^r$$

may be evaluated by the multinomial theorem. For $r=n$ and $n+1$, the formulas in Theorem 3 yield:

$$(A - B_0)^n = (n!)B_n \neq 0 \quad \text{and} \quad (A - B_0)^{n+1} = 0.$$

Thus, if a quasi-idempotent matrix A is associated with an exponential matric polynomial of degree n, then there exists an idempotent matrix B_0 which is an identity for A and such that $A-B_0$ is nilpotent of index $n+1$.

On the other hand, let A be a matrix for which there is an idempotent matrix B_0, which is an identity for A, and such that $A-B_0$ is nilpotent of index $n+1$. Consider the matric polynomial $F^*(X)$ given below:

$$(1)\quad B_0 + (A - B_0)X + [(A - B_0)^2/2!]X(X - E) + \cdots + [(A - B_0)^n/n!]X(X - E)\cdots(X - nE + E).$$

Since $A^r = [B_0 + (A - B_0)]^r$, it follows from the binomial theorem and the nilpotency of $A - B_0$ that $F^*(rE) = A^r$ for every positive integer r. Hence, A is quasi-idempotent and $F^*(X)$ is the unique exponential matric polynomial associated with A. The unicity of $F^*(X)$ assures that a quasi-idempotent matrix A has associated with it a unique matrix B_0 which is an identity for A and such that $A - B_0$ is nilpotent.

THEOREM 4. *A matrix A is quasi-idempotent if and only if there is an idempotent matrix B_0 which is an identity for A and such that $A - B_0$ is nilpotent. The associated exponential matric polynomial $F^*(X)$ is computed by* (1) *and its degree is one less than the index of nilpotency of $A - B_0$.*

Of course, the associated exponential polynomial matrix $F(x)$ is the image of the associated exponential matric polynomial $F^*(X)$ in the natural isomorphism. For a given matrix A, it might be difficult to establish the existence or nonexistence of an idempotent matrix B_0 satisfying the conditions of Theorem 4. This may be remedied as follows:

If the matrix B_0 is an identity for the matrix A, consider the identity

$$(*) \qquad (B_0 - A)^k - (E - A)^k = B_0 - E, \qquad \text{for any positive integer } k.$$

If the matrix A is quasi-independent and B_0 is the associated idempotent matrix such that $(A - B_0)^{n+1} = 0$, then it follows from (*) that $B_0 = E - (E - A)^{n+1}$. If both sides of this equation are multiplied on the left by A, the resulting equation is equivalent to $A(E - A)^{n+1} = 0$. Hence, if the matrix A is quasi-idempotent and is associated with an exponential matric polynomial of degree n, then $A(E - A)^{n+1} = 0$.

Conversely, let A be a matrix such that $A(E - A)^k = 0$ for some positive integer k and define B_0 by the equation $B_0 = E - (E - A)^k$. If a matrix C has the property that it can be written in the form AD, where D commutes with A, then it is easily verified that $CB_0 = B_0C = C$. Since the matrix B_0 has the property required of C, then $B_0B_0 = B_0$ and B_0 is idempotent. Since A also has the property required of C, then B_0 is an identity for A. Finally, it follows from the identity (*) and $B_0 = E - (E - A)^k$ that $(B_0 - A)^k = 0$ and hence, by Theorem 4, that A is quasi-idempotent.

THEOREM 5. *A matrix A is quasi-idempotent if and only if $A(A - E)^k = 0$ for some positive integer k. If n is a positive integer such that $A(A - E)^n \neq 0$* and *$A(E - A)^{n+1} = 0$, then $B_0 = E - (E - A)^{n+1}$ is the constant term in the exponential matric polynomial $F^*(X)$ associated with A and $F^*(X)$ may be computed by* (1).

The integer n of Theorem 5 will be called the *index* of the quasi-idempotent matrix A. It is the degree of the associated matric polynomial and is one less than the index of nilpotency of $A - B_0$.

3. Computation of the exponential matric polynomial associated with a quasi-idempotent matrix. If A is quasi-idempotent of index n, the constant term

in the associated exponential matric polynomial is given by the formula $B_0 = E-(E-A)^{n+1}$. The coefficient of X in $F^*(X)$ is determined from (1) to be:

$$(2)\quad B_1 = (A - B_0) - (A - B_0)^2/2 + (A - B_0)^3/3 + \cdots + (-1)^{n-1}(A - B_0)^n/n.$$

It follows directly from Theorem 3 that $B_i = B_1^i/i!$ for $i>0$. These give explicit formulas for computing the associated matric polynomial $F^*(X)$ and also lead naturally to the following notions. Since $A = F^*(E)$,

$$(3)\qquad A = B_0 + B_1 + B_1^2/2! + \cdots + B_1^n/n!.$$

Except for the appearance of B_0, which is an identity for A, in place of the universal identity matrix E and for the fact that these are finite sums, the formulas of (2) and (3) are the same as the usual series for the logarithmic and exponential functions of matrices. This formal resemblance suggests investigating the following definitions in light of the present ideas:

Definition 1. *If A is quasi-idempotent, then* $\ln A$ *is the coefficient of X in the unique exponential matric polynomial associated with A.*

Definition 2. *If N is a nilpotent matrix of index $n+1$ and B_0 is an identity for N, then* $\exp_{B_0} N = B_0+N+N^2/2!+ \cdots +N^n/n!$.

In this language, (2) and (3) now take the form:

Theorem 6. *If A is a quasi-idempotent matrix of index n and B_0 is the idempotent matrix associated with A, then*

$$\ln A = (A - B_0) - (A - B_0)^2/2 + \cdots + (-1)^{n-1}(A - B_0)^n/n$$
$$\text{and } A = \exp_{B_0}(\ln A).$$

Two examples will be given to show how the ideas in sections 1 and 2 are sufficient to prove that the functions defined have appropriate properties.

If B_0 is an identity for a nilpotent matrix N of index $n+1$, then the matric polynomial $G^*(X) = B_0+NX+(N^2/2!)X^2+ \cdots +(N^n/n!)X^n$ satisfies the characteristic conditions of Theorem 3 and is hence an exponential matric polynomial. The exponential matric polynomial $G^*(X)$ is associated with the quasi-idempotent matrix $G^*(E) = \exp_{B_0} N$, and the matrix $\ln(\exp_{B_0} N)$ is the coefficient of X in $G^*(X)$ or N.

Theorem 7. *If B_0 is an identity for a nilpotent matrix N of index $n+1$,* then $\exp_{B_0} N$ *is quasi-idempotent of index m and* $\ln(\exp_{B_0} N) = N$.

If A and C are commuting quasi-idempotent matrices associated with the exponential matric polynomials $F^*(X) = B_0+B_1X+ \cdots$ and $G^*(X) = D_0+D_1X+ \cdots$, respectively, then the assumption of commutativity implies that any B_i commutes with any D_j. Then for all positive integers r,

$$(AC)^r = A^rC^r = F^*(rE)G^*(rE) = [F^*(X)G^*(X)]_{X=rE}.$$

This means that $F^*(X)G^*(X)$ is the exponential matric polynomial associated with AC and hence that AC is quasi-idempotent. Moreover, B_0D_0 is the constant term in $F^*(X)G^*(X)$ and the coefficient of X is $B_0D_1+D_0B_1$ or $B_0 \ln C+D_0 \ln A$.

THEOREM 8. *If A and C are commuting quasi-idempotent matrices with associated idempotent matrices B_0 and D_0, then AC is a quasi-idempotent matrix associated with B_0D_0 and* $\ln AC=D_0 \ln A+B_0 \ln C$.

There is the immediate corollary:

COROLLARY. *If A and C are commuting quasi-idempotent matrices associated with the same idempotent matrix, then* $\ln AC$ *exists and* $\ln AC=\ln A+\ln C$.

It follows from Theorem 5 that the idempotent matrix associated with a non-singular quasi-idempotent matrix is the identity matrix E. Thus, any pair of commuting non-singular quasi-idempotent matrices satisfy the conclusion in the Corollary.

Concluding remarks. If A is a quasi-idempotent matrix with the associated idempotent matrix B_0, it follows directly from the ideas above that the set of matrices $G=\{\exp_{B_0} (s \ln A) \mid s \text{ is a complex number}\}$ is a multiplicative group containing A and that $\exp_{B_0} [(1/n) \ln A)]$ is an n-th root of A in G.

The proper role of quasi-idempotent matrices in the classical theory may be indicated as follows: If C is a group element in a multiplicative group, then the minimum function of C is of the form:

$$m(z) = \prod_i (z - c_i)^{e_i}, \quad i = 1, 2, \cdots, s; \qquad \text{with } c_i \neq 0,$$

or of the form $zm(z)$. In either case, C has a decomposition into a direct sum of the form

$$C = C_1 \oplus C_2 \oplus \cdots \oplus C_s,$$

where $z(z-c_i)^{e_i}$ is the minimum function of C_i. Hence, if $A_i=c_i^{-1}C_i$, then

$$C = c_1A_1 \oplus c_2A_2 \oplus \cdots \oplus c_sA_s,$$

where each A_i is quasi-idempotent of index e_i-1 and $A_iA_j=0$ for $i\neq j$. That is, any group element is a linear combination of mutually orthogonal quasi-idempotent matrices. The associated idempotent matrices are the principal idempotent elements (Frobenius covariants) of C.

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APPLICATIONS OF FAÀ DI BRUNO'S FORMULA IN MATHEMATICAL STATISTICS

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1. History of the problem solved by Faà di Bruno. The problem of finding an explicit expression for the p-th derivative of a function of a function is an old one. Let $z=G(y)$ and $y=f(x)$ be two functions such that all the derivatives of $G(y)$ and $f(x)$ up to order p exist. The problem consists then in finding a formula for

$$\frac{d^p z}{dx^p} = \frac{d^p}{dx^p} G[f(x)].$$

The need for such a formula was already stated in old treatises on calculus such as S. F. Lacroix's [11]; however no satisfactory solution was given. O. Schloemilch [16] gave a formula for the case where $f(x)=x^\lambda$ or where $f(x)=e^x$. By specializing his formulae he obtains many of the expressions for successive derivatives of algebraic functions which were known at his time and also some formulae ascribed to Euler and Laplace for the successive derivatives of $1/(e^x \pm a)$.

Reinhold Hoppe realized that the problem could be solved for all functions $f(x)$ for which the higher derivatives of $[f(x)]^k$ (k a positive integer) are explicitly known. In 1845 he published a monograph [7] on this subject and, at the request of the editor of Crelle's Journal, he also wrote a paper [8] indicating the contents of his book.

In 1870 R. Götting [5] considered the case $G(y)=y^k$ and derived a number of formulae for suitably specialized choices of $f(x)$. This induced Hoppe to restate his more general result in a brief note [9]. Hoppe's book was not accessible to the present writer. His information about Hoppe's work is based on the two papers [8], [9] mentioned above.

The first solution of the problem which carried no restrictions as to the functions $G(y)$ and $f(x)$ was given by Cavaliere Francesco Faà di Bruno in 1855. He published two almost identical papers, one in the *Annali di Tortolini* [2], the second in the *Quarterly Journal of Mathematics* [3]. In these papers he gave his formula essentially in the form in which we quote it in the next section. He also wrote the formula in the form of a determinant. About twenty years later Faà di Bruno published a book [4] on binary forms. The formula is given on page 4 while its proof is found in an appendix (pp. 304–305). R. Most [15] also wrote a brief paper on this subject. He quotes [5] and [9] but was apparently not aware of Faà di Bruno's work. He obtained a general formula. His method could also be applied in case G is a function of several variables each of which depends on x. The proof in Faà di Bruno's book is rather complicated. A simple proof by induction was given by L. Koenigsberger [10]. Another simple proof, assuming that both $G(y)$ and $f(x)$ are analytic in a neighborhood of the origin, is due to Fr. Meyer [14] who discussed also the algebraic character

of Faà di Bruno's formula. A good survey of the problem and its history may be found in section 24 of A. Voss' article [**17**] on calculus in the *Enzyklopädie*.

Faà di Bruno's formula is nowadays almost forgotten and rarely occurs in modern textbooks of Calculus. It may, however, be found in Goursat's *Cours d'Analyse* [**6**]. The purpose of the present paper is to revive interest in Faà di Bruno's formula and to show that it has worthwhile applications in mathematical statistics. Some of the formulae discussed in the subsequent sections are well known but usually presented without proof. One of the results (Theorem 3) has not yet been published.

2. The formula of Faà di Bruno. We assume now that $z=G(y)$ and $y=f(t)$ have derivatives of all orders up to order p and write for simplicity

$$f_v = \frac{1}{v!}\frac{d^v}{dt^v}f(t), \qquad f = f_0 = f(t). \tag{2.1}$$

Faà di Bruno's formula is then

$$\frac{d^pz}{dt^p} = \frac{d^p}{dt^p}G[f(t)] = \sum \frac{d^kG(y)}{dy^k}\,\frac{p!}{i_1!i_2!\cdots i_s!}f_{k_1}^{i_1}f_{k_2}^{i_2}\cdots f_{k_s}^{i_s}, \tag{2.2}$$

where the summation is to be extended over all partitions of p such that

$$\begin{cases} i_1 + i_2 + \cdots + i_s = k \\ i_1k_1 + i_2k_2 + \cdots + i_sk_s = p. \end{cases} \tag{2.3}$$

We do not intend to prove (2.2) here. The reader can easily establish this formula by induction or he may consult the references [**10**] or [**14**]. However, we list two particular cases which occur in the statistical applications. We put first $G(y)=\ln y$ so that

$$\frac{d^kG(y)}{dy^k} = (-1)^{k-1}(k-1)!y^{-k}$$

and obtain from (2.2) and (2.3)

$$\frac{d^p}{dt^p}\ln f(t) = \sum (-1)^{k-1}\frac{p!(k-1)!}{i_1!i_2!\cdots i_s!}(f_{k_1}/f)^{i_1}(f_{k_2}/f)^{i_2}\cdots(f_{k_s}/f)^{i_s}. \tag{2.4}$$

We next put $G(y)=e^y$ and write $\phi(t)$ instead of $f(t)$; then

$$\frac{d^p}{dt^p}e^{\phi(t)} = \sum e^{\phi(t)}\frac{p!}{i_1!i_2!\cdots i_s!}\phi_{k_1}^{i_1}\phi_{k_2}^{i_2}\cdots\phi_{k_s}^{i_s}. \tag{2.5}$$

In (2.4) and (2.5) the summation is to be taken over all partitions of p for which (2.3) holds.

3. Cumulants and moments. In the following sections we denote by $F(x)$ a distribution function, that is, a never decreasing function which is continuous to

the right and for which $F(-\infty)=0$, $F(+\infty)=1$. The Fourier transform

$$f(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x)$$

exists for any distribution function $F(x)$ and is called its characteristic function. It is easy to see that $f(t)$ is a continuous function of t and that $f(0)=1$. It follows then that a characteristic function does not vanish in some neighborhood of $t=0$. We may, therefore, consider its logarithm

$$\phi(t) = \ln f(t)$$

at least in this neighborhood. This function is sometimes called the cumulant generating function of the distribution $F(x)$. The number

$$\alpha_k = \int_{-\infty}^{+\infty} x^k dF(x)$$

is called the moment of order k of the distribution $F(x)$. It is known (see [1] chapter 10) that the existence of the moment of order k of a distribution implies that its characteristic function may be differentiated k times and that

$$\alpha_v = \frac{1}{i^v} f^{(v)}(0) \qquad v = 0, 1, 2, \cdots k. \tag{3.1}$$

If the moment of order k exists, then the cumulant generating function $\phi(t)$ is also k times differentiable. The quantities

$$\kappa_v = \frac{1}{i^v} \phi^{(v)}(0) \qquad v = 1, 2, \cdots k \tag{3.2}$$

are called the cumulants (sometimes semi-invariants) of the distribution $F(x)$.

As a first application of Faà di Bruno's formula, we state and prove the following proposition.

THEOREM 1. *Let $F(x)$ be a distribution function and assume that its moment of order m exists. The moments and cumulants of $F(x)$ are then connected by the relations*

$$\kappa_p = \sum (-1)^{k-1} \frac{p!(k-1)!}{i_1!(k_1!)^{i_1} \cdots i_s!(k_s!)^{i_s}} \alpha_{k_1}^{i_1} \cdots \alpha_{k_s}^{i_s} \tag{3.3}$$

and

$$\alpha_p = \sum \frac{p!}{i_1!(k_1!)^{i_1} \cdots i_s!(k_s!)^{i_s}} \kappa_{k_1}^{i_1} \cdots \kappa_{k_s}^{i_s} \tag{3.4}$$

for $p=1, 2, \cdots, m$. The summations are extended over all partitions of p satisfying (2.3).

The relation (3.3) follows immediately from (2.4), (3.1) and (3.2). Relation (3.4) is obtained from (2.5) if we let $\phi(t)$ be the cumulant generating function of $F(x)$. Then $f(t)=e^{\phi(t)}$ and we derive (3.4) by setting $t=0$ and considering again (3.1) and (3.2).

4. The k-statistics. The next application of Faà di Bruno's formula deals with certain sampling characteristics of populations. We consider a random variable and suppose that we have taken n independent observations $x_1, x_2, \cdots, x_n$ forming a sample of n. The $x_1, x_2, \cdots, x_n$ are then by assumption n independently and identically distributed random variables; we denote their common distribution function by $F(x)$ and call it the distribution function of the population under consideration. Populations are frequently studied by means of functions of a sample. It is customary to call a function of the observations a statistic.

We define the k-statistic of order p as a symmetric polynomial statistic which has the property that its expectation equals the p-th cumulant κ_p for any population distribution possessing the necessary moments. We denote the k-statistics of order p by $k_p=k_p(x_1, x_2, \cdots, x_n)$ and will use in the following the symbol $\mathbf{E}(\cdots)$ for the expectation.

This definition must be justified by demonstrating that it is possible to determine uniquely for every positive integer p a k-statistic k_p.

As a first step we show by means of an indirect proof that for each p at most one statistic can satisfy the requirements of the definition. Let us, therefore, assume tentatively that two statistics k_p and k_p^* satisfy the conditions of the definition. Then $k_p-k_p^*$ is a symmetric polynomial statistic such that $\mathbf{E}(k_p-k_p^*)=0$. If k_p and k_p^* are not identical, then this is a relation between the moments which is valid for any population distribution function. However, such a relation can not hold for an arbitrary population distribution. Thus the assumption that k_p and k_p^* are two different k-statistics of the same order leads to a contradiction.

Before giving an explicit expression for k_p we introduce some notations, which are used in combinatorial analysis for work with symmetric polynomials.

We write $(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})$ for the symmetric polynomial of the n variables $x_1, x_2, \cdots, x_n$ whose terms consist of the product of i_1 powers of degree k_1, i_2 powers of degree $k_2, \cdots, i_s$ powers of degree k_s, that is

$$(4.1)\qquad (k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}) = \sum x_1^{k_1}\cdots x_{i_1}^{k_1}x_{i_1+1}^{k_2}\cdots x_{i_1+i_2}^{k_2}\cdots x_k^{k_s},$$

where $i_1+i_2+\cdots+i_s=k$ is the number of factors in each term and where $i_1k_1+i_2k_2+\cdots+i_sk_s=p$ is the degree of the symmetric polynomial. Clearly there are

$$\frac{n(n-1)\cdots(n-k+1)}{i_1!i_2!\cdots i_s!} \text{ terms in the sum } (k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}),$$

and we have

$$(4.2)\qquad \mathbf{E}\{(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})\} = \frac{n(n-1)\cdots(n-k+1)}{i_1!i_2!\cdots i_s!}\alpha_{k_1}^{i_1}\cdots\alpha_{k_s}^{i_s}.$$

If we introduce

$$(4.3)\qquad [k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}] = i_1!i_2!\cdots i_s!(k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s})$$

then we have

$$(4.4)\qquad \mathbf{E}\{[k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}]\} = n(n-1)\cdots(n-k+1)\alpha_{k_1}^{i_1}\cdots\alpha_{k_s}^{i_s}.$$

From (4.4) and (2.4) we obtain easily an explicit expression for the k statistic of order p

$$(4.5)\qquad k_p = \sum(-1)^{k-1}\frac{p!(k-1)!\,[k_1^{i_1}k_2^{i_2}\cdots k_s^{i_s}]}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s}n(n-1)\cdots(n-k+1)}.$$

The summation is to be extended over all partitions of p satisfying (2.3).

It would have been possible to define the statistics k_p by (4.5); such a definition would depend only on the observations and would therefore be valid also for population distributions which do not have moments of order p. However, this generalization of the concept of k-statistics would be of little value since these statistics are interesting mainly on account of the relation

$$(4.6)\qquad \mathbf{E}k_p = \kappa_p$$

which we used in our original definition and which has meaning only if the p-th moment of $F(x)$ exists.

We show now an interesting property of the k-statistics which we shall use in the next section.

THEOREM 2. *The k-statistics are invariant under translations, that is for any real a we have*

$$(4.7)\qquad k_p(x_1+a, x_2+a, \cdots, x_n+a) = k_p(x_1, x_2, \cdots x_n).$$

To prove Theorem 2 we introduce the random variable $y=x+a$ and remark that for all orders exceeding one, x and y have the same cumulants. We consider now the sample $y_1\cdots y_n$ where $y_j=x_j+a$. This has a population distribution funtion $G(y)=F(y-a)$. Let $p>1$, then

$$\mathbf{E}\{k_p(x_1+a,\cdots,x_n+a)\} = \mathbf{E}\{k_p(y_1,\cdots,y_n)\} = \kappa_p.$$

Equation (4.7) follows then immediately from the last relation and from the uniqueness of the k-statistics.

5. A characterization of the normal distribution. In this section we characterize a population with a normal population distribution function by a property

of the k-statistics.

The distribution function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(y-\alpha_1)^2/2\sigma^2} dy$$

is called the normal distribution. Its characteristic function is given by $e^{it\alpha_1}e^{-\sigma^2 t^2/2}$. A population with a normal population distribution is called a normal population and can be characterized in the following manner:

THEOREM 3. *Let $x_1, x_2, \cdots, x_n$ be n independent observations taken from a population with distribution function $F(x)$ and denote by p an integer greater than one. Assume that the p-th moment of $F(x)$ exists. The population is normal if, and only if, the k-statistic of order p is independent of the sample mean.*

We first prove the sufficiency of the condition. The independence of k_p and the sample mean $(x_1+x_2+\cdots+x_n)/n$ means that the two statistics $Y=k_p$ and $L=x_1+x_2+\cdots+x_n$ are independent. This fact may be expressed in terms of the characteristic functions of these statistics. The characteristic function of Y and L can be written as $\mathbf{E}\{e^{itL+iuY}\}$; it is well known that the independence of L and Y may be expressed by means of the relation

$$\mathbf{E}\{e^{itL+iuY}\} = \mathbf{E}\{e^{itL}\}\mathbf{E}\{e^{iuY}\}. \tag{5.1}$$

The right side of (5.1) is the product of the characteristic functions of the random variables L and Y respectively. We remark that the relation (5.1) may be differentiated with respect to u because the p-th moment of $F(x)$ exists. We differentiate (5.1) with respect to u and put afterwards $u=0$ and obtain

$$\mathbf{E}\{Ye^{itL}\} = \mathbf{E}\{Y\}\mathbf{E}\{e^{itL}\} = \mathbf{E}\{Y\}[f(t)]^n. \tag{5.2}$$

We must next compute $\mathbf{E}\{Ye^{itL}\}$. Substituting for $Y=k_p$ the expression (4.5), we obtain

$$\mathbf{E}\{Ye^{itL}\} = \mathbf{E}\left\{\sum (-1)^{k-1} \frac{p!(k-1)!\,[k_1^{i_1}\cdots k_s^{i_s}]e^{itL}}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s} n(n-1)\cdots(n-k+1)}\right\}.$$

The sum is to be extended over all partitions of p satisfying (2.3). We interchange the operations of summation and expectation and obtain

$$\mathbf{E}\{Ye^{itL}\} = \sum (-1)^{k-1} \frac{p!(k-1)!\,\mathbf{E}\{[k_1^{i_1}\cdots k_s^{i_s}]e^{itL}\}.}{i_1!(k_1!)^{i_1}\cdots i_s!(k_s!)^{i_s}\, n(n-1)\cdots(n-k+1)}. \tag{5.3}$$

The existence of the p-th moment of $F(x)$ implies that the characteristic function $f(t)$ may be differentiated p times and that

$$\frac{d^r f(t)}{dt^r} = i^r \int_{-\infty}^{\infty} x^r e^{itx} dF(x), \qquad (r = 0, 1, \cdots, p).$$

From this it is easily seen that the typical term in the sum $[k_1^{i_1} \cdots k_s^{i_s}]$ has the same expectation

$$i^{-p}[f(t)]^{n-k}[f^{k_1}(t)]^{i_1} \cdots [f^{k_s}(t)]^{i_s}.$$

Since the $n(n-1) \cdots (n-k+1)$ terms of this sum have the same expectation, we obtain

$$\mathbf{E}\{[k_1^{i_1} \cdots k_s^{i_s}]e^{itL}\}$$
$$= n(n-1) \cdots (n-k+1) i^{-p}[f(t)]^{n-k}[f^{k_1}(t)]^{i_1} \cdots [f^{k_s}(t)]^{i_s}.$$

We substitute this into (5.3) and use again the notation introduced by (2.1) and obtain

$$(5.4) \quad \mathbf{E}\{Ye^{itL}\} = i^{-p}[f(t)]^n \sum (-1)^{k-1} \frac{p!(k-1)!}{i_1! \cdots i_s!} [f_{k_1}/f]^{i_1} \cdots [f_{k_s}/f]^{i_s}$$

where the summation is to be extended over all partitions of p for which (3.2) holds.

From (5.2), (5.4) and Faà di Bruno's formula (2.4) we obtain easily the differential equation

$$(5.5) \qquad \frac{d^p}{dt^p} \ln f(t) = i^p \kappa_p.$$

To obtain this differential equation we had to divide by $[f(t)]^n$. It is, therefore, valid in any region in which $f(t)$ is different from zero. The fact that $f(0) = 1$ and that $f(t)$ is continuous assures the existence of a neighborhood of the origin in which this condition is satisfied. We restrict our considerations first to such a neighborhood of $t=0$. Using the initial conditions suggested by (3.2), this equation can be integrated and we obtain

$$(5.6) \qquad f(t) = \exp\left\{\sum_{j=1}^{p} \kappa_j (it)^j / j!\right\}.$$

This solution is valid in any neighborhood of the origin in which $f(t)$ is different from zero. We show by an indirect proof that $f(t)$ has no real zero so that (5.6) is valid for all t.

Let us, therefore, assume that $f(t)$ has a zero and that t_0 is the zero closest to the origin. Then (5.6) holds for $t < t_0$ and we conclude from this and from the continuity of $f(t)$ that

$$\lim_{t \to t_0} f(t) = \exp\left\{\sum_{j=1}^{p} \kappa_j (it_0)^j / j!\right\}.$$

This, however, contradicts the assumption $f(t_0)=0$ so that the validity of (5.6) for all real t is proven.

All the solutions of the differential equations (5.5) are given by (5.6); however, not all these solutions are characteristic functions. It is still necessary to select from the solutions (5.6) those functions which are characteristic functions. This can be accomplished by means of the following result due to Marcinkiewicz [**13**].

THEOREM *of Marcinkiewicz. No function of the form* $\exp[a_0+a_1z+\cdots+a_rz^r]$ *with* $r>2$ *can be a characteristic function.*

The degree of the polynomial in (5.6) can, therefore, not exceed two, *i.e.*, $\kappa_j=0$ for $j>2$ and

$$f(t) = e^{i\kappa_1 t-\kappa_2 t^2/2}. \tag{5.7}$$

This is the characteristic function of the normal distribution and we conclude from the uniqueness theorem (see [**1**]) that the sufficiency of the condition of Theorem 2 is established.

The necessity of the condition follows from Theorem 2 and the well known fact that in a normal population every translation invariant statistic is independent of the mean.

Finally we should like to note that a remark in a paper by R. G. Laha [**12**] indicates that D. Basu and R. G. Laha obtained independently the result of Theorem 3 and that it will be published in *Sankhya*. The author of the present paper learned from Mr. Basu that their proof of Theorem 3 is in many respects similar to the one given above, but that it makes no use of Faà di Bruno's formula.

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A NOTE ON THE EVEN PERFECT NUMBERS

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1. The harmonic mean. Following Ore,* we define $H(n)$ to be the harmonic mean of the divisors of n, so that

$$\frac{1}{H(n)} = \frac{1}{\tau(n)} \sum_{d|n} \frac{1}{d} = \frac{1}{n\tau(n)} \sum_{d'|n} d' = \frac{\sigma(n)}{n\tau(n)},$$

where $\tau(n)$ is the number of divisors of n, $\sigma(n)$ is the sum of the divisors of n, and $dd'=n$. Thus we have

$$H(n) = \frac{n\tau(n)}{\sigma(n)}, \tag{1}$$

from which it is clear that $H(n)$ is multiplicative.

Computation shows that when p is a prime

$$H(p^a) = \frac{p^a(a+1)}{1+p+\cdots+p^a}, \qquad a \geqq 1. \tag{2}$$

When p is an odd prime and $a \geqq 2$, we may use (2) and an induction on a to show $H(p^a) > 2$. Similarly, when $p=2$ and $a \geqq 3$, we have $H(2^a) > 2$. When p and q are distinct primes, we have

$$H(pq) = \frac{4pq}{(p+1)(q+1)} \geqq 2,$$

for this inequality reduces to $(p-1)(q-1) \geqq 2$; we note that the equality holds only when $p=2$, $q=3$. When p is an odd prime,

* On the averages of the divisors of a number, O. Ore, this MONTHLY, vol. 55, 1948, pp. 614–19.

$$H(4p) = \frac{24p}{7(p+1)} > 2.$$

Combining these results with the fact that $H(n)$ is multiplicative, we find that $H(n) > 1$ when $n > 1$, and that $H(n) > 2$, except when $n = p$, a prime, and when $n = 1$, 4 and 6. In particular, we need the result that $H(n) > 2$ for every odd composite number.

2. Even perfect numbers. After Euclid and Euler we know that the even perfect numbers (defined by requiring $\sigma(n) = 2n$) are exactly those numbers of the form $n = 2^{q-1}(2^q - 1)$ where $p = 2^q - 1$ is prime. Ore has already noted that for such an even perfect number $H(n) = n(2q)/2n = q$.

The object of this note is to show the following converse.

THEOREM: *If a given integer n is even and has the form*

$$n = 2^{H(n)-1}(2^{H(n)} - 1), \tag{3}$$

then n must be a perfect number.

Proof: In view of the Euclid-Euler theorem it will suffice to prove in (3) that $P = 2^{H(n)} - 1$ is prime. Since n is assumed even, we know $H(n) > 1$. From (3) we compute

$$H(n) = \frac{2^{H(n)-1}H(n)}{2^{H(n)} - 1} H(P) > \frac{1}{2} H(n)H(P).$$

Hence $H(P) < 2$ and by the remark at the end of the first section, we know that P, although odd and greater than 1, is not composite.

MATHEMATICAL NOTES

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SUMS OF POWERS OF NUMBERS HAVING A GIVEN PERIOD MODULO m

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Raymond Moller [3, Th. 1] showed that if the numbers a_i are the numbers less than a prime p having period e' (mod p), then the sum of the nth powers

$$\sum a_i^n \equiv \frac{\phi(e')}{\phi(e'/(n, e'))} \mu(e'/(n, e')) \pmod{p},$$

where ϕ is the Euler ϕ-function and μ is the Möbius function. We proceed to

establish similar theorems for composite moduli.

THEOREM 1. *The sum of the positive integers less than p^s having period $e \equiv p^r e'$* (mod p^s), *where* $0 \leqq r < s$, $e' \mid (p-1)$, $s \geqq 1$, *and p is an odd prime is congruent to* $\phi(p^r)\mu(e')$ (mod p^s).

*Proof.** Let S_e denote the sum (mod p^s) of those residues (mod p^s) whose period is a divisor of e. Let $\sum_e$ denote the sum of the residues (mod p^s) whose period equals e. Then

$$S_e = \sum_{d \mid e} \sum_d. \tag{1}$$

By the Möbius inversion formula [2]

$$\sum_e = \sum_{d \mid e} \mu(e/d) S_d. \tag{2}$$

Let a be a primitive root (mod p^s). If $(d, p-1) \neq 1$,

$$S_d \equiv \sum_{k=0}^{d-1} a^{k\phi(p^s)/d} \equiv \frac{1 - a^{\phi(p^s)}}{1 - a^{\phi(p^s)/d}} \equiv 0 \pmod{p^s}. \tag{3}$$

If $d = p^t$, then

$$S_d = \sum_{k=0}^{p^t-1} a^{k\phi(p^{s-t})} \equiv \sum_{k=0}^{p^t-1} (1 + m_k p^{s-t}) \equiv p^t + p^{s-t} \sum_{k=0}^{p^t-1} m_k \pmod{p^s}$$

and $0 \leqq m_k < p^t$. But $m_j \neq m_k$, $j \neq k$, since assuming the contrary $a^{j\phi(p^{s-t})} \equiv a^{k\phi(p^{s-t})}$ (mod p^s) implies $a^{(j-k)\phi(p^{s-t})} \equiv 1$ (mod p^s) contrary to the hypothesis that a is primitive (mod p^s). Thus the m_k are the integers 0 to $p^t - 1$ in some order. Thus

$$S_d \equiv p^t + p^{s-t} p^t \frac{p^t - 1}{2} \equiv p^t \pmod{p^s}. \tag{4}$$

Substituting (3) and (4) in (2) and setting $e = e'p^r$ we have

$$\begin{aligned} \sum_e &= \sum_{d \mid e} \mu(e/d) S_d = \sum_{t \leqq r} \mu(e' p^{r-t}) S_{p^t} \\ &= \mu(e') p^r + \mu(e'p) p^{r-1} \equiv \mu(e')\phi(p^r). \end{aligned} \tag{5}$$

THEOREM 2. *The sum of the nth powers of the positive integers less than p^s having period $e = p^r e'$* (mod p^s), *where* $0 \leqq r < s$, $e' \mid (p-1)$, $s \geqq 1$, *and p is an odd prime, is congruent to*

$$\frac{\phi(e')}{\phi(e'/(n, e'))} \phi(p^r) \mu(e'/(n, e')) \pmod{p^s}.$$

Proof. The proof of Moller's Theorem 1 [3] holds as well for this case.

LEMMA 1. *For $s > 2$ and $0 < r < s-1$, the integers a_i belonging to* 2^r (mod 2^s),

* This proof was suggested by a referee.

in addition to 2^s-1 *for* $r=1$, *are* $a_i=\pm 1+2^{s-r}\sigma_i$, *where the* σ_i *are the positive odd numbers less than* 2^r.

Proof. For $r=1$, $(\pm 1+2^{s-1}\sigma_i)^{2^0}\not\equiv 1 \pmod{2^s}$ and

$$(\pm 1 + 2^{s-1}\sigma_i)^{2^1} = 1 \pm 2^s\sigma_i + 2^{2s-2}\sigma_i^2 \equiv 1 \pmod{2^s},$$

so that a basis is established.

State as induction hypothesis that the $\pm 1+2^{s-(r-1)}\sigma_i$ belong to $2^{r-1} \pmod{2^s}$. Now

$$(\pm 1 + 2^{s-r}\sigma_i)^2 = 1 \pm 2^{s-r+1}\sigma_i + 2^{2s-2r}\sigma_i^2 = 1 \pm 2^{s-(r-1)}(\sigma_i \pm 2^{s-r-1}\sigma_i^2)$$

$$\equiv 1 + 2^{s-(r-1)}\sigma_{i'}$$

or

$$1 - 2^{s-(r-1)}(2^{r-1} - \sigma_{j'}) \equiv 1 + 2^{s-(r-1)}\sigma_{j'}.$$

By the induction hypothesis, $1+2^{s-(r-1)}\sigma_{j'}$ belongs to $2^{r-1} \pmod{2^s}$. Then $(\pm 1+2^{s-r}\sigma_i)^{2^r}\equiv(1+2^{s-(r-1)}\sigma_{j'})^{2^{r-1}}\equiv 1 \pmod{2^s}$ and $(\pm 1+2^{s-r}\sigma_i)^{2^t}\equiv 1$ for $t<r$ would imply that $(1+2^{s-(r-1)}\sigma_{j'})^{2^{t-1}}\equiv 1$ for $t-1<r-1$. Therefore $\pm 1+2^{s-r}\sigma_i$ belongs to $2^r \pmod{2^s}$.

Since we have found 2^r integers belonging to $2^r \pmod{2^s}$ for each positive r less than $s-1$ and since, in addition, 1 belongs to 2^0 and 2^s-1 to $2^1 \pmod{2^s}$, we have accounted for all the 2^{s-1} odd positive integers less than 2^s.

THEOREM 3. *For* $s>2$, $1<r<s-1$, *the sum of the nth powers of the positive integers less than* 2^s *having period* 2^r *is congruent to zero for n odd and to* 2^r *for n even. For* $r=1$ *the sum is congruent to* -1 *for n odd and* $s>1$, *to* 1 *for n even and* $s=2$, *and to* 3 *for n even and* $s>2$. *For* $s>0$, *the sum of the nth powers of* 1, *which belongs to* 1, *is congruent to* 1.

Proof. Using the representation obtained in Lemma 1, we have for odd n the pairing

$$(\pm 1 + 2^{s-r}\sigma_i)^n \equiv -(\mp 1 + 2^{s-r}(2^r - \sigma_i))^n,$$

giving $\sum a_i^n\equiv 0$ for $s>2$ and $1<r<s-1$. For $s>2$ and $r=1$ we have in addition $a_3\equiv -1$, so that $\sum a_i^n\equiv -1$.

For n even, say $n=2y$, we have

$$\sum_i a_i^{2y} = \sum_i (1 + 2^{s-r}\sigma_i)^{2y} + \sum_i (-1 + 2^{s-r}\sigma_i)^{2y}$$

$$= \sum_i 1 + \sum_i 2y2^{s-r}\sigma_i + \sum_i \sum_{j=2}^{2y} C_j^{2y} 2^{j(s-r)}\sigma_i^j$$

$$+ \sum_i 1 - \sum_i 2y2^{s-r}\sigma_i + \sum_i \sum_{j=2}^{2y} C_j^{2y}(-1)^{2y-j}2^{j(s-r)}\sigma_i^j.$$

The two sums $\sum_i 1$ sum to 2^r. The following two sums cancel. It can be shown by induction on r that $\sum \sigma_i^j$ for $j>1$ is divisible by 2^{r-1}. Thus the term $\pm \sum_i C_j^{2y} 2^{j(s-r)} \sigma_i^j$ for any $j>1$ is divisible by $2^{j(s-r)+r-1} = 2^{s+(j-1)(s-r)-1}$. Then, since $s-r>1$, this term is congruent to zero. Therefore, $\sum a_i^n \equiv 2^r$ for n even, $s>2$, and $1<r<s-1$. Including $a_3 \equiv -1$ for $r=1$, we obtain $\sum a_i^n \equiv 3$.

THEOREM 4. *Let the modulus m be a product of distinct prime powers $m=m_1m_2 \cdots m_k$, where $m_r=p_r^{\alpha_r}$. Let the a_i be the distinct positive integers less than m having period e* (mod m). *Let $e_1, e_2, \cdots, e_k$ be divisors of e such that $e=[e_1, e_2, \cdots, e_k]$. Let the b_{i_r} be the N_r distinct positive integers less than m_r having period e_r* (mod m_r). *(If e_r does not divide $\phi(m_r)$, then there are no b_{i_r} and N_r equals zero.) Then the sum of the nth powers of the distinct positive integers less than m with period e* (mod m) *is a solution* (mod m) *of the k simultaneous congruences*

$$\sum a_i^n \equiv \sum_D N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k \sum_{i_r} b_{i_r}^n \pmod{m_r}$$

$$\textit{for} \quad r = 1, 2, \cdots, k,$$

the sums $\sum_D$ taken over all decompositions $e_1, e_2, \cdots, e_k$ of e.

Note that $\sum_{i_r} b_{i_r}^n$ is evaluated by Theorems 2 and 3. Whenever N_r is nonzero, that is when $e_r|\phi(m_r)$, N_r is known to equal $\phi(e_r)$. (This can be deduced from [1] arts. 85–89.)

Proof. Let the residues of a_i (mod m_r) for $r=1, 2, \cdots, k$ be $b_1(i), b_2(i), \cdots, b_k(i)$. We shall show that each a_i is in correspondence with a set of $b(i)$'s whose periods form a decomposition of e such that $e=[e_1, e_2, \cdots, e_k]$.

For a given a_i consider the $b_r(i)$. We shall show that their periods constitute a set $e_1, e_2, \cdots, e_k$ such that $e=[e_1, e_2, \cdots, e_k]$. Since $a_i^e \equiv 1$ (mod m) and therefore $a_i^e \equiv 1$ (mod m_r), each of the periods is a divisor of e. Let L be the least common multiple of the periods. Then $a_j^L \equiv 1$ (mod m_r), so that $a_i^L = 1+f_rm_r$ for some integer f_r, for each $r=1, 2, \cdots, k$. Then $f_1m_1=f_2m_2= \cdots =f_km_k$. But since the m_r are relatively prime, each f_rm_r is divisible by m so that $a_i^L \equiv 1$ (mod m), and $e|L$. Thus the periods of the b_r (mod m_r) constitute a set $e_1, e_2, \cdots, e_k$ such that $e=[e_1, e_2, \cdots, e_k]$.

Conversely, every such set of exponents gives rise to an a_i, namely a solution (mod m) of the simultaneous congruences $a_i \equiv b_r$ (mod m_r), where b_r has period e_r (mod m_r), $r=1, 2, \cdots, k$.

'Among the $b_r(i)$ for any one r the set of N_r distinct b_{i_r}'s having period e_r (mod m_r) is repeated $N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k$ times. Now $\sum a_i^n \equiv \sum b_r(i)^n$ (mod m_r) for each $r=1, 2, \cdots, k$. Dividing the $b_r(i)$'s into like sets of distinct b_{i_r}'s, we have

$$\sum a_i^n \equiv \sum_D N_1N_2 \cdots N_{r-1}N_{r+1} \cdots N_k \sum_{i_r} b_{i_r}^n \pmod{m_r}.$$

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ON THE INFINITUDE OF PRIMES

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In this note we would like to offer an elementary "topological" proof of the infinitude of the prime numbers. We introduce a topology into the space of integers S, by using the arithmetic progressions (from $-\infty$ to $+\infty$) as a basis. It is not difficult to verify that this actually yields a topological space. In fact, under this topology, S may be shown to be normal and hence metrizable. Each arithmetic progression is closed as well as open, since its complement is the union of other arithmetic progressions (having the same difference). As a result, the union of any finite number of arithmetic progressions is closed. Consider now the set $A = \bigcup A_p$, where A_p consists of all multiples of p, and p runs through the set of primes $\geqq 2$. The only numbers not belonging to A are -1 and 1, and since the set $\{-1, 1\}$ is clearly not an open set, A cannot be closed. Hence A is not a finite union of closed sets which proves that there are an infinity of primes.

A STATISTICAL DERIVATION OF A PAIR OF TRIGONOMETRIC INEQUALITIES

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The following inequalities and a particular generalization of them can be obtained by comparing the variances of a pair of minimum variance estimators with the corresponding variances of certain less efficient estimators:† Given $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ such that $0 \leqq \theta_i < \pi$ and $\theta_j \neq \theta_k$ for $j \neq k$. Then

$$\text{(1)} \quad \frac{\sum_{i=1}^{n} \cos^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \leqq \binom{n}{2}^{-2} \sum_{i=1}^{n} \left\{ \frac{\cos \theta_1}{\sin (\theta_i - \theta_1)} + \cdots + \frac{\cos \theta_{i-1}}{\sin (\theta_i - \theta_{i-1})} + \frac{\cos \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\cos \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2$$

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† So far as the author has been able to determine, the inequalities stated in this paper appear to be new. The statistics involved can be considered as respective pairs of estimators of the coordinates of a fixed point in a plane, based on line-of-sight observations. The minimum variance estimators (under the conditions stated) are the coordinates of the point from which the sum of squares of the distances to each line-of-sight is a minimum. The less efficient estimators correspond to the estimate (of the location of the fixed point) obtained by averaging arithmetically the vectors determined by the intersections of all pairs of lines of sight.

and

$$\frac{\sum_{i=1}^{n} \sin^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \leqq \binom{n}{2}^{-2} \sum_{i=1}^{n} \left\{ \frac{\sin \theta_1}{\sin (\theta_i - \theta_1)} + \cdots + \frac{\sin \theta_{i-1}}{\sin (\theta_i - \theta_{i-1})} \right.$$

$$\left. + \frac{\sin \theta_{i+1}}{\sin (\theta_i - \theta_{i+1})} + \cdots + \frac{\sin \theta_n}{\sin (\theta_i - \theta_n)} \right\}^2 . \tag{2}$$

Proof: Let $\epsilon_i = z_i - \xi \sin \theta_i - \eta \cos \theta_i$, for $i = 1, 2, \cdots, n$, represent n random independent deviates, with $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, random components z_i, known parameters θ_i subject to the restrictions stated above, and with unknown parameters ξ and η.

If we let $(\hat{\xi}, \hat{\eta})$ represent the least squares estimate of (ξ, η), *i.e.*, the estimate of (ξ, η) obtained by minimizing $\sum(z_i - \hat{\xi} \sin \theta_i - \hat{\eta} \cos \theta_i)^2$ (*cf.* [1]), then $\hat{\xi}$ and $\hat{\eta}$ are unbiased linear estimates of ξ and η respectively with variances

$$\sigma^2(\hat{\xi}) = \frac{\sum_{i=1}^{n} \cos^2 \theta_i}{\sum_1^n \sin^2 \theta_i \sum_1^n \cos^2 \theta_i - \left(\sum_1^n \sin \theta_i \cos \theta_i \right)^2} = \frac{\sum_1^n \cos^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} \tag{3}$$

and

$$\sigma^2(\hat{\eta}) = \frac{\sum_{i=1}^{n} \sin^2 \theta_i}{\sum\sum_{j>i} \sin^2 (\theta_i - \theta_j)} . \tag{4}$$

Further, in accordance with the generalized Markoff theorem on least squares (see [2]), $\hat{\xi}$ and $\hat{\eta}$ are minimum variance unbiased estimates of ξ and η in the class of unbiased linear estimates.

Thus any other pair of unbiased linear estimators (x, y) of (ξ, η) will in general have variance greater than (3) and (4), respectively.

Substituting $z_k = \epsilon_k + \xi \sin \theta_k + \eta \cos \theta_k$ for $k = i, j$ in

$$x_{ij} = \frac{z_i \cos \theta_j - z_j \cos \theta_i}{\sin (\theta_i - \theta_j)} \tag{5}$$

shows that, for any i and j, with $i \neq j$, x_{ij} is an unbiased linear estimate of ξ. Therefore the linear function

$$\bar{x} = \binom{n}{2}^{-1} \sum\sum_{j>i} x_{ij} \tag{6}$$

is also an unbiased estimate of ξ. It follows from (5) and (6) by inspection that $\bar{x}=\sum_{i=1}^{n}\delta_i z_i$, where

$$\delta_i = \binom{n}{2}^{-1}\sum_{j\neq i}^{n}\frac{\cos\theta_j}{\sin(\theta_i-\theta_j)}.$$

Hence

$$\sigma^2(\bar{x}) = \sum_{i=1}^{n}\delta_i^2\sigma^2(z_i) = \sum_{i=1}^{n}\delta_i^2. \tag{7}$$

Since (3), the variance of $\hat{\xi}$, is identical to the left side of (1), and (7) is identical to the right side, the inequality (1) is established.

Analogously, starting with

$$y_{ij} = \frac{z_j\sin\theta_i - z_i\sin\theta_j}{\sin(\theta_i-\theta_j)}$$

and

$$\bar{y} = \binom{n}{2}^{-1}\sum_{j>i}\sum y_{ij},$$

we arrive at the inequality (2).

To obtain a generalization of (1) consider the set of those estimators of ξ, each of which is obtained by averaging arithmetically some combination of the x_{ij}'s as defined by (5). Any such average is an unbiased linear estimate of ξ, and is in general distinct from $\hat{\xi}$. Next corresponding to *each such estimate* define the characteristic function $\alpha_{ij}(=\alpha_{ji})$ as follows:

$\alpha_{ij}=1$ if x_{ij} is included in the averaging;

$\alpha_{ij}=0$ otherwise.

The arithmetic average corresponding to a given combination of x_{ij} can then be written as

$$x(\alpha) = \left\{\binom{n}{2} - r\right\}^{-1}\sum_{j>i} x_{ij}\alpha_{ij}$$

where $\binom{n}{2}-r$ is the number of α_{ij} which do not vanish. The variance of $x(\alpha)$ is then given by the right side of

$$\begin{aligned}\frac{\sum\cos^2\theta_i}{\sum_{j>i}\sum\sin^2(\theta_i-\theta_j)} \leqq \left[\binom{n}{2}-r\right]^{-2}\sum_{i=1}^{n}\Bigg\{&\frac{\alpha_{i1}\cos\theta_1}{\sin(\theta_i-\theta_1)} \\ &+\frac{\alpha_{i2}\cos\theta_2}{\sin(\theta_i-\theta_2)}+\cdots+\frac{\alpha_{i,i-1}\cos\theta_{i-1}}{\sin\theta_i-\theta_{i-2}} \\ &+\frac{\alpha_{i,i+1}\cos\theta_{i+1}}{\sin(\theta_i-\theta_{i+1})}+\cdots+\frac{\alpha_{in}\cos\theta_n}{\sin(\theta_i-\theta_n)}\Bigg\}^2,\end{aligned} \tag{8}$$

and thus (8) holds for all possible combinations of zeros and ones in the α_{ij}, (provided at least one α_{ij} is not zero) with appropriate r.

Analogously we have

$$\frac{\sum \sin^2\theta_i}{\sum\sum_{j>1} \sin^2(\theta_i-\theta_j)} \leqq \left[\binom{n}{2}-r\right]^{-2}\sum_{i=1}^{n}\left\{\frac{\alpha_{i1}\sin\theta_1}{\sin(\theta_i-\theta_1)}+\cdots+\frac{\alpha_{i,i-1}\sin\theta_{i-1}}{\sin(\theta_i-\theta_{i-1})}\right.$$

$$\left.+\frac{\alpha_{i,i+1}\sin\theta_{i+1}}{\sin(\theta_i-\theta_{i+1})}+\cdots+\frac{\alpha_{in}\sin\theta_n}{\sin(\theta_i-\theta_n)}\right\}^2.$$

References

1. Whittaker and Robinson, The Calculus of Observations, 4th Edition, Chapter IX, London, Blackie & Sons, 1940.

2. J. Neyman and F. N. David, Extension of the Markoff theorem on least squares, Statistical Research Memoirs, II, p. 105.

ON THE ISOSCELES TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. THEOREM. *If the centers I_a, I_b, I_c, I_d of the escribed spheres of the truncs opposite the vertices A, B, C, D of a tetrahedron T are situated on the circumsphere of T, then T is isosceles and conversely.*

Proof. Let a, a', b, b', c, c', be the lengths of the edges BC, DA, CA, DB, AB, DC; S_a, S_b, S_c, S_d, R_a, R_b, R_c, R_d be the areas and the radii of the circumcircles of the faces BCD, CDA, DAB, ABC. With reference to T the barycentric coordinates of I_a, I_b, I_c, I_d are

$$(-S_a, S_b, S_c, S_d), \quad (S_a, -S_b, S_c, S_d), \quad (S_a, S_b, -S_c, S_d), \quad (S_a, S_b, S_c, -S_d).$$

The equation of the circumsphere of T in barycentric coordinates (x, y, z, t) is

$$c^2xy+b^2xz+a'^2xt+a^2yz+b'^2yt+c'^2zt=0. \tag{1}$$

Hence the conditions that I_a, I_b, I_c, I_d be on this circumsphere are

$$-c^2S_aS_b-b^2S_aS_c-a'^2S_aS_d+a^2S_bS_c+b'^2S_bS_d+c'^2S_cS_d=0 \tag{2}$$

$$-c^2S_aS_b+b^2S_aS_c+a'^2S_aS_d-a^2S_bS_c-b'^2S_bS_d+c'^2S_cS_d=0 \tag{3}$$

$$c^2S_aS_b-b^2S_aS_c+a'^2S_aS_d-a^2S_bS_c+b'^2S_bS_d-c'^2S_cS_d=0 \tag{4}$$

$$c^2S_aS_b+b^2S_aS_c-a'^2S_aS_d+a^2S_bS_c-b'^2S_bS_d-c'^2S_cS_d=0. \tag{5}$$

Linear combinations of (2)–(5) give

$$\frac{S_a}{ab'c'}=\frac{S_b}{a'bc'}=\frac{S_c}{a'b'c}=\frac{S_d}{abc}. \tag{6}$$

* Translated from the French by W. E. Byrne, Virginia Military Institute.

From the relations $abc=4R_dS_d, \cdots$, and (6) it follows that

$$R_a = R_b = R_c = R_d, \tag{7}$$

and the circumcenter and incenter of T coincide. Hence T is isosceles [1].

Conversely, if T is isosceles, the points I_a, I_b, I_c, I_d are on the circumsphere of T [2]. Or (1) may be used with $a=a'$, $b=b'$, $c=c'$, *etc.*

Another proof. For an arbitrary tetrahedron T the incenter and the centers of the seven escribed spheres form with the given tetrahedron a desmic system [3]. According to a known theorem [4] if the points I_a, I_b, I_c, I_d are on the circumsphere of T, then the lines AI_a, BI_b, CI_c, DI_d meet at the incenter I, which coincides with the point L whose normal coordinates are proportional to R_a, R_b, R_c, R_d. Thus (7) holds. Consequently T is isosceles and conversely.

COROLLARY. *The tetrahedron $T' \equiv I_aI_bI_cI_d$ is isosceles if T is isosceles, and corresponding edges of these two tetrahedrons meet in pairs at their midpoints.*

Proof. T and T' are symmetric with respect to the center of the circumsphere of T. The circumcenter coincides with their common centroid [5].

COROLLARY. *The tetrahedrons I_aBCD, I_bCDA, I_cDAB, I_dABC are trirectangular at I_a, I_b, I_c, I_d and conversely.*

Proof. A given tetrahedron T may be bordered on the outside (inside) by trirectangular tetrahedrons if and only if all its faces are right triangles and this takes place when T is isosceles. The points I_a, I_b, I_c, I_d are diametrically opposite on the circumsphere of T to the vertices A, B, C, D. Let R be the radius of the circumsphere of T. Then

$$\begin{aligned}\overline{I_dB}^2 + \overline{I_dC}^2 &= \overline{I_dD}^2 - \overline{DB}^2 + \overline{I_dD}^2 - \overline{DC}^2 \\ &= 8R^2 - b'^2 - c'^2 = 8R^2 - b^2 - c^2 = a^2;\end{aligned}$$

and likewise,

$$\overline{I_dC}^2 + \overline{I_dA}^2 = b^2, \qquad \overline{I_dA}^2 + \overline{I_dB}^2 = c^2$$

since it is known [6] that $8R^2=a^2+b^2+c^2$.

Conversely, if I_dABC is trirectangular at I_d, its altitude drawn from I_d goes through the orthocenter H of the triangle ABC; H is the point of contact of the sphere (I_d) with the face ABC of T [7]. According to a known theorem [8] the inscribed sphere (I) touches the face ABC at the isogonal conjugate of H, *i.e.*, at the circumcenter of ABC. The same conclusions hold for the tetrahedrons I_aBCD, I_bCDA, I_cDAB, so the spheres $ABCD$, (I) are concentric and T is isosceles.

2. THEOREM. *In an arbitrary tetrahedron $T \equiv ABCD$ the orthogonal projections on the planes BCD, CDA, DAB, ABC of the isogonal conjugates A_2, B_2, C_2, D_2 of the vertices A_1, B_1, C_1, D_1 of the trirectangular tetrahedrons A_1BCD, B_1CDA, C_1DAB, D_1ABC which border T on the outside (inside) are the vertices of four tri-*

rectangular tetrahedrons. (Pedal tetrahedrons of A_2, B_2, C_2, D_2 with respect to T).

Proof. If we designate, as in plane geometry [9], by A_3, B_3, C_3, D_3 the orthogonal projections of D_2 on the planes BCD, CDA, DAB, ABC, the lines D_1A, D_1B, D_1C are perpendicular to the planes $B_3C_3D_3$, $C_3D_3A_3$, $D_3A_3B_3$ which are therefore perpendicular in pairs. Hence tetrahedron $D_3A_3B_3C_3$ is trirectangular at D_3. The same argument applies to the others.

COROLLARY. *The pedal tetrahedrons of the centers I_a, I_b, I_c, I_d with respect to the isosceles tetrahedron T are trirectangular at the orthocenters of the faces BCD, CDA, DAB, ABC and conversely.*

Proof. In this case I_a, I_b, I_c, I_d coincide with their isogonal conjugates with respect to T and the pedal tetrahedrons of these points are trirectangular at the orthocenters of the faces BCD, CDA, DAB, ABC. Conversely, if the pedal tetrahedron of I_d is trirectangular at H, then I_dABC is trirectangular at I_d and the conclusion follows from the corollary of paragraph 1.

References

1. N. A. Court, Modern Pure Solid Geometry, p. 97, art. 304.
2. *Ibid*, p. 100, art. 313.
3. *Ibid*, p. 235, art. 722 (definition of desmic system).
4. V. Thébault, Mathesis, t. 56, question 3370 and t. 61, p. 324
5. N. A. Court, *loc. cit.*, p. 95, art. 298; p. 100, art. 313.
6. *Ibid*, p. 102, ex. 18.
7. *Ibid*, p. 100, art. 315.
8. J. Neuberg, Mémoire sur le Tétraèdre, 1884, p. 23.
9. R. A. Johnson, Modern Geometry, p. 153, art. 213.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

L'HÔPITAL'S RULE AND EXPANSION OF FUNCTIONS IN POWER SERIES

M. R. SPIEGEL, Rensselaer Polytechnic Institute

When a student is first introduced to the concept of expansion of functions in power series, it is instructive to provide methods for evaluating the coefficients preliminary to using the conventional one of computing successive derivatives according to the rules of Maclaurin and Taylor. One such method, which will be described here, is useful when the student has had L'Hôpital's rule for the eval-

THE CIRCUMSCRIBED CIRCLE

ARTHUR PORGES, Los Angeles City College

A common exercise in most analytic geometry textbooks is the determination of the equation of the circle circumscribing the triangle formed by three given straight lines. The method implied usually involves first finding the points of intersection of the lines, after which those points are used with some standard device like a determinant.

An approach suggested by the device often used for a conic through five given points, however, makes pairwise solution of the three linear equations unnecessary, and has some interest as a classroom variant.

Given the three straight lines

$$L_i \equiv a_iX + b_iY + c_i = 0, \qquad (i = 1, 2, 3), \tag{1}$$

we construct the function

$$U(X, Y) \equiv L_1L_2 + hL_1L_3 + kL_2L_3, \tag{2}$$

where h and k are parameters to be determined later.

Obviously, $U(X, Y) = 0$ is satisfied by the points of intersection of the lines (1). Further, $U(X, Y)$ is a second degree function of X and Y of the form

$$\begin{aligned} U(X, Y) \equiv{} & A(h, k)X^2 + B(h, k)XY + C(h, k)Y^2 + D(h, k)X \\ & + E(h, k)Y + F(h, k). \end{aligned} \tag{3}$$

If we now impose the conditions

$$A(h, k) = C(h, k), \tag{4}$$

and

$$B(h, k) = 0, \tag{5}$$

it follows that the values of h and k derived from the simultaneous solution of the linear system (4) and (5) will reduce $U(X, Y) = 0$ to the equation of the desired circle.

THE FUNDAMENTAL THEOREM OF THE DIFFERENTIAL CALCULUS

W. R. RANSOM, Tufts College

That every equation of a certain type has a root, and that the limit of a certain type of sum is an integral, are agreed to be *the* Fundamental Theorems of Algebra and of the Integral Calculus. What is there entitled to that designation in the Differential Calculus?

It should be something about differentials, since we are talking about the differential calculus. In spite of the diversity of opinion about these curious symbols or quantities (see recent controversies in this MONTHLY, vol. 58, p. 336 (1951), and vol. 59, pp. 392–406 (1952)), there seems to be a fact about them of

wide generality that is universally exhibited. This fact is usually presented in a paragraph with the phrase "function of a function" in its title: it is embodied in this equation,

$$\operatorname*{Limit}_{\Delta z\to 0} \frac{\Delta y}{\Delta z} = \frac{dy}{dz}\cdot$$

The qualifications usually specified for this require that y shall be a function of x, that x shall be a function of z, and that certain derivatives exist and are not zero. Instead of the dy/dz, we find sometimes the product $(dy/dx)(dx/dz)$ or the quotient $(dy/dx)/(dz/dx)$.

If we wish to make a very useful choice of what to take as the fundamental theorem of the differential calculus, we do well to reconsider what the differential calculus is about for the stage at which it is first taught.

The student comes to the differential calculus from a study of analytical geometry and physics. In these subjects he is familiar with quantities whose measurements yield varying values, but for which *natura non agit per saltum.*

With average rates he has long been familiar, and the first task of the calculus is to familiarize him with the new concept of an "instantaneous rate." The average rate is a convenience in everyday life: it represents no reality in nature, but is merely an expedient in simplifying calculations. The limiting values, however, have important physical meanings: the overturning of a car on a curve does not depend upon its average speed, $\Delta s/\Delta t$, but upon a critical value, $ds/dt=\sqrt{F\cdot R}$, (appropriate units being assumed), where F is the force required for overturning and R the radius of the curve.

The first chapter in many textbooks develops Cauchy's idea of "function," and introduces freakish possibilities, which will be quite exceptional and usually ignored. We have been told that it is common practice to omit this chapter and proceed to the next. This omission is good practice. It is well to build up the idea of differentiation not so much on Cauchy's abstraction as upon the ideas which the student already understands, and concepts which will be readily applicable to the problems for which he is to use the differential calculus.

In his problems, he will be concerned with groups of quantities, some constant, others variable. These quantities are connected by equations with which he is familiar, $y=x^2$, $A=Bh$, $pv=c$, *etc.* Among these variables there is no "independent variable": independence is an attribute imposed *temporarily* by the mathematician—there is nothing in nature or geometry that dictates its choice. The idea of independence should be relegated to the background as soon as possible. Instead of a group of n functions of an independent variable, we have $n+1$ variables, connected by M equations: from these equations M differential equations can be obtained, and from these, $M(n+1)$ equations can be obtained (by division) that connect the $n(n+1)$ possible rates with the variables.

In such a group of variables we may consider two states, and by subtractions determine relations involving the increments (positive, negative, or zero) by

which the variables are affected. Among these variables we may arbitrarily select one variable, x, and compute the derivatives of the other variables with respect to x. Selecting at pleasure a number to be denoted by "dx" (in no way dependent upon the size of Δx), called "differential of x," we may express the derivatives of all the other variables as fractions with dx as denominator, and denote their numerators by da, db, dc, *etc.* This gives us as definitions of the other differentials

$$da/dx = \operatorname*{Limit}_{\Delta x \to 0} (\Delta a/\Delta x), \qquad db/dx = \operatorname*{Limit}_{\Delta x \to 0} (\Delta b/\Delta x), \quad etc.$$

The differentials having been defined in this way, by means of an arbitrarily chosen variable, we are now able to abandon the distinction between the variables, and make an important and useful statement about any one of the instantaneous rates. This statement deserves to be called *The Fundamental Theorem of the Differential Calculus.*

With the implied reservation that there can be no division by zero, the fundamental theorem is that whatever variables, y and z, are selected from the list, a, b, c, *etc.*, we shall have the relation that

$$\operatorname*{Limit}_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \frac{dy}{dz} \cdot$$

The proof is very simple, requiring only an understanding about the limit of a product, and is so familiar as given in paragraphs about the function of a function, that we need not give it here.

This theorem is of such a fundamental nature, and of such frequent and useful applicability, that it should be brought to the very beginning of the study of the differential calculus, and recognized as of importance equal to that of the fundamental theorem of the integral calculus.

TRANSFORMATION OF STANDARD INTEGRALS

V. Punga, Rensselaer Polytechnic Institute

In every textbook of integral calculus we can find the tables of standard integrals. The problem of integration consists in reducing the given integral to one of the standard type.

Probably not everybody is aware of the fact that there exist transformations transforming any one of the standard integrals into any other one. The transformation formulas can be found in the following way:

If $\int f_1(v)dv = \phi_1(v)$ and $\int f_2(u)du = \phi_2(u)$, then, in order to transform $\int f_1(v)dv$ into $\int f_2(u)du$, we have to solve $\phi_1(v) = \phi_2(u)$ for v in terms of u, say $v = \psi(u)$, which is the required transformation. The transformation is impossible if we can not solve $\phi_1(v) = \phi_2(u)$ for v in terms of u (in elementary functions).

For example, in order to transform $\int \sec v\, dv (= \ln (\sec v + \tan v))$ into $\int e^u du (= e^u)$, setting $C = 0$, we set up: $\ln (\sec v + \tan v) = e^u$ or $\sqrt{1 + \tan^2 v} + \tan v$

$=e^{(e^u)}$. Solving this equation for v, we obtain

$$v = \arc \tan \tfrac{1}{2}(e^{eu} - e^{-eu}).$$

Hence,

$$dv = \frac{2e^u du}{e^{eu} + e^{-eu}}, \qquad \sec v = \frac{e^{eu} + e^{-eu}}{2}$$

and therefore:

$$\int \sec v dv = \int \frac{e^{eu} + e^{-eu}}{2} \cdot \frac{2e^u du}{e^{eu} + e^{-eu}} = \int e^u du.$$

As a curiosity I suggest the following method of integration (without taking it too seriously). Let the students memorize only one standard integral, say $\int e^u du = e^u + C$, and teach them how to reduce any other integral to this one. In this way we can reduce the table of standard integrals to only one formula.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1166. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let DE be a variable chord perpendicular to diameter AB of a given circle (O). The maximum circle (ω_0) inscribed in the smaller segment, DEB, touches chord DE in C. The circle (ω_1) is tangent to (ω_0), (O), and DC, and another circle (ω_2) is tangent to (ω_1), (O), and DC. Find the ratio BC/CA for which the radius of circle (ω_2) is a maximum.

E 1167. *Proposed by C. D. Olds, San Jose State College*

Prove by mathematical induction that if a is a real number $\neq 1$, n a positive integer, then

$$\alpha_n = \frac{a^{2n+2} - 1}{a(a^{2n} - 1)} > \frac{n+1}{n}.$$

E 1168. *Proposed by R. R. Phelps, U.C.L.A.*

In an analogy with perfect numbers, let us define a *perfect triangle* as one whose integer valued sides add up to twice its area. An example is the (3, 4, 5) triangle. Find all perfect triangles.

E 1169. *Proposed by Joseph Lehner, Los Alamos Laboratory*

Let $0<x_1<x_2< \cdots$ and let $\sum_{n=1}^{\infty} x_n^{-h}$ converge. Then, if $0<h\leqq 1$ and $\alpha>0$, we have $x_{n+1}-x_n>\alpha$ for infinitely many indices n. If $h>1$, there are sequences $\{x_n\}$ with $x_{n+1}-x_n\to 0$.

E 1170. *Proposed by Viktors Linis, University of Ottawa*

Show that there exists a centrally-symmetric hexagon inscribed in any closed convex curve such that the ratio of the respective areas is at least 2/3.

SOLUTIONS

Calculating an Average Value

E 1136 [1954, 642]. *Proposed by W. A. Bowers, University of North Carolina*

An elementary physics text, wishing to avoid using calculus, and also hoping (unsuccessfully!) to avoid the appearance of pulling the answer out of a hat, gives the following recipe for calculating the work done in moving a positive unit charge from r_1 to r_2 against the attraction of a negative unit charge at the origin. "Since the force varies with distance, we cannot simply multiply 'the force' by the distance; instead we must take the geometric mean of the initial and final values of the force, and multiply it by the distance." Noting with amazement that this does indeed work, we ask: what is the most general function for which this is true, that is, for which the average value over an arbitrary interval equals the geometric mean of the values at the end points?

Solution by L. A. Ringenberg, Eastern Illinois State College. Let $F(x)$ have a differentiable positive square root $f(x)$ and satisfy the condition of the problem. Then

$$\int_x^y [f(t)]^2 dt = f(x)f(y)(y-x).$$

Taking partial derivatives we get

$$[f(y)]^2 = f(x)f'(y)(y-x) + f(x)f(y)$$

and

$$-[f(x)]^2 = f'(x)f(y)(y-x) - f(x)f(y).$$

From these relations we obtain

$$[f(y)-f(x)]/(y-x) = f(x)f'(y)/f(y) = f(y)f'(x)/f(x),$$

whence

$$f'(x)/[f(x)]^2 = f'(y)/[f(y)]^2,$$

and, since x and y are arbitrary,

$$f'(x)/[f(x)]^2 = c, \qquad \text{a constant.}$$

Therefore

$$F(x) = [f(x)]^2 = (hx + k)^{-2}, \qquad h \text{ and } k \text{ arbitrary.}$$

Conversely, we may easily verify by substitution that any function of the form $F(x) = (hx+k)^{-2}$ satisfies the desired relation.

Also solved by P. M. Anselone, H. E. Bray, G. B. Charlesworth, I. A. Dodes, Virginia Hanly, W. W. Hooker, E. S. Keeping, P. G. Kirmser, C. S. Ogilvy, Azriel Rosenfeld, J. P. Scholz, R. P. Tapscott, Chih-yi Wang, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Editorial Note. It is to be observed that the attraction function of the proposal is essentially already the most general one.

An Easy Inequality

E 1137 [1954, 642]. *Proposed by Simon Green, Philander Smith College, Arkansas*

Let $N=1+1/2+1/3+ \cdots +1/n$. Prove that $e^N>n+1$.

I. *Solution by N. J. Fine, University of Pennsylvania.*

$$\begin{aligned} e^N &= e^1 e^{1/2} e^{1/3} \cdots e^{1/n} \\ &> (1 + 1)(1 + 1/2)(1 + 1/3) \cdots (1 + 1/n) \\ &= (2)(3/2)(4/3) \cdots (n + 1)/n = n + 1. \end{aligned}$$

II. *Solution by W. E. Briggs, University of Colorado.* The inequality holds for $n=1$. Assume it also holds for $n=2, 3, \cdots, k$. Then

$$e^{1+1/2+\cdots+1/k} > k + 1$$

and

$$\begin{aligned} e^{1+1/2+\cdots+1/(k+1)} &> e^{1/(k+1)}(k + 1) \\ &> [1 + 1/(k + 1)](k + 1) \\ &= k + 2, \end{aligned}$$

and the inequality holds for all n by induction.

III. *Solution by Bernard Greenspan, Drew University.* Since

$$N = 1 + 1/2 + \cdots + 1/n > \int_1^{n+1} dx/x = \ln (n + 1),$$

we have $e^N > n+1$.

Also solved by A. N. Aheart, Arlo Anderson, P. M. Anselone, J. W. Baldwin, Leon Bankoff, P. T. Bateman, H. W. Becker, Julian Braun, K. A. Bush, C. N. Campopiano, W. B. Carver, N. A. Childress, R. J. Cormier, Hüseyin Demir, I. A. Dodes, Edgar Dougherty, F. J. Duarte, H. M. Feldman, Edward Fleisher, Calvin Foreman, J. E. Freund, Joyce Friedman, Ruth Frisch, Zoffmann Fudita, Laurence Glasser, A. M. Glicksman, Sidney Glusman, E. S. Grable and J. W. Sawyer (jointly), Nathaniel Grossman, D. S. Greenstein, Vern Hoggatt, W. W. Hooker, P. F. Hultquist, A. R. Hyde, P. G. Kirmser, Sidney Kravitz, M. J. Mansfield, D. C. B. Marsh, W. E. Mientka, J. D. Miller, Leo Moser, T. F. Mulcrone, J. B. Muskat, C. S. Ogilvy, C. D. Olds, S. Parameswaran, F. D. Parker, L. L. Pennisi, W. J. Pervin, R. B. Plymale, O. J. Ramler, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, David Sachs, C. M. Sandwick, Sr., Frank Saunders, M. Schwartz, Berthold Schweizer, F. A. Sherk, Nathan Shklov, Augustus Sisk, O. E. Stanaitis, D. D. Strebe, D. R. Sudborough, A. V. Sylwester, R. P. Tapscott, J. A. Tierney, Alan Wayne, Roy Westwick, Howard Wicke, K. B. Williams, David Zeitlin, and the proposer. Late solutions by A. E. Anderson, M. S. Klamkin, C. F. Pinzka, and D. C. Russell.

Sharper bounds for e^N were given by Bush and Rhoades. Thus Bush showed that $e^N \geqq e(n+1)/2$, equality holding only if $n=1$, and Rhoades showed that $e^N > n+17/10$.

Bisosceles and Trisosceles Triangles

E 1138 [1954, 642]. *Proposed by J. P. Ballantine, University of Washington*

For any triangle prove that: (1) if $B=2A$, then $b^2=a^2+ac$, (2) if $B=3A$, then $b^3-ab^2-a^2b-ac^2+a^3=0$.

I. *Solution by D. C. B. Marsh, Texas Technological College.* (1) Since $B=2A$, $C=\pi-3A$ and (by a Mollweide equation)

$$(a+c)/b = \cos(2A-\pi/2)/\sin A = \sin 2A/\sin A = b/a.$$

The result follows at once.

(2) Here $C=\pi-4A$ and (again by a Mollweide equation)

$$\begin{aligned} c/(b-a) &= \cos(\pi/2-2A)/\sin A = \sin 2A/\sin A \\ &= 2\cos A = (b^2+c^2-a^2)/bc. \end{aligned}$$

The result follows at once.

II. *Solution by Joyce Friedman, ACF Electronics, Alexandria, Va.* (1) Draw BD, the bisector of B. Then triangles BDC and ABC are similar, and

$$a/b = BD/c = (b-AD)/a.$$

Since triangle ABD is isosceles, $AD=BD$, whence

$$AD = ac/b \quad \text{and} \quad a/b = (b-ac/b)/a.$$

That is, $b^2=a^2+ac$.

(2) Draw BD such that angle $ABD=2A$, angle $DBC=A$. Then triangles BDC and ABC are similar, and

$$BD/c = a/b = (b - AD)/a.$$

By part (1), $(AD)^2=(BD)^2+(BD)c$. But $BD=ac/b$, and therefore

$$a/b = (b - \sqrt{(ac/b)(ac/b + c)})/a.$$

Simplifying this gives the desired result, $b^3-ab^2-a^2b-ac^2+a^3=0$.

Also solved by A. N. Aheart, R. V. Andree's freshman mathematics class, Norman Anning, Leon Bankoff, H. H. Berry, Margaret Blue, W. B. Carver, R. L. Caskey, G. B. Charlesworth, P. L. Chessin, N. A. Childress, Shannon Clark, K. W. Crain, J. E. D'Atri, Hüseyin Demir, I. A. Dodes, F. J. Duarte, A. L. Epstein, H. M. Feldman, Russell Godard, Bernard Greenspan, Vern Hoggatt, Douglas Holdridge, W. W. Hooker, Raymond Huck, P. F. Hultquist, A. R. Hyde, John Jones, Jr., Sam Kravitz, Sidney Kravitz, R. A. Miller, George Millman, Lawrence Miner, J. B. Muskat, C. S. Ogilvy, S. Parameswaran, M. J. Pascual, W. O. Pennell, P. W. A. Raine, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, F. W. Saunders, J. W. Sawyer, F. A. Sherk, Augustus Sisk, O. E. Stanaitis, E. P. Starke, J. A. Tierney, Alan Wayne, Roy Westwick, Dale Woods, Roscoe Woods, Hazel S. Wilson, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, Josef Langr, C. F. Pinzka, and D. C. Russell.

Editorial Note. Miller pointed out that this problem is the converse of his Problem 123, *National Mathematics Magazine*, vol. 2 (1936), p. 58.

Miss Friedman's method (solution II above) may be successively used to obtain relations for triangles having $B=4A$, $5A$, $\cdots$. This raises the problem of finding a general formula, involving the sides of the triangle, equivalent to $B=nA$, where n is a positive integer. In problem E 620 [1945, 46], Wayne pointed out that the identity

$$\sin (n+1)A = r \sin nA - \sin (n-1)A,$$

where $r=2 \cos A$, together with the law of sines, yields the equality

$$\frac{b}{c} = \frac{1}{r -} \frac{1}{r -} \cdots \frac{1}{r}$$

(to n components), where $B=nA$ and $r=(b^2+c^2-a^2)/bc$.

The relation of part (1) is interestingly applicable to the three familiar triangles with angles $(A, A, 2A)$, $(A, 2A, 2A)$, $(A, 2A, 4A)$.

An Identity Projectivity

E 1139 [1954, 642]. *Proposed by N. A. Court, University of Oklahoma*

Three collinear points P, Q, R are marked on the sides BC, CA, AB of a triangle ABC. Starting with an arbitrary point X of the line BC, the following

points are constructed successively:

$$Y = (XR, CA), \qquad Z = (YP, AB), \qquad X' = (ZQ, BC);$$
$$Y' = (X'R, CA), \qquad Z' = (Y'P, AB), \qquad X'' = (Z'Q, BC).$$

Show that points X and X'' coincide.

I. *Solution by Chih-yi Wang, University of Minnesota.* It is not necessary to assume that P, Q, R are collinear. By Menelaus' theorem we have the following six relations:

$$\begin{aligned}
&(1) \quad (AR/RB)(BX/XC)(CY/YA) = -1,\\
&(2) \quad (AZ/ZB)(BP/PC)(CY/YA) = -1,\\
&(3) \quad (AZ/ZB)(BX'/X'C)(CQ/QA) = -1,\\
&(4) \quad (AR/RB)(BX'/X'C)(CY'/Y'A) = -1,\\
&(5) \quad (AZ'/Z'B)(BP/PC)(CY'/Y'A) = -1,\\
&(6) \quad (AZ'/Z'B)(BX''/X''C)(CQ/QA) = -1,
\end{aligned}$$

Forming the product of the three ratios (1) to (2), (3) to (4), (5) to (6), we obtain

$$BX/XC = BX''/X''C,$$

whence X and X'' coincide.

II. *Solution by Hüseyin Demir, Zonguldak, Turkey.* From

$$X \underset{\wedge}{\overset{R}{=}} Y \underset{\wedge}{\overset{P}{=}} Z \underset{\wedge}{\overset{Q}{=}} X' \underset{\wedge}{\overset{R}{=}} Y' \underset{\wedge}{\overset{P}{=}} Z' \underset{\wedge}{\overset{Q}{=}} X''$$

we get the projectivity $X \overline{\wedge} X''$, in which B, C, P are self-corresponding elements. Hence all the points of BC are self-corresponding elements. It is to be noted that the property holds even if the points P, Q, R are not collinear.

III. *Solution by W. B. Carver, Cornell University.* Using a homogeneous coordinate system, let the vertices A, B, C be the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and let the points P, Q, R be, respectively, $(0, a, 1)$, $(1, 0, b)$, $(c, 1, 0)$, and let X be $(0, x, 1)$. We then find successively $Y(-cx, 0, 1)$, $Z(cx, a, 0)$, $X'(0, a, -bcx)$, $Y'(a, 0, bx)$, $Z'(1, -bx, 0)$, $X''(0, x, 1)$. Thus X'' coincides with X, whether the points P, Q, R are collinear or not.

IV. *Solution by O. J. Ramler, The Catholic University of America.* A proof can be given based on Sylvester's residuation theorem, "All cubics through eight points pass through a unique ninth point."

Let $p=0$, $q=0$, $r=0$, $p'=0$, $q'=0$, $r'=0$ be the equations of the lines PYZ, $QX''Z'$, $RX'Y'$, RXY, $PY'Z'$, $QX'Z$, and let $a=0$, $b=0$, $c=0$ be the equations of the lines BC, CA, AB. Then $pqr=0$, $p'q'r'=0$, $abc=0$ are three composite cubics each passing through the eight points P, Q, R, X', Y', Z', Y, Z. The first

two cubics also pass through the ninth point W common to $QX''Z'$ and RXY, whence the third cubic also must pass through this ninth point. But W cannot lie on $b=0$ or on $c=0$, since these lines contain QY and $Z'R$ respectively. Hence W lies on $a=0$, and X, X'', W all coincide.

Note that we have not assumed that P, Q, R are collinear.

Also solved by R. L. Caskey, G. B. Charlesworth, K. W. Crain, Carolyn A. Denslow and Claire A. Pinney (jointly), N. J. Fine, D. C. B. Marsh, Geoffrey Mathews, T. F. Mulcrone, M. W. Oliphant, S. Parameswaran, David Sacks, F. A. Sherk, Roy Westwick, Roscoe Woods, and the proposer. Late solutions by Josef Langr and C. F. Pinzka.

Related Sequences

E 1140 [1954, 642]. *Proposed by Albert Wilansky, Lehigh University*

Let sequences $\{x_n\}$, $\{y_n\}$ be called related if $\sum_{k=1}^{n}(x_k-y_k)$ is a bounded function of n. For example, $\{(-1)^n\}$ and $\{0\}$ are related. Is there a bounded sequence not related to any convergent sequence?

Solution by A. Cesàro, University of Wisconsin. If $\{x_n\}$ and $\{y_n\}$ are "related," then $(1/n)\left|\sum_1^n (x_k-y_k)\right|\to 0$. That is, C_1-lim $(x_n-y_n)=0$. If $\{y_n\}$ is convergent, it is C_1 summable, and it is therefore necessary that $\{x_n\}$ be C_1 summable. Thus no bounded sequence $\{x_n\}$ which is non-C_1 summable can be "related" to any convergent sequence $\{y_n\}$. Such will be the case, for example, for the sequence

$$1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, \cdots.$$

Also solved by K. A. Bush, N. J. Fine, M. S. Klamkin, C. F. Pinzka, George Piranian, and H. H. Wicke.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4638. *Proposed by Paul Erdös, University of Notre Dame*

Let $k>2$. Does the following equation have any solutions in integers:

$$n(n+1)\cdots(n+k-1)=2m(m+1)\cdots(m+k-1),\quad m+k-1<n?$$

For $k=2$ there are infinitely many solutions, easily determined.

4639. *Proposed by L. B. Rall, Oregon State College*

Given a set of distinct numbers $a_0, a_1, \cdots$, such that $a_j \neq a_k$ if $j \neq k$, prove that, for all positive integers n,

$$\sum_{j=0}^{n} \prod_{k=0,k\geqq j}^{n} \frac{(a_k - a_{n+1})}{(a_k - a_j)} = 1.$$

4640. *Proposed by M. L. Keedy, University of Nebraska*

Let H be a proper subgroup of a group G and $\overline{H}$ its complement in G. Show that $H' = \overline{H}\,\overline{H}$ is a normal subgroup of G, and that it equals either G or H. Characterize those subgroups for which $H' = H$.

4641. *Proposed by E. P. Starke, Rutgers University*

Given N, however large, prove there exist primes P such that each of the numbers $1, 2, 3, \cdots, N$ is a quadratic residue (mod P).

4642. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, New York*

Let $a_2, a_3, \cdots, a_n$ be arbitrary. Prove there must be an x in $(0, 1)$ such that

$$\left| x + a_2x^2 + \cdots + a_nx^n \right| \geqq \frac{1}{n}\tan\frac{\pi}{4n}\cdot$$

SOLUTIONS

Powers of Infinite Matrices

4576 [1954, 126]. *Proposed by G. A. Garreau, Northampton Polytechnic College, London, England*

Construct an infinite matrix A such that A^{m-1} exists, whereas A^m does not.

Solution by the Proposer. Let m be positive, and $p<m$. Let $c>1$. Define the infinite matrix $A^{(p)}$ as follows: all its elements are zero except those lying in the first row, the second column, or the principal diagonal. In addition $a_{1,1}^{(p)} = a_{2,2}^{(p)} = 0$, so that the first column and the second row consist entirely of zeros. The nonzero elements are

$$a_{1,2}^{(p)} = \sum_{i=3}^{\infty} c^{-i(m-p)}; \qquad a_{1,k}^{(p)} = c^{k(p-1)}, \qquad k = 3, 4, \cdots;$$

$$a_{n,2}^{(p)} = c^{-n(m-1-p)}; \qquad a_{n,n}^{(p)} = c^{np}, \qquad n = 3, 4, \cdots.$$

We now show that, if p, q are two numbers such that $p+q<m$, then $A^{(p)}\times A^{(q)}=A^{(p+q)}$. Writing $A^{(p)}\times A^{(q)}=B=(b_{n,k})$ we have

$$b_{1,2}=\sum_{k=1}^{\infty} a_{1,k}^{(p)}a_{k,2}^{(q)}=\sum_{k=3}^{\infty} c^{k(p-1)}c^{-k(m-1-q)}=\sum_{k=3}^{\infty} c^{-k(m-p-q)}=a_{1,2}^{(p+q)}.$$

If $k>2$,

$$b_{1,k}=\sum_{i=1}^{\infty} a_{1,i}^{(p)}a_{i,k}^{(q)}=a_{1,k}^{(p)}a_{k,k}^{(q)}=c^{k(p-1)}c^{kq}=a_{1,k}^{(p+q)}.$$

If $n>2$,

$$b_{n,2}=\sum_{i=1}^{\infty} a_{n,i}^{(p)}a_{i,2}^{(q)}=a_{n,n}^{(p)}a_{n,2}^{(q)}=c^{np}c^{-n(m-1-q)}=a_{n,2}^{(p+q)},$$

$$b_{n,n}=\sum_{i=1}^{\infty} a_{n,i}^{(p)}a_{i,n}^{(q)}=a_{n,n}^{(p)}a_{n,n}^{(q)}=c^{np}c^{nq}=a_{n,n}^{(p+q)}.$$

All further elements are zero. Hence $A^{(p)}\times A^{(q)}=A^{(p+q)}$, as required.

We now write A for $A^{(1)}$, and obtain by induction $A^{(p)}=A^p$, $p<m$, for p a positive integer, the product on the right being associative whenever it exists. We can define A^p as $A^{(p)}$ even if p is not an integer, and the index laws are obeyed if $p<m$.

However, $A^{(m)}$ does not exist, since $a_{1,2}^{(m)}$ is defined as the sum of an infinite series of 1's, and, *a fortiori*, $A^{(p)}$ does not exist if $p>m$.

Nested Neighborhoods in a Hausdorff Space

4577 [1954, 126]. *Proposed by Albert Wilansky, Lehigh University*

Given a point x in a Hausdorff space, does there exist a set of neighborhoods of x whose intersection contains only x, and such that of any two neighborhoods in the set one includes the other (*i.e.*, the neighborhoods are "nested")?

Solution by M. K. Fort, Jr., University of Georgia. It is possible that no such set of neighborhoods exists. Let A be an uncountable set, and let x be a point of A. We define a subset G of A to be open if and only if either G does not contain the point x or $A-G$ is a finite set. It is easy to verify that the resulting topology is Hausdorff.

Now suppose that there exists a "nested" set N of neighborhoods of x whose intersection contains only x. We let C be a countably infinite subset of $A-\{x\}$. For each point $c\in C$ there exists $G_c\in N$ such that $c\notin G_c$.

If $G\in N$, then there exists $c\in C\cap G$. It follows that $G\not\subset G_c$ and hence $G_c\subset G$. Thus the intersection of all of the neighborhoods in N is the same as the intersection of the sets G_c, $c\in C$. However, the intersection of the sets G_c, $c\in C$, contains all but a countable number of points of A and hence contains points other than x. This is a contradiction.

Also solved by H. D. Block, L. C. Dean, Jr., David Ellis, Leonard Gillman,

P. R. Halmos, J. R. Schoenfield and L. R. Ford, Jr., H. E. Vaughan, and L. E. Ward, Jr.

Canasta Hands

4578 [1954, 198]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Find the number of essentially different eleven card Canasta hands which can be dealt from a 104 card pack. (We ignore red threes since they are always replaced in a hand. The pack consists of 8 aces, 8 twos, 4 threes, 8 fours, 8 fives, $\cdots$, 8 kings and 4 jokers. All cards of a given denomination are considered identical.)

I. *Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J.* The number of hands is the coefficient of x^{11} in

$$(1 + x + \cdots + x^4)^2(1 + x + \cdots + x^8)^{12} = (1 - x^5)^2(1 - x^9)^{12}(1 - x)^{-14},$$

where terms of the form $x^a x^b \cdots x^n$ are seen to correspond to hands containing $a, b, \cdots, n$ cards of the respective denominations.

Neglecting powers of x not involved, the desired result is the coefficient of x^{11} in $(1-2x^5-12x^9+x^{10})\sum\binom{k+13}{k}x^k$, which is

$$\binom{24}{11} - 2\binom{19}{6} - 12\binom{15}{2} + \binom{14}{1} = 2{,}440{,}634.$$

II. *Solution by Leo Moser, University of Alberta.* Let $C_r[a^i, b^j]$ denote the coefficient of x^r in

$$f(x) = (1 + x + x^2 + \cdots + x^a)^i(1 + x + x^2 + \cdots + x^b)^j.$$

The solution of the given problem is then

$$C_{11}[8^{12}, 4^2] = 2{,}440{,}634.$$

In "A problem in combinatory analysis," (*Trans. Roy. Sci. Can.* 48 (1953), pp. 21–26), the proposer gives the relatively efficient formula

$$C_r[a^i, b^j] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m}\binom{i+j+r-1-(a+1)k-(b+1)m}{r-(a+1)k-(b+1)m}.$$

A simple proof of this formula is given by the author, "On a combinatorial formula of Mendelssohn," (*Trans. Roy. Sci. Can.* 48(1953), p. 27).

Also solved by T. N. E. Greville, Bart Park, R. E. Shafer, and the Proposer.

Factors of e^z

4579 [1954, 199]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the relation

$$e^z = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

hold for $|z|<r$, where $a_n \geqq 0$, $b_n \geqq 0$ $(n=0, 1, \cdots)$. Show that the two factors on the right side must be entire functions of the form

$$e^{az+c}, \qquad e^{bz-c}, \qquad (a \geqq 0,\ b \geqq 0,\ a+b=1).$$

Solution by Leonard Carlitz, Duke University. We shall prove the following stronger result. Assume $\alpha_0=\beta_0=1$,

$$\sum_{r=0}^{n} \binom{n}{r} \alpha_r \beta_{n-r} = 1, \qquad \alpha_n,\ \beta_n \geqq 0 \qquad (n = 1, 2, \cdots).$$

Then $\alpha_n = a^n$, $\beta_n = b^n$, $a \geqq 0$, $b \geqq 0$, $a+b=1$. (The assumption $\alpha_0=\beta_0=1$ involves no loss in generality.)

Proof. It follows immediately from the hypothesis that $0 \leqq \alpha_n \leqq 1$, $0 \leqq \beta_n \leqq 1$ for $n=0, 1, 2, \cdots$. Therefore

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha_n z^n}{n!}, \qquad g(z) = \sum_{n=0}^{\infty} \frac{\beta_n z^n}{n!}$$

are entire functions of z such that

$$e^z = f(z)g(z).$$

Moreover

$$|f(z)| \leqq e^{|z|}, \qquad |g(z)| \leqq e^{|z|}.$$

Thus the functions $f(z)$, $g(z)$ are of order 1 at most and have no zeros; consequently by Hadamard's theorem (Titchmarsh, *Theory of Functions*, p. 250)

$$f(z) = e^{az}, \qquad g(z) = e^{bz},$$

where $a+b=1$, $a \geqq 0$, $b \geqq 0$. This proves our theorem.

Also solved by M. Aissen, I. N. Baker, George Brauer, Paul Erdös and Michael Golomb, Viktors Linis, Edgar Reich, H. S. Shapiro, O. E. Stanaitis, and the Proposer.

Editorial Note. Erdös and Golomb show that the proposed conclusion follows from the weaker hypothesis

$$a \leqq \arg a_n \leqq A, \qquad b \leqq \arg b_n \leqq B, \qquad A + B - (a+b) < \pi.$$

Several contributors noted the probabilistic interpretation: If the convolution of two discrete probability distributions, $\sum_0^\infty a_n = 1$, $\sum_0^\infty b_n = 1$, is a Poisson distribution, then also the factor distributions are Poisson distributions. For the analogous result concerning the normal distribution see H. Cramer, Über eine Eigenschaft der normalen Verteilungsfunktion, *Math. Zeitschrift*, 41 (1936), pp. 405–414. See also H. Cramer, *Random Variables and Probability Distributions*, p. 52.

Series with Non-negative Coefficients

4580 [1954, 199]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

For what real values of a does the power series expansion of the function $(1-ax+ax^2-x^3)^{-1}$ have all its coefficients non-negative?

Solution by L. Carlitz, Duke University. Put

$$(1 - ax + ax^2 - x^3)^{-1} = (f(x))^{-1} = \sum_{n=0}^{\infty} c_n x^n.$$

We shall prove the following

THEOREM. *$c_n \geqq 0$ for all $n \geqq 0$ if and only if $a=0$ or $a \geqq 3$ or $a=1+2\cos 2\pi/k$, where k is an integer >2.*

Since $c_1=a$, $c_2=a^2-a$, it is clear that $a \geqq 0$ is a necessary condition; indeed, if $a \neq 0$, it is necessary that $a \geqq 1$. Write

$$f(x) = (1 - x)(1 + (1 - a)x + x^2) = (1 - x)(1 - \beta x)(1 - \gamma x);$$

the factors are distinct unless $a=-1$ (which may be ignored) or $a=3$. For $a=3$ we get

$$(1 - x)^{-3} = \sum_{n=0}^{\infty} \frac{1}{2}(n + 1)(n + 2)x^n,$$

so that $c_n > 0$ for all $n \geqq 0$. Excluding this case we put

$$\frac{1}{f(x)} = \frac{A}{1 - x} + \frac{B}{1 - \beta x} + \frac{C}{1 - \gamma x}$$

and find (using $\beta+\gamma=a-1$, $\beta\gamma=1$) that

$$A = \frac{1}{(\beta - 1)(\gamma - 1)}, \qquad B = \frac{\beta^2}{(\beta - 1)(\beta - \gamma)}, \qquad C = \frac{\gamma^2}{(\gamma - 1)(\gamma - \beta)}.$$

It follows that

$$(1)' \quad c_n = A + B\beta^n + C\gamma^n = \frac{1}{(\beta - 1)(\gamma - 1)} + \frac{1}{\beta - \gamma}\left(\frac{\beta^{n+2}}{\beta - 1} - \frac{\gamma^{n+2}}{\gamma - 1}\right)$$

$$= \frac{1}{\beta - \gamma}\left(\frac{\beta^{n+2} - \beta}{\beta - 1} - \frac{\gamma^{n+2} - \gamma}{\gamma - 1}\right)$$

$$= \sum_{r=1}^{n+1} \frac{\beta^r - \gamma^r}{\beta - \gamma} = \sum_{r=0}^{n} (\beta^r + \beta^{r-1}\gamma + \cdots + \gamma^r).$$

The discriminant of the quadratic $1+(1-a)x+x^2$ is $(a-3)(a+1)$. We accordingly have two cases to consider: (i) $a>3$, (ii) $1<a<3$. In case (i) it is easily verified that β and γ are real and positive and therefore $c_n>0$ for all $n\geqq 0$.

In case (ii) we find that β and γ are complex, $|\beta|=|\gamma|=1$. We put

$$u = e^{i\theta}, \qquad \beta = u^2, \qquad \gamma = u^{-2}, \qquad \beta + \gamma = 2\cos 2\theta = a - 1,$$

so that $0<\theta<\pi/4$. Then (1) reduces to

$$c_n = \frac{\sin (n+1)\theta \sin (n+2)\theta}{\sin\theta \sin 2\theta}\,.$$

The condition $c_n\geqq 0$ is therefore equivalent to

$$\sin (n+1)\theta \sin (n+2)\theta \geqq 0 \qquad (n\geqq 0). \tag{2}$$

Now (2) will fail to hold if and only if there is an integer s such that $(n+1)\theta<s\pi<(n+2)\theta$, that is $n+1<s\pi/\theta<n+2$. In other words (2) fails provided that $s\pi/\theta$ is not an integer for some s. Clearly this will happen unless $\theta=\pi/k$, where k is integral. This completes the proof of the theorem.

Also solved by M. I. Aissen, C. E. Buell, Fritz Herzog, J. B. Kelly, the Proposer, and one whose solution is unsigned.

Contraction Point of a Transformation

4581 [1954, 199]. *Proposed by Ky Fan, University of Notre Dame*

Let M be a metric space in which every closed spheroid, *i.e.*, set of the form

$$S(z_0; r) = \{z\in M \mid d(z_0, z) \leqq r\}, \qquad (z_0\in M, r>0)$$

is compact (d denotes the distance). Let f be a continuous transformation from M into itself. Suppose that there is a point $x_0\in M$ such that

$$d(x_0, f(x)) < d(x_0, x) \tag{1}$$

for every $x\neq x_0$ of M. Prove

(i) If $f(x_0)=x_0$, then

$$\lim_{n\to\infty} f^n(x) = x_0 \tag{2}$$

for all $x\in M$; where $f^1(x)=f(x)$ and $f^n(x)=f(f^{n-1}(x))$.

(ii) If $f(x_0)\neq x_0$, then for every point $x\in M$, there is a positive integer k, depending on x, such that $f^k(x)=x_0$.

Solution by R. A. Struble, Illinois Institute of Technology

We show first that for each $x\in M$, if $f^n(x)\neq x_0$ for $n=1, 2, \cdots$, then (2) holds. In fact, the monotonically decreasing sequence $\{d(x_0, f^n(x))\}$ converges to some $R\geqq 0$. Further, since the sequence $\{f^n(x)\}$ is contained in the compact spheroid $S(x_0; d(x_0, x))$, it possesses at least one limit point y. Certainly,

$d(x_0, y)=R$. If $R>0$, then by (1), $d(x_0, f(y))<R$. However, this is impossible since f is continuous at y and every neighborhood of y contains some of the $f^n(x)$ for which $d(x_0, f^n(x))\geqq R$. Hence $R=0$ and (2) is true.

(i) If $f(x_0)=x_0$, then in any event we have (2).

(ii) If $f(x_0)\neq x_0$, then since f is continuous at x_0, (2) cannot be satisfied by any $x\in M$ and hence for each $x\in M$ there is a positive k, depending on x, such that $f^k(x)=x_0$.

Also solved by Joseph Auslander, J. D. Baum, R. R. Bernard, B. F. Bryant, Helen F. Cullen, D. O. Ellis, C. D. Gorman, R. H. Kasriel, P. R. Kelly, A. E. Livingston, M. D. Marcus, H. F. Mattson, O. W. Rechard, A. I. Rosenfeld, W. R. Smythe, Jr., G. H. M. Thomas, L. E. Ward, Jr., G. L. Weiss and R. B. Kellogg, and the Proposer.

Editorial Note. C. D. Gorman calls any point $x_0\in M$ for which (1) holds when $x\neq x_0$, a *contraction point* of M. He proves that there cannot be more than two contraction points of M; he proves also that if $f(x_0)\neq x_0$, M contains at least as many components as there are distinct points in the sequence $\{f^n(x_0)\}$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

1. *Technical Mathematics.* Harold S. Rice and Raymond M. Knight. McGraw-Hill. 1954. 14+748 pages. $6.50.
2. *Functional Mathematics.* Book 1. W. Gager, C. Carlton, C. Shuster and F. Kokomoor. Charles Scribner's Sons. 14+434 pages.
3. *Functional Mathematics.* Book 2. W. Gager, C. Carlton, C. Shuster and F. Kokomoor. Charles Scribner's Sons. 15+447 pages.

Technical Mathematics, written "especially for junior colleges, technical institutes and other schools where students are being prepared for employment as engineering technicians equipped with the ability to apply fundamental mathematics to the problems of industry," covers arithmetic and methods of computation, intuitive geometry, introductory algebra, trigonometry, analytic geometry, and vector algebra.

The book is divided into seven teaching units. In addition, there are three appendices on Computation Aids and Approximations, Formulas of Geometry and Mensuration, and Solution of Higher Degree Equations. The selection of topics and manner of presentation are traditional, as would be expected in a text of this kind, and it contains materials for two or three semesters. An early introduction to the slide rule and methods of approximations should be of great benefit to any student who will but make use of these aids. An excellent chapter

on tables and interpolation is made somewhat confusing by the absence of the tables (squares and circumferences) referred to. A large number of applied problems is a feature of this text, the problems being of wide selection and appeal. Teachers using this text should have no difficulty in answering the usual question: "But what is the *use* of this?"

The two volumes in the *Functional Mathematics* series are apparently intended for the first two years of high school, although there seems to be no explicit statement to that effect. The reviewer found mention of a *Book 3* in the preface, and feels that the number of volumes, and the grades they are intended for, should appear somewhere on the cover or the title page.

The use of the word "functional" in two different senses is unfortunate. For example, on page iv of the preface is the statement, "Mathematics is the study of functional relationships," and a few sentences later, "It is evident, therefore, that functional mathematics is mathematics which functions in the daily life of the student."

The contents of these two volumes include the materials of the traditional courses for the ninth and tenth grades, with such additional topics as installment buying, life insurance, saving and investing, taxes, and considerable material on computation with approximate numbers. Features of these books are a "maintenance program" of spiral learning, a high degree of applications to life situations, and an easy, informal style, written to the student. Teachers in high school who have broken away from the traditional compartmentalized courses will welcome these books.

M. D. EULENBERG
Wright Junior College
Chicago, Illinois

Infinity. By Lillian R. Lieber. New York, Rinehart & Company, Inc. 1953. 10+359 pages. $5.00.

It was both a genuine pleasure and a profitable experience to read this book, and nothing that may be said farther along in this review is to be taken as unfavorable criticism. Every student of mathematics from the elementary calculus level up should read it, and many of the keener students below that level will find it useful.

Printed in the usual Lieber style of short lines of varying length to "make reading easier," the book first introduces the reader to SAM; the S standing for science, making observations, testing, gathering facts from the outside world, and realism in general; the A representing intuition, hunches, emotions, imagination, and postulations; and the M signifying logical reasoning, mathematics. The author appeals to the reader in a sort of refrain, as the chapters develop, to make full and well-balanced use of the SAM within him, not only in his study of mathematics itself, but also in all of life.

After a series of illustrations given to show that infinity is not "something very large," the author deals with the mathematical point of view, showing how

mathematicians first began with the idea of a "potential" infinity and later developed the notion of "actual" infinity in the form of sets, each having an infinite "number" of members.

Potential infinity is illustrated by means of "division" by zero, the extension of a line without limit, and a study of conic sections in which it is shown that the parabola has just one ideal point and the hyperbola two. Non-Euclidean geometries are clearly and interestingly treated, and the terms hyperbolic, elliptic, and parabolic as applied to geometry are explained.

In the treatment of "actual" infinity, Cantor's theory of sets is discussed at great length and with good effect. Different kinds of sets are defined, described, and illustrated by means of examples. After introducing transfinites, the author goes into a discussion of operations with them. Some of the postulates governing transfinite cardinals and ordinals are explained and illustrated.

There is at about this point somewhat of a digression to acquaint the reader with the concepts of the fourth and higher dimensions. This is done by means of the binomial theorem, cutting a square to show that $(a+b)^2=a^2+2ab+b^2$, a cube to illustrate $(a+b)^3$, and thus raising the question as to how $(a+b)^4$, *etc.*, could be presented geometrically. The bare statement that the binomial theorem was originated by Newton is in the opinion of this reviewer hardly correct.

The calculus of Newton and Leibnitz is nicely done, including the relation between differentiation and integration and an explanation of the fundamental theorem of integral calculus. The convenience of the calculus for finding areas bounded partly or entirely by curves is taken up after an exposition of the difficulties the ancient Greeks had with this type of problem. Archimedes' quadrature of the parabola is used to illustrate.

It would have been appropriate and extremely interesting, especially when explained in the Lieber style, to have included what is usually called "The Method" of Archimedes for discovering new theorems. Archimedes wasn't far from the integral calculus when he applied his method to the segment of a parabola, for his scheme of summing lines to get an area was very similar to our idea of slices in integration.

The author concludes her work with a discussion of modern integration and some of the tough problems in the theory of transfinites. The various types of integrals developed by Riemann, Stieltjes, Lebesgue, and Denjoy are mentioned, with the most attention given to functions that are Riemann integrable.

There are some 25 or 30 pages of art work that adds a great deal to make the book interesting and challenging. To this reviewer, some of the sketches were also quite bewildering—as was probably intended.

It was not the purpose of this review to search for minor errors, but if there had been many they would certainly have been noticed. One slip was observed on page 20 where in a footnote the length of an inch in centimeters was given as 2.4.

F. W. Kokomoor
University of Florida

NEW BOOKS RECEIVED

Algebra for College Students. By W. W. Whyburn and P. H. Daus. New York, Prentice-Hall. 1955. xi+290 pages.

Mathematics of Engineering Systems. By D. F. Lawden. Published in America by John Wiley and Sons, New York. 1955. viii+380 pages. $5.75.

Les Fondements Logiques des Mathematiques. By E. W. Beth. Paris, Gauthier-Villars. 1955. xv+241 pages. $7.38.

Axiomatique Intuitionniste Sans Negation de la Geometrie Projective. (No. 6 in Collection de Logique Mathematique, Series A). Paris, Gauthier-Villars. 1954. 108 pages. $3.72.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York, New York University Press. 1954. x+154 pages. $5.00.

Traite de Mecanique Rationnelle. By Paul Appell. Paris, Gauthier-Villars. 1955. 202 pages.

Memorial des Sciences Mathematiques, Le Calcul Symbolique a deux variables et ses applications. By L. Poli and P. Delerue. Paris. 1954. 77 pages.

Cours de Geometrie Infinitesimale. By Gaston Julia. Paris, Gauthier Villars. 1955. 80 pages.

Functional Mathematics. Book 3. By W. A. Gager, L. C. Lyle, C. N. Shuster and F. W. Kokomoor. New York, Charles Scribner's Sons. 1955. 13+481 pages. $3.20.

Introduction to Modern Algebra and Matrix Theory. By R. A. Beaumont and R. W. Ball. New York, Rinehart and Co. 1954. xii+331 pages. $6.00.

Contributions to the Theory of Partial Differential Equations. Edited by L. Bers, S. Bochner, and F. John. Princeton, New Jersey, Princeton University Press. 1955. 257 pages. $4.00.

Psychological Statistics, Second Edition. By Quinn McNemar. New York, John Wiley and Sons, Inc. 1955. vii+408 pages. $6.00.

Algêbre. By Paul Dubreil. Paris, Gauthier-Villars. 1954. 467 pages. $11.71.

Trigonometry. By Roy Dubisch. New York, The Ronald Press Company. 1955. 395 pages. $5.00.

Oeuvres de Marie Sklodowska Curie. Edited by Irene Joliot Curie, Warsaw, Academie Polonaise des Sciences, 1954. xii+685 pages.

Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues. U. S. Department of Commerce, National Bureau of Standards Applied Mathematics Series 39. Edited by Olga Taussky. Washington 25, D.C., Government Printing Office. 139 pages. 1954. $2.00.

Exploring Mars. By Robert S. Richardson. New York, McGraw-Hill Book Company, Inc. $4.00.

Engineering Cybernetics. By H. S. Tsien. New York, McGraw-Hill Book Company. 1954. xii+289 pages. $6.50.

La Theorie Harmonique. By Andre Lamouche. *Le Principe de Simplicite dans les Mathematiques et dans les Sciences Physiques*. Paris, Gauthier-Villars. 1954. $5.43.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

ANNUAL MEETING OF THE ASSOCIATION FOR COMPUTING MACHINERY

The annual general meeting of the Association for Computing Machinery will take place at the Moore School of Electrical Engineering, University of Pennsylvania, September 14–16, 1955. As in the past, this meeting is intended to serve both as a place for the reporting of new ideas and developments in the applications of computing machinery and as a place for renewing old friendships and making new ones.

CONFERENCE FOR TEACHERS OF MATHEMATICS AT UNIVERSITY OF CALIFORNIA AT LOS ANGELES

The Fifth Annual Conference for Teachers of Mathematics, together with a Fourth Annual Mathematics Laboratory, will be held at the University of California at Los Angeles, July 5 to 15, 1955. The University Extension will present the sessions in cooperation with the Departments of Mathematics and Education of the University, the California Mathematics Council, and the National Council of Teachers of Mathematics.

The purpose of the Los Angeles conference is to bring together teachers interested in mathematics—arithmetic through calculus—to study problems in the teaching of mathematics and to learn new uses of mathematics in various fields of endeavor. For further information, write to University of California Extension, Los Angeles 24.

FELLOWSHIPS FOR SECONDARY SCHOOL MATHEMATICS TEACHERS

Case Institute of Technology has announced that thirty fellowships for secondary school mathematics teachers are available. These fellowships are designed to pay the expenses of attending a program which will be held at the Institute, June 19 to July 29, 1955. This program is sponsored by the E. I. du Pont de Nemours Company.

Courses of study in the program together with faculty members are as follows: Professor Morris, modern ideas in geometry; Professors McCuskey and Crowder, the calculus and its applications; Professor Brown and assistants, modern methods of computation. Inquiries should be addressed to: Dean Elmer Hutchisson, Case Institute of Technology, 10900 Euclid Avenue, Cleveland 6, Ohio.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1955:

State University of Iowa. June 14 to August 10: Professor Conkwright, differential equations; Professor Dye, introduction to topology; Professor Hogg, elements of statistics; Professor Muhly, survey of elementary college mathematics; Professor Oberg, elementary theoretical mechanics, numerical integration of differential and integral equations; Professor Woods, studies in secondary algebra, constructive geometry; additional courses scheduled are advanced calculus, matrices and determinants, elementary theory of numbers, reading in mathematics, and research.

University of California at Los Angeles. June 20 to July 30: Professor Sokolnikoff, tensor analysis; Staff, seminar in mathematics, seminar in numerical analysis.

University of Delaware. June 20 to July 29: Professor Remage, fundamentals of geometry. (Graduate credit available for teachers.)

University of Michigan. A number of courses in applied statistics and statistical methods will be given under the auspices of the Departments of Mathematics (see this MONTHLY, April, 1955), Economics, Sociology, Psychology, and the School of Business Administration. Also, the following special programs will be offered: July 18 to August 12, Program in Survey Research Techniques; June 20 to August 12, Summer Institute in Mathematics for Social Scientists; August 1 to 12, Program on Digital Computers; August 17 to 27, Program on Quality Control. Inquiries regarding courses in statistics should be addressed to Professor C. C. Craig, Department of Mathematics, and inquiries regarding allied programs should be addressed to the Office of the Summer Session.

The University of Michigan announces also its Third Annual Workshop for College Professors, June 20 to July 8, 1955. There will also be a one-week Institute on College Administration, July 11 to 15. The University will also offer a six-week course entitled, The College Teacher. The Workshop and the Institute are directed by Professor A. D. Henderson.

University of Minnesota. June 13 to July 16: Professor Roy of University of North Carolina, multivariate analysis, experimental designs for research workers; Professor Loud, methods of applied mathematics; Professor Nering, advanced algebraic theory, analytic projective geometry. July 18 to August 20: Professor Hatfield, development of the number system, special functions; Professor Loud, methods of applied mathematics.

University of Pittsburgh. June 13 to July 22 and July 25 to August 2: Professor Laush (first session) and Professor Blumberg (second session), differential equations; Professor Christiano, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable; Professor Bryson, partial differential equations, Fourier series; Professor Levine, topology; Professor Bompiani, non-euclidean geometry. July 5 to August 12: Professor

Knipp, algebra for teachers, solid analytic geometry; Professor Myers, recreational mathematics for teachers; Professor Levine, theory of equations; Mr. Kachun, navigation for teachers. August 15 to August 26: Mr. Sebesta, geometry for teachers. June 20 to August 12 (evenings): Professor Bryson, Laplace transform theory and applications.

College teachers of mathematics may work for advanced degrees at the University of Pittsburgh by taking course work in the summer and doing reading and research during the academic year at their own institutions.

University of Texas. June 7 to August 31: Staff, advanced calculus with engineering applications, differential equations with engineering applications. June 7 to July 18: Professor Ettlinger, advanced calculus, differential equations and applications; Professor Greenwood, interpolation and graphical methods; Professor Guy, Fourier and Laplace transforms; Professor Moore, introduction to the foundations of geometry, theory of sets; Dr. Nicol, introduction to abstract algebra and number theory; Staff, conference course, thesis, dissertation. July 19 to August 31: Professor Cooper, theory of functions of a complex variable; Professor Lane, advanced calculus, introduction to the applications of continued fractions; Professor Lubben, introduction to modern projective geometry, topics in modern algebra; Professor Wall, analytic functions, infinite processes; Staff, differential equations with applications, thesis, dissertation.

University of Virginia. June 20 to August 13: Professor Botts, differential equations, applied mathematics; Professor Ball, transformation theory; Professor McShane, advanced analysis. July 5 to August 13: Professor Botts, foundations of algebra.

Wayne University. The Computation Laboratory announces four special summer courses: June 6 to June 11, electronic computers, business and engineering applications; June 13 to June 18, automatic data processing; June 20 to June 25, mathematical programming of management problems; June 27 to July 2, numerical methods and advanced programming techniques.

PERSONAL ITEMS

Rutgers University reports: Dr. R. K. Brown, formerly in military service, has been appointed to an instructorship; Professor O. H. Alisbah of the University of Ankara, Turkey, has been Visiting Professor for the academic year 1954–1955.

Professor A. A. Albert of the University of Chicago has been reappointed as a member of the General Sciences Panel Advisory to the Assistant Secretary of Defense for Research and Development, D. A. Quarles, for the year 1955.

Mr. H. W. Baker, formerly a student at Nebraska Wesleyan University, is now a teaching assistant in the Department of Chemistry, Purdue University.

Mr. W. R. Brittenham, previously a graduate student at the University of Wisconsin, Milwaukee, has a position as a mathematician at the A. O. Smith Corporation, Milwaukee, Wisconsin.

Mr. R. L. Brooks, formerly a mathematician with the National Bureau of Standards, Washington, D. C., is now employed as a flight test analysis engineer by Lockheed Aircraft Corporation, Burbank, California.

Mr. G. C. Bush, previously a student at McMaster University, has been appointed to a research assistantship at Massachusetts Institute of Technology.

Dr. K. H. Carlson of Michigan State College has been appointed to an assistant professorship at Valparaiso University.

Professor D. R. Carpenter of Roanoke College has retired with the title of Professor Emeritus.

Professor J. C. Cothran of the Department of Chemistry, University of Minnesota, Duluth, has been appointed Lecturer at Kansas State Teachers College, Emporia.

Dr. M. D. Davis of the Institute for Advanced Study has been appointed to an assistant professorship at the University of California, Davis.

Dr. W. A. Horning, formerly a theoretical physicist for the Hanford Works, Richland, Washington, is now a physicist for the Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. R. L. Huckins, previously a teaching fellow at the University of Wichita, has been promoted to an instructorship.

Mr. P. M. Moskowitz, recently a dynamics engineer for Sikorsky Aircraft, Bridgeport, Connecticut, has a position as a programmer for Remington-Rand Corporation, New York City.

Mr. L. R. Norwood, formerly a senior engineer at the Electronic Defense Laboratory, Sylvania Electric Products, Mountain View, California, has a position as an administrative engineer and member of the technical staff, Hughes Research and Development Laboratories, Culver City, California.

Dr. R. H. Owens, previously in the Office of Naval Research, Pasadena, California, has been transferred to Washington, D. C.

Dr. T. D. Riney of Purdue University has accepted a position with the Bell Telephone Laboratories, Allentown, Pennsylvania.

Professor C. E. Rusch of Mission House College has been appointed to an assistant professorship at Wisconsin State College, Eau Claire.

Mr. W. D. Serbyn, formerly a graduate assistant at the Carnegie Institute of Technology, is now a part-time instructor at the Institute of Technology, University of Minnesota.

Dr. R. G. Stoneham of the University of California has been appointed a mathematician with Logistics Research, Inc., Redondo Beach, California.

Professor Irving Sussman of the University of San Francisco is on leave of absence for the year 1954–55 and is an associate professor at California State Polytechnic College.

Mr. C. J. Vanderlin, Jr., formerly a teaching assistant at the University of Wisconsin, Extension Division, has been promoted to an instructorship.

Dr. R. S. Varga, previously a teaching fellow at Harvard University, has a

position as a senior scientist for Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Professor Emeritus G. W. Gorrell of the University of Denver died on January 9, 1954. He was a member of the Association for thirty-three years.

Professor Emeritus J. M. Kinney of Wilson Junior College, Chicago, Illinois, died on January 19, 1955. He was a charter member of the Association.

Professor Emeritus H. F. Price of Pacific University died on January 12, 1955.

Mr. A. C. Washburne, Actuary Emeritus of the Berkshire Life Insurance Company, died on August 26, 1954. He was a charter member of the Association.

Assistant Professor Harold Weintaub of Tufts College died on November 7, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE JOINT COMMITTEE OF THE AMERICAN SOCIETY FOR ENGINEERING EDUCATION AND THE MATHEMATICAL ASSOCIATION OF AMERICA ON ENGINEERING MATHEMATICS

This Committee was appointed in the fall of 1953 and charged with:

1. Consideration of how mathematics can be taught in the most effective way for engineers;
2. Acting as liaison between ASEE and the MAA;
3. Keeping in close contact with the MAA Committee on the undergraduate mathematical program under the Chairmanship of W. L. Duren.

The matter of liaison and of keeping in contact with Duren's committee have been effected in the following ways. The first meeting of the Committee was in Baltimore on December 30, 1953 in conjunction with the annual meeting of the MAA. The second meeting of the Committee took place in Urbana on June 17, 1954 at the annual meeting of the ASEE. The MAA Committee on the Undergraduate Program was represented at our first meeting by W. L. Duren and at the second by G. B. Price. At the invitation of R. S. Burington, Chairman of the Mathematics Division of the ASEE, Price also presented to the Division a report on current efforts to revise the undergraduate program in mathematics.

The liaison between the MAA and the ASEE is probably best maintained through the Mathematics Division of the ASEE. Mathematicians engaged in teaching engineers should be encouraged to join and to support the activities of both the ASEE and the MAA. Continued publicity about the activities of the Mathematics Division of the ASEE in the MONTHLY should be encouraged.

Mathematics for Engineers. The Committee has directed its attention to the following three questions in this connection.

1. What mathematics should be taught?
2. Who should teach it?
3. How should it be taught?

What Mathematics should be taught to engineering students? This question is the subject of study and debate from many sides.

The ASEE's Committee on Evaluation of Engineering Education in its Interim Report of June 15, 1954 (p. 10) states: "At the undergraduate level, competence in the theory and use of simple ordinary differential equations and their application to the solution of physical problems lies close to the boundary of minimum acceptability of mathematics in any satisfactory engineering curriculum. For students whose interests will be centered in research, development, or the higher phases of analysis and design, or who contemplate subsequent graduate study in engineering, additional mathematics may be both desirable and necessary."

The MAA's Committee on the Undergraduate Mathematics program has endeavored to formulate a first year mathematics course which would "try to incorporate into collegiate mathematical teaching the valuable results of modern researches in mathematics and logic and to eliminate that which is unnecessary or faulty." While they have in mind a program for *all* students who study mathematics in college, they are thoroughly aware of the fact that "the revision must concentrate upon the needs of the vast majority ('95 per cent') of students who take mathematics to increase their powers for solving problems and grasping the ideas of science. Thus the approach to mathematics at this stage (*i.e.* during the first two years of college) shall be *operational* rather than merely *appreciative*." They are experimenting with, and currently engaged in writing text materials for, a basic three-hour course for all students who take mathematics for the first year. In the first semester this would take up: graphs; functions; limits; introductory calculus based on polynomials, exponentials, and logarithms. The material in the second semester would include: sets, mathematical language, groups, combinatorics, distributions, approximate sums, probability. The basic course would be supplemented for engineering and physical science students with a two-hour course as follows:

First semester

Numerical trigonometry, graphical analysis, Newton's methods, numerical integration, tables of logarithms and exponential functions, slide rule, log and semi-log paper, rate problems.

Second semester

Worded problems, binary representation, computing machines, centroids and moments of inertia, averages, precision in measure and computation, finite differences, tables of normal probability density, sampling.

For engineering students, this would be continued in the second year into a course in calculus and analytic geometry with geometric and mechanical applications using vector methods, which would also include linear differential equations with constant coefficients. The program has been tried, with success, at Tulane.

There are other experimental mathematics programs being tried at the University of Washington, Haverford College, Carleton College, The Illinois Institute of Technology, Brown University, Yale, and no doubt at many other institutions.

High School Preparation. So far as the undergraduate curriculum in engineering is concerned, the crucial question seems to be "how much useful mathematics can be taught to the engineers in the time now allotted to mathematics in the undergraduate curriculum?" The answer to this question obviously depends upon the previous mathematical preparation of the entering freshmen. Where this preparation is meager, it is now customary for the students to take college algebra, trigonometry, and analytic geometry in the freshman year and to follow this with a course in calculus in the sophomore year. Where the high school preparation includes trigonometry as well as a strong background in algebra, it is possible to give a combined course in calculus and analytic geometry beginning in the freshman year.

But not all entering freshmen engineering students will have had strong preparation in high school mathematics. There is reason to hope that significant improvements can be made in our methods of taking care of these students. Williams College has been successful in taking students without trigonometry directly into the unified calculus, analytic geometry, and differential equations course and developing the necessary analytic trigonometry as it is needed in the calculus. The universal freshman program envisioned by the MAA Committee on the Undergraduate Program will presuppose only one year of high school algebra and one year of geometry. A program in mathematics and physics was devised at Lafayette College for students enrolled in the preinduction program sponsored by the Ford Foundation. Most of these men entered college after two years of high school. They had studied neither physics nor trigonometry and their mathematical background was primarily limited to one year of algebra and to one year of plane geometry. The two year program at Lafayette included the essential parts of trigonometry, analytic geometry, calculus, and differential equations. Experiments along these lines while not conclusive do at least suggest the possibility of by-passing some of the traditional prerequisites for the calculus with a resulting acceleration and expansion of the useful mathematics that can be included in the normal engineering curriculum. However, some serious questions have been raised in connection with these experimental programs. The major items of concern seem to be:

1. that the existence of the experimental programs, with the low level of mathematical pre-requisites for some of them, may result in a further

weakening of the high school mathematics preparation of future engineers and scientists. It is taken as axiomatic that high school students will not learn as much mathematics in two years as they would in four. In the face of increasing demands for mathematical competence, the colleges and universities (so the argument goes) should, if anything, be upgrading their entrance requirements in mathematics for students in engineering and the physical sciences. And the University of Illinois, for one, seems to be moving in this direction.

2. that the experimental courses may attempt to survey so many topics that they will attempt to teach "a little about everything and not much about anything." This may be an exaggerated fear, but it nevertheless exists.
3. that "by far and away the largest portion of engineers get along on and *use* a fair amount of the elements of algebra, plane, solid, and analytic geometry, trigonometry in particular, plus the fundamental concepts of the calculus and copious use of handbooks and tables. Moreover, whatever the shortcomings of the traditional first year mathematics courses in engineering may be, they have at least made a serious attempt to meet the student where his knowledge left off and usually provide time for review and filling in gaps."

This Committee has not had the opportunity to debate the issues raised here. Accordingly, we make no recommendation either for or against any of the experimental programs now under discussion. We do, however, endorse experimentation as such. Moreover, we feel that the prospect for curriculum improvement is enhanced by a lively and serious debate of these issues.

We also wish to call attention to an experiment of a somewhat different nature, designed to stimulate gifted students to make more progress in mathematics at an early age. The Ford Foundation through its Fund for the Advancement of Education has recently sponsored a study called *The School and College Study of Admission with Advanced Standing*. The Mathematics Committee of that study proposed a program in mathematics for the more able students which would include a strong introduction to differential and integral calculus in the *twelfth grade*. And, in a similar vein, we wish to call attention to a recent talk by R. S. Burington on "Mathematics for our time," reported in the Mathematics Teacher, vol. XLVII, No. 5, pp. 295–298, May 1954.

What conclusions, then, has this Committee reached in answer to the question "What mathematics should be taught to engineers"? We wrote personal letters to about forty mathematicians, scientists, and engineers soliciting their comments and advice on the question of what mathematics should be taught to engineers, and how it could be taught most effectively. From the replies received we deduced that:

a. There was a consensus of opinion that all engineers would benefit by studying mathematics through at least the elementary course in ordinary differential equations.

On the question of specific course content we are not prepared to take a stand either for or against the program proposed by the MAA Committee; largely because it is still in the experimental stage and because of certain doubts stated above.

b. Statistics and probability deserve more attention than they now receive. It is not at all obvious how this is to be managed within the present time limitations. One possible solution to the problem has been suggested by the MAA Committee on undergraduate curriculum. This problem deserves further study.

c. It would be highly desirable to provide the opportunity for students to take further electives in mathematics; for example advanced calculus, complex variables, numerical analysis, matrix algebra, and so on.

Who should teach mathematics to engineers? There is, of course, the possibility of a dichotomy with some people answering "the engineering staff, in order to increase motivation" and with others answering "the mathematics staff because of their deeper insight into the logical structure of the subject." In his talk at Urbana, L. W. Cohen referred to mathematics as a language. In order to learn to use this language effectively engineers need teachers who know the language thoroughly and know how to use it. In this connection we quote from the interim report of the Committee on Evaluation of Engineering Education of June 15, 1954: "A minimum level of performance in mathematics should be established whether it be obtained in required mathematics or in engineering courses. However, few engineering courses are taught in a manner to make a significant contribution to the students' knowledge of basic mathematics, nor is time available for this purpose. The engineering sciences and subsequent professional subject matter should be developed by making effective use of such mathematical proficiency, and should be taught by staff members competent to do so."

We are in complete agreement that, in these days when some engineers need far more mathematics than the minimum now required, it is important that mathematics be taught by mathematicians who by their inspiration and enthusiasm can stimulate an interest in mathematics *per se*. Unfortunately, however, the impression exists that some engineering students are taught by mathematicians who have little or no appreciation for the interests and needs of the engineers. We believe that those mathematicians who are responsible for curriculum planning should consult with their engineering colleagues from time to time to try to determine the engineers' current and anticipated needs in mathematics. Then they should try to meet those needs by providing appropriate courses taught by competent, sympathetic teachers.

One experiment which has been tried, and which appears to be worth wider consideration, is for the mathematics department to invite a member of the engineering staff to teach one or more regular mathematics courses. This gives him a chance to become familiar with the structure and timing of the mathe-

matics course, and also puts him in an excellent position to make criticisms and offer suggestions on the course from the point of view of an engineer. This might also be done on a reciprocal basis with a member of the mathematics staff teaching one of the engineering sciences in order to become familiar with the applications of mathematics in it. One member of our Committee has reported that he took part in such an experiment at the University of Illinois and found the experience valuable, particularly the discussions among these involved in the exchange teaching. This sort of an exchange arrangement would be one way of helping a teacher to avoid getting into a rut. Other ways include: individual or group research projects, consulting on mathematics problems arising in industry or governmental laboratories, sabbatical leaves, and so on.

There are two aspects of motivation that bear further mention. On the one hand, mathematics should continually draw upon the physical and engineering sciences for illustrative problems. But, of equal or perhaps greater importance is the fact that motivation is tremendously enhanced if liberal use is made of mathematical techniques in physics and the engineering sciences, both in the first two years and in the later years. We quote again from the interim report previously cited (p. 11): "In the engineering sciences, full use should be made of the prerequisite mathematics, physics, and chemistry, recognizing that repetition is a normal pedagogical necessity, but that it can be most effective only when consciously and purposefully used. Perhaps nowhere else can the qualities of a scholarly engineering faculty be employed so effectively as in the presentation of these engineering sciences with an appropriate mathematical understanding."

How shall mathematics be taught to engineering students? These students are interested in mathematics as a tool. The majority of them are not content to study mathematics for its own sake. They usually can be induced to become interested in "why" a technique works, but only after they are convinced that: (a) it is a useful technique, and (b) they understand how to apply it. Teachers recognize this fact by using a large amount of problem material. In this connection, we should like to call attention to the collection of *Engineering Problems Illustrating Mathematics*, prepared by a committee of the ASEE under the chairmanship of John W. Cell, and published by the McGraw-Hill Book Company in 1943. These problems are divided into categories illustrating the mathematics usually covered in college algebra, trigonometry, analytic geometry, and differential and integral calculus. It has been suggested that more material of this kind would be helpful. In particular, more examples where the emphasis is on the analysis of the problem and the methods for setting it up in mathematical form, would be desirable. A fund of such problems requiring statistical techniques would be particularly welcomed by some teachers. Our deliberations and the replies to our inquiries have indicated needs for:

a. More emphasis on numerical methods.
b. More emphasis on graphical methods.

c. More emphasis on fundamental concepts.
d. More emphasis on translation from physical problems to mathematical problems, and on the interpretation of the results of the solution of the mathematical problems in terms of the given physical problems.
e. More use of mathematics in junior and senior engineering courses.

Much has been published on the question of improvement of teaching methods. The Committee feels it is appropriate to list here, for convenient reference, the following books and articles which are related to the work of the Committee:

BOOKS AND ARTICLES RELEVANT TO THE WORK OF THE COMMITTEE

J. W. Cell, *Engineering Problems Illustrating Mathematics*, McGraw-Hill (1943).
Effective Teaching—McGraw-Hill (1950) (Fred C. Morris, Editor)
Report of the Sub-Committee on Minimum Essentials in Mathematics for Engineering Instruction-California Committee for the Study of Education.
Report of the Committee on Adequacy and Standards of Engineering Education, J. of Engineering Education, vol. 42, No. 5 (1952) pp. 249–254.
Proceedings of the A.S.E.E.

Vol. 39, 1931–32	pp. 299–310
42, 1934–35	149–152, 292–296
45, 1937–38	122–131, 190–194, 548–558
46, 1938–39	716–724
47, 1939–40	394–401, 699–703
49, 1941–42	57– 66, 346–352
50, 1942–43	432–437
51, 1943–44	664–668
52, 1944–45	407–413
54, 1946–47	330–335, 531–535, 536–539, 641–652
55, 1947–48	175–180, 300–307, 308–312, 358–365, 366–373
58, 1950–51	308–310
59, 1951–52	170–172
60, 1952–53	33– 46, 136–144, 472–475

Conclusion. One imagines that the question of "how to teach (anything)" most effectively confronts every conscientious teacher almost constantly. One also feels that the question of *motivation* on the part of the teacher is at least as important as it is for his students. If he feels that *what* he is teaching is interesting, important, esthetic, and useful—then his enthusiasm will almost certainly stimulate his students to *learn*. But no teacher can long maintain a fictitious enthusiasm for subject matter which, to him, is dull, trivial, non-esthetic, and largely useless. It is for this reason that the committee chose to include a consideration of *what* mathematics to teach as well as *how* to teach it. We do not propose any definite answers to either question. But we have pointed out some

experiments which seem to us to be significant. In doing so, we hope not only to recognize and encourage those who are conducting these experiments, but also to stimulate others to devise their own or to try some of the existing ones.

Recomendations. We recommend the following to the ASEE and MAA and their respective members involved in the teaching of mathematics to engineers:

1. Use all appropriate means to encourage better mathematics preparation in the high schools.
2. Teach "operational" rather than merely "appreciative" mathematics to engineers in the first two years of college.
3. Understand the engineers' needs and try to meet them; whatever the level of preparation.
4. Debate vigorously the issues involved in curriculum structure.
5. Make a serious attempt to introduce statistics and probability into the curriculum in the first two years.

Respectfully submitted:
R. S. Burington
J. W. Cell
R. P. Dilworth
W. E. Restemeyer
S. E. Warschawski
G. B. Thomas, Jr., Chairman

FORD FOUNDATION GRANT

The Ford Foundation has awarded a grant of $25,000 to the Mathematical Association of America for a study by the Association's Committee on the Undergraduate Program in Mathematics. This study will review the status of teaching and research utilization in mathematics, and the relationships of mathematics to the sciences and other fields of knowledge.

The grant will be used by the Committee to facilitate communication among those who are working on these problems. The Committee plans to bring together writing teams which will seek to bridge the gap between modern mathematics and the undergraduate curriculum by writing pilot text materials. However the Committee does not plan to use its funds to subsidize the writing of actual text books.

The present chairman of the Committee on the Undergraduate Program is Professor E. J. McShane of University of Virginia. Other members of the Committee are: J. G. Kemeny, Dartmouth College; G. B. Price, University of Kansas; A. L. Putnam, University of Chicago; A. W. Tucker, Princeton University; W. L. Duren Jr., Tulane University, *ex officio*.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The seventeenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, on January 15, 1955. Professor J. G. Herriot, Chairman of the Section, presided at both the morning and the afternoon sessions.

One hundred three persons attended the meeting including the following fifty-eight members of the Association:

H. L. Alder, H. A. Arnold, H. M. Bacon, R. A. Barnett, G. C. Barton, Alice K. Bell, Stoughton Bell, II, M. T. Bird, R. L. Blair, D. W. Blakeslee, W. E. Bleick, S. G. Bourne, Randolph Church, T. H. Dewey, Roy Dubisch, D. G. Duncan, Hazel E. Eggett, G. C. Evans, E. J. Farrell, F. D. Faulkner, Ruth A. Fish, Harley Flanders, L. C. Graue, Hazel M. Hadley, W. R. Hanson, J. G. Herriot, Marjorie L. Hoffman, V. E. Hoggatt, Jr., H. D. Huskey, Walter Jennings, J. L. Kelley, R. M. Lakness, C. M. Larsen, D. H. Lehmer, A. R. Lovaglia, R. B. Merkel, A. B. Mewborn, E. D. Miller, F. R. Morris, W. H. Myers, Andrewa R. Noble, C. D. Olds, C. L. Perry, Jr., J. P. Pierce, M. H. Protter, F. M. Pulliam, C. H. Rawlins, Jr., R. M. Robinson, E. B. Roessler, J. P. Roth, Abraham Seidenberg, Mary V. Sunseri, Irving Sussman, J. V. Talacko, C. C. Torrance, H. G. Tucker, L. A. Walker, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, Professor C. C. Torrance, U. S. Naval Postgraduate School, Monterey; Vice-Chairman, Professor H. L. Alder, University of California, Davis; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

By invitation of the Section, Professor David Gilbarg, Indiana University and Stanford University, delivered an address at the morning session entitled *Free Surface Flows.* Abstract of this address follows:

The hydrodynamical problem of flows with free boundaries is surveyed from the point of view of the mathematical methods used in solving both special flow problems and problems of general theory. In particular, the method of conformal mapping is described and applied to the determination of the Helmholtz flow past a flat plate, and the comparison method initiated by Lavrentieff is applied to the proof of uniqueness of symmetric Helmholtz flows past curved obstacles.

The following papers were presented:

1. *On mean values and harmonic polynomials*, by Professor C. L. Perry, Jr., U. S. Naval Postgraduate School, Monterey.

The Gauss mean value property (average over circles or spheres) was discussed (Kellogg, *Potential Theory*, p. 224). The results of Beckenbach and Reade (*Trans. A.M.S.*, vol. 53, and *Duke Math. Jour.*, vol. 12), Perkins (*Proc. Int. Cong. of Math.*, 1954), and Walsh (*Bull. A.M.S.*, vol. 42), who investigated mean values over regular polygons and polyhedra, were reviewed. Properties of functions having similar mean value relations for irregular configurations were also considered.

2. *A classroom note on the carpenter's square*, by Professor M. T. Bird, San Jose State College.

The locus described by the interior angle of the carpenter's square as the outer edges of the arms remain in contact with two fixed points B $(-a, 0)$ and A $(a, 0)$ is sought. Parametric, rectangular coordinate, and polar equations may not identify the locus as a limaçon. A mechanical model, produces a curve which has an axis of symmetry oblique to the coordinate axes. This inductive approach suggests a coordinate system with pole at the double point and polar axis on the axis of symmetry. The polar equation of the limaçon is established by elementary geometry of the circle.

3. *A remark on Hilbert's nullstellensatz*, by Professor Abraham Seidenberg, University of California, Berkeley.

Let (1): $F_1(X_1, \cdots, X_n)=0, \cdots, F_s(X_1, \cdots, X_n)=0, G(X_1, \cdots, X_n)\neq 0$ be a system of polynomial equations and inequalities over a field K. By regarding all the X_i but one as parameters,

one sees how to eliminate that one from (1), hence how to eliminate all the X_i, and thus decide whether (1) has a solution. If (1) has no solution, the decision method shows how to find an integer r and polynomials $A_1, \cdots, A_s$ such that $G^r = A_1F_1 + \cdots + A_sF_s$. This proof of Hilbert's theorem is so trivial that it can be applied with little modification to algebraic partial differential systems.

4. *Dyadic treatment of rotation*, by Professor F. M. Pulliam, U. S. Naval Postgraduate School, Monterey.

In a combined treatment of vector algebra and solid analytic geometry, the rotation of axes can be more completely covered than in the standard texts on analytic geometry. A simple introduction to dyadics is shown to be sufficient to get the axis of rotation and the angle of rotation from the direction angles of the new cartesian axes and conversely. The dyadic method is natural and easy and might at least be given in the appendix of such a combined treatment.

5. *Analytic geometry from the vector point of view*, by Professor S. G. Bourne, University of California, Berkeley.

From a mathematical point of view, the vector approach to coordinate geometry is superior to the conventional approach, in that concepts do not depend on the dimensionality of the space, and proofs are simplified. From a practical point of view, its superiority rests in the early introduction of vectors into the student's mathematical education, thereby enabling him to utilize the algebra of vectors in his study of the physical sciences and engineering. This treatment is suitable either for a first course or a second course in analytic geometry. The needs of the modern physical sciences are such that an early introduction of these mathematical methods is deemed indispensable.

6. *Fourier kinematic analysis of the reciprocating engine by contour integration*, by Professor W. E. Bleick, U. S. Naval Postgraduate School, Monterey.

The exact Fourier kinematic analyses of the motion of the piston and connecting rod of a reciprocating engine were obtained by contour integration. The simplicity of this exact method compares favorably with that of the usual approximate truncated power series approach. The proper selection of branch cuts is involved. This may be a useful classroom example illustrating the application of contour integration to a simple problem in engineering mechanics.

7. *On the reduction of a matrix to diagonal form*, by Professors M. P. Epstein and Harley Flanders, University of California, Berkeley, presented by Professor Flanders.

This paper has been published in this MONTHLY, vol. 62, pp. 168–171.

8. *The classification of problems in the calculus of variations*, by Professor F. D. Faulkner, U. S. Naval Postgraduate School, Monterey.

The problems usually studied in a beginning course in calculus of variations belong to a class known as the problem of Bolza [G. A. Bliss, *Lectures on the Calculus of Variations*, Chicago, 1946, §68, 69]. If these are expressed as a problem of Mayer, and the differential equations written in the particular form $du_i = K_i(x, y, u_j)$, $i, j = 1, 2, \cdots n$, where x is a preferred variable in the sense that it is monotonic, it becomes clear what type of end conditions can be imposed: they cannot, in general, involve y. The *order* of the system is defined as the order of the resulting differential equation for y and is generally equal to n, but may be less. It determines the number of parameters in the family of extremals through a point.

C. D. OLDS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 7, 1955.
ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.
INDIANA, Butler University, Indianapolis, May 7, 1955.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Teresa, Winona, Minnesota, May 7, 1955.
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA
PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.
WISCONSIN, Cardinal Stritch College, Milwaukee, May 7, 1955.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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CONTENTS

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THE EQUIVALENCE OF DEFINITIONS OF A MATRIC FUNCTION

R. F. RINEHART, Case Institute of Technology

1. Introduction. The general field of matric analysis, which is finding ever-increasing applicability in systems of linear differential equations [**3**], [**13**] and in quantum mechanics, rests upon the extension of the function concept to matrices. Two different basic approaches are available for such an extension. One may consider matrices whose elements are functions of a complex parameter z. From this standpoint one has a mapping of a set of complex numbers into the set of all matrices of a given order. On the other hand one may consider mappings of a set of matrices into a set of matrices. This second concept is the one with which this paper will be concerned.

We shall consider square matrices with elements in the complex field and define a general matric function to be a correspondence f which relates to each admissible matrix Z of order n a matrix W, denoted by $f(Z)$, of order n with elements in the complex field. What constitutes an admissible matrix Z depends of course upon the nature of the correspondence f. If $f(A)$ is uniquely determined by A, then f is said to be *single-valued* at A; otherwise it is multiple-valued at A.

Matric polynomials with scalar coefficients provide familiar examples of such matric functions. Any scalar polynomial $p(x) = \sum_{i=0}^{k} c_i x^i$ with complex coefficients gives rise to a matric polynomial $p(Z)$ by "substitution" of the matrix Z for the indeterminate x. By "substitution" we mean that Z^i is determined by matric multiplication, x^0 is replaced by the identity matrix of order n, $c_i Z^i$ is interpreted as scalar multiplication of Z^i by c_i and the addition as matric addition. Such matric polynomials are matric functions in the sense defined above. They further obey the same rational integral laws of combination as their scalar brothers.

It is natural to inquire whether non-polynomial functions (or more generally non-algebraic functions) of a complex variable similarly admit a matric counterpart or analogue.* Whether or not such a counterpart or extension of scalar functions is useful and interesting depends on what combinatorial properties of scalar functions are preserved in the matric function. Fantappiè [6] was the first writer to state explicitly a desirable set of combinatorial requirements to be fulfilled by a definition of such a matric function. These are the following, where $f(A)$ denotes the matric function arising from the scalar function $f(z)$:

I. *If* $f(z) = k$,	*then* $f(A) = k\ I$,
II. *If* $f(z) = z$,	*then* $f(A) = A$,
III. *If* $f(z) = g(z) + h(z)$,	*then* $f(A) = g(A) + h(A)$,
IV. *If* $f(z) = g(z) \cdot h(z)$,	*then* $f(A) = g(A)\ h(A)$,

* From now on the term "matric function" will be confined to mean a function arising from a scalar function of a complex variable.

where A is an admissible matrix for each of the functions, and where in III *and* IV *matric addition and multiplication are implied in $f(A)+g(A)$ and $f(A)\ h(A)$.**

These requirements will insure (1) that the definition, when applied to a polynomial $p(z)$, will yield the usual matric polynomial $p(A)$, and (2) that any rational identity in scalar functions of a complex variable will be fulfilled by the corresponding matric functions; *e.g.*, if sin A and cos A can be defined in accordance with the above, then $(\sin A)^2+(\cos A)^2=I$.

A fifth requirement that would be highly desirable would be

$$\text{V. If } f(z)=h(g(z)), \text{ then } f(A)=h(g(A)),$$

for all admissible A. This requirement, if attainable, would insure that each *functional* identity among scalar functions would be preserved in their matric function counterparts; *e.g.*, from $e^{\log z}=z$, would follow $e^{\log A}=A$.

The extension of the concept of a function of a complex variable to matric functions has occupied the attention of a number of mathematicians since 1883. The history of the field is unusual in that many of the writers seem to have been unaware of what had been done by their predecessors. It seems that almost every mathematician who became intrigued by the idea proceeded to frame his own definition of a matric function, with little or no attention to connections with earlier definitions.

As a result there have been proposed in the literature since 1880 eight distinct definitions of a matric function, by Weyr, Sylvester and Buchheim, Giorgi, Cartan, Fantappiè, Cipolla, Schwerdtfeger and Richter. Attention has been given in only a few cases to the equivalence, or non-equivalence, with other definitions, or to what combinatorial properties of scalar functions were preserved. The casual reader in the field thus gains the impression that a considerable number of essentially distinct extensions of scalar functions to matrices has been achieved.

The principal purposes of this paper are to show that:

(a) All of the definitions except those of Weyr and Cipolla are essentially equivalent.
(b) Weyr's definition is less general than these six, but coincides with them when it is applicable.
(c) Cipolla's definition is more general than these six.
(d) The requirements I–IV of Fantappiè are fulfilled by Cipolla's definition, and V is fulfilled provided $f(z)$ and $g(z)$ are single-valued.

2. The Definitions of Matric Function. Since a polynomial in a complex variable z yields a well-known and satisfactory matric function, it is natural to expect that attempts to extend the concept of a general scalar function would be strongly motivated by properties of matric polynomials. This, we shall see, is the case for most of the eight definitions now to be summarized.

* Fantappiè also imposed a fifth requirement which he needed in order to utilize the theory of linear functionals in arriving at a definition of a matric function.

2.1 *Sylvester, Buchheim*

Sylvester [14], in 1883, gave as definition of a matric function corresponding to the scalar function $f(z)$

$$f(A) = \sum_{j=1}^{n} \prod_{i \neq j} \frac{A - \lambda_i I}{\lambda_j - \lambda_i} f(\lambda_j)$$

where A is a matrix with distinct characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. This definition, a direct extension of the Lagrange interpolation formula for a polynomial $p(z)$ of degree n, is applicable only when A has distinct characteristic roots.

Buchheim [4], in 1886, generalized Sylvester's definition to the case where the characteristic roots are not necessarily distinct. The generalized interpolation formula of Lagrange

$$p(z) = \sum_{j=1}^{t} \prod_{i \neq j} (z - a_i)^{s_i} \sum_{k=0}^{s_j-1} \frac{1}{k!} q_j^{(k)}(a_j)(z - a_j)^k, \tag{2.1}$$

where

$$q_j(z) = \frac{p(z)}{\prod_{i \neq j} (z - a_i)^{s_i}},$$

as is well-known, defines uniquely the polynomial of degree $n-1 = \sum_{i=1}^{t} s_i - 1$ or less which, together with its derivatives, assumes the set of prescribed values $p(a_i), p'(a_i), \cdots, p^{(s_i-1)}(a_i)$ at each of the points $z = a_i$, $i = 1, \cdots, t$. If $p(z)$ is a given polynomial of degree $n-1$ or less, then (2.1) is an identity in z, and hence (2.1) will remain valid if z is replaced by any square matrix A.

If, however, A is a square matrix of order n, and the a_i are its distinct characteristic roots λ_i, and the s_i are the multiplicities of those roots, then (2.1) holds with z replaced by A *for any polynomial* $p(z)$ *whether of degree less than* n *or not.*

To see this let $p(z)$ be an arbitrary polynomial. Let $c(z) = |A - zI|$. Then by the division algorithm, there exist polynomials $h(z)$ and $r(z)$ such that $p(z) = h(z)\, c(z) + r(z)$ where $r(z)$ is either zero or of degree less than n. It follows that $p(A) = r(A)$. Now the polynomial $p(z) - r(z) = h(z)c(z)$ and also its first $s_j - 1$ derivatives have the value zero at $z = \lambda_j$, for each j. But since the right member of (2.1) is completely determined, for a given A, by the values of the polynomial p and its first $s_j - 1$ derivatives at each λ_j, we have

$$\begin{aligned} p(A) = r(A) &= \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{r(z)}{\prod_{h \neq j} (z - \lambda_h)^{s_h}} \right]_{z=\lambda_j} (A - \lambda_j I)^k \\ &= \sum_{j=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{p(z)}{\prod_{h \neq j} (z - \lambda_h)^{s_h}} \right]_{z=\lambda_j} (A - \lambda_j I)^k \end{aligned} \tag{2.2}$$

which is the right member of (2.1) formed for $p(z)$ and A.

Exactly the same argument could be applied if the s_j represent the multiplicities of the λ_j as roots of the minimum polynomial $m(z)$ of A, using $m(z)$ in place of $c(z)$. Hence (2.2) holds for any polynomial $p(z)$ and any square matrix A with distinct roots $\lambda_1, \cdots, \lambda_t$ of multiplicities in the minimum equation $s_1, \cdots, s_t$.

Now the right member of (2.2), valid for any polynomial function $p(z)$, has meaning if $p(z)$ is any function which is analytic at the repeated zeros of the minimum function and defined at the non-repeated zeros. Hence it is natural to use (2.2) as the definition of any such function of the matrix A. This is Buchheim's definition.

$$(2.3) \qquad f(A) = \sum_{i=1}^{t} \prod_{i \neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j - 1} \frac{d^k}{dz^k} \left[\frac{f(z)}{\prod_{h \neq j} (z - z_h)^{s_h}} \right]_{z=\lambda_j} (A - \lambda_j I)^k$$

where s_i is the multiplicity of λ_i as a root of the minimum polynomial of A.

2.2 *Power series definition—Weyr*

The power series definition of a matric function probably occurred to a number of mathematicians prior to Sylvester's paper. However, E. Weyr [15], in 1887, appears to have been the first one to give a convergence criterion for a matric power series. The power series definition is a natural extension of polynomial functions of a matrix.

Let $f(z)$ be analytic at $z = z_0$, and

$$(2.4) \qquad f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \cdots .$$

Then one can define the corresponding matric function

$$(2.5) \quad f(A) = f(z_0)I + f'(z_0)(A - z_0 I) + \cdots + \frac{f^{(k)}(z_0)}{k!}(A - z_0 I)^k + \cdots$$

provided the series converges.*

By a translation $y = z - z_0$ of the complex plane, series (2.4) is converted into a Maclaurin series in y with unchanged radius of convergence. At the same time, all of the characteristic roots of $B = A - z_0 I$ are, by the theorem of Frobenius, those of A reduced by z_0, so that the characteristic roots of A are subjected to the same translation. Hence the following theorem of K. Hensel [9] originally stated and proved for a Maclaurin series, is valid for (2.5):

* A sequence of matrices Z_n is said to be convergent if each sequence of corresponding elements is convergent, and the limit matrix is the matrix of the limits of the element sequences. An infinite series of matrices is said to be convergent, if the sequence of partial sums S_n converges.

The power series (2.5) *converges if, and only if, every characteristic root* λ_i *of* A *lies within or on the circle of convergence of* $f(z)$, *and for every root* λ_i *of multiplicity* ν_i *in the minimum polynomial of* A, *the* (ν_i-1)*th derivative* $f^{(\nu_i-1)}(\lambda_i)$ *converges.**

This theorem, an extension of a result of Weyr, settles the question of convergence of matric power series. The condition that all characteristic roots lie within, or on some circle of convergence of $f(z)$ is clearly a more stringent demand than that required by the Buchheim definition, hence the power series definition is less broadly applicable. Further, this defect cannot be overcome by a process of analytic continuation.

To try to sidestep this difficulty Ferrar [7] has employed the following device. Let t be any complex number such that the characteristic roots $t\lambda_i - tz_0$ of $tA - tz_0I$ are all less in absolute value than the radius of convergence of

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \cdots .$$

Then

$$f(tA) = f(z_0)I + f'(z_0)t(A - z_0I) + \cdots + \frac{f^{(k)}(z_0)}{k!}t^k(A - z_0I)^k + \cdots$$

will converge, as will be shown in §2.5, to a polynomial in $t(A - z_0I)$ with coefficients which are polynomials in t and $f^{(j)}(t\lambda_i)$, $j = 1, \cdots, s_i - 1$. This polynomial has a perfectly well-defined value for $t = 1$, which is defined to be $f(A)$.

2.3 *Cartan*

E. Cartan, *circa* 1928, proposed in a letter to G. Giorgi, that the matric function $f(A)$ corresponding to a scalar function $f(z)$ analytic at the characteristic roots $\lambda_1, \cdots, \lambda_n$ of A be defined, by analogy with Cauchy's integral theorem, by

$$f(A) = \frac{1}{2\pi i}\int \frac{f(z)dz}{zI - A} \tag{2.6}$$

where the integral is taken for each element of the matrix $f(z)(zI - A)^{-1}$ around a set of admissible closed paths enclosing each of the distinct characteristic roots of A.

Cartan had in mind that this definition would be more readily extendable to infinite matrices, but apparently did not elaborate his proposed definition further. Giorgi made no observations regarding any relationship of this definition to his own.

* Hensel's proof of this theorem is given in [11]. Note, however, that the theorem is slightly misstated there.

The Cartan definition is more restrictive than that of Buchheim in that $f(z)$ is required to be analytic at each characteristic root of A.

2.4 *Fantappiè*

L. Fantappiè [6] in 1928, imposed conditions I–IV of §1, and in addition the condition:

V′. *If $f(z, t)$ is analytic in a parameter t, then the elements of $f(A, t)$ depend analytically upon t.*

He then employed the theory of linear functionals to deduce that the elements f_{rs} of $f(A)$ must be given by the sum of the residues of $-(D_{sr}(t)/D(t))f(t)$ at its singularities, where $D_{sr}(t)$ is the cofactor of the r,s element of $A-tI$, and $D(t)=|A-tI|$.

This definition is, as noted by Cipolla, evidently exactly equivalent to that suggested by Cartan, except for the apparent added requirement V′, which, however, follows as a consequence from Cartan's definition. Fantappiè did not discuss the relationship of his definition to those of Giorgi and Cartan which were probably not known to him at the time, nor to those of Weyr and Buchheim.

2.5 *Giorgi*

G. Giorgi [8] in 1928 proposed a definition of a matric function which can be considered to be motivated by a property of polynomial functions $p(z)$, namely that if P is any non-singular matrix then

$$P^{-1}p(A)P = p(P^{-1}AP), \quad \text{or} \quad p(A) = Pp(P^{-1}AP)P^{-1}.$$

Thus if A is reduced to its Jordan normal form C by a similarity transformation, the polynomial $p(C)$ when re-transformed by the inverse similarity transformation will be $p(A)$.

If this principle is postulated for defining a non-algebraic matric function, then the problem of defining $f(A)$ is reduced to that of defining $f(C)$.

Now C is a direct sum $C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k$ of blocks of the form,

$$C_i = \begin{pmatrix} \lambda_i & 1 & 0 \cdots 0 \\ 0 & \lambda_i & 1 \cdots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots \lambda_i \end{pmatrix},$$

and $p(C)=p(C_1)\dotplus \cdots \dotplus p(C_k)$,* for any polynomial function $p(z)$. To find $p(C_i)$, of order ν_i, we express $p(z)$ in a Taylor expansion about $z=\lambda_i$, getting

$$p(C_i) = p(\lambda_i)I + p'(\lambda_i)(C_i - \lambda_i I) + \cdots + \frac{p^{(\nu_i-1)}(\lambda_i)}{(\nu_i - 1)!}(C_i - \lambda_i I)^{\nu_i-1}$$

since $(C_i-\lambda_i I)^k$ vanishes for $k \geqq \nu_i$. Here I stands for the identity matrix of order ν_i. This yields for the polynomial $p(C_i)$ the matrix

* Note that here the λ_i occurring in different C_i are not necessarily distinct.

$$p(C_i) = \begin{bmatrix} p(\lambda_i) & p'(\lambda_i) & \dfrac{p''(\lambda_i)}{2!} & \cdots & \dfrac{p^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!} \\ 0 & p(\lambda_i) & p'(\lambda_i) & \cdots & \dfrac{p^{(\nu_i-2)}(\lambda_i)}{(\nu_i-2)!} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p'(\lambda_i) \\ 0 & 0 & 0 & \cdots & p(\lambda_i) \end{bmatrix}. \tag{2.7}$$

Now this definition of a polynomial function of C_i has meaning for a non-algebraic function as well. Hence it is natural to define, with Giorgi, corresponding to a scalar function $f(z)$,

$$f(A) = P[f(C_1) \dotplus f(C_2) \dotplus \cdots \dotplus f(C_k)]P^{-1} \tag{2.8}$$

where $f(C_i)$ is defined as in (2.7) with f in place of p, $C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k$ being the Jordan normal form of A yielded by the transforming matrix P.

This definition is clearly meaningful under exactly the same circumstances as Buchheim's definition, since the maximum order of the blocks C_i corresponding to a common characteristic root is equal to the multiplicity of that root in the minimum polynomial of A. In particular, if λ_i is a simple root of the minimum polynomial, then all blocks C containing λ_i will be of order one, and Giorgi's (or Buchheim's) definition does not require analyticity of $f(z)$ at $z=\lambda_i$.

The value $f(A)$ given by Giorgi's definition is independent of the matrix P chosen to transform A into canonical form. This may be seen as follows. Let $C=P^{-1}AP$ be a Jordan canonical form of A. By applying a further similarity transformation $R^{-1}CR$, one can bring together as consecutive blocks all those C_i which correspond to a given characteristic root λ. Call these block sets K_j, $(j=1, \cdots, r)$. Then, clearly,

$$f(R^{-1}CR) = R^{-1}f(C)R = R^{-1}[f(K_1) \dotplus f(K_2) \dotplus \cdots \dotplus f(K_r)]R$$

where the characteristic roots of K_i are different from those of K_j, $i \neq j$.

Now each $f(K_i)$ is a polynomial $p_i(K_i)$ in K_i, namely

$$f(\lambda_i)I + f'(\lambda_i)(K_i - \lambda_i I) + \cdots + \frac{f^{(\nu_i-1)}(\lambda_i)}{(\nu_i-1)!}(K_i - \lambda_i I)^{\nu_i-1} \tag{2.9}$$

where I is an identity matrix of order equal to that of K_i and ν_i is the order of the largest Jordan block in K_i.

By a theorem of M. H. Ingraham [**10**], there exists a single polynomial $u(z)$, of degree less than the degree of the minimum polynomial of A, such that

$$u(K_i) = p_i(K_i), \qquad i = 1, \cdots, r.$$

Hence

$$\begin{aligned} f(R^{-1}CR) &= R^{-1}f(C)R = R^{-1}(u(K_i) \dotplus u(K_2) \dotplus \cdots \dotplus u(K_r))R \\ &= R^{-1}(u(C))R = u(R^{-1}CR). \end{aligned}$$

Now the polynomial u does not depend on the matrix P chosen to reduce A to Jordan form, nor on the matrix R, since it is determined entirely by the blocks K_i which are independent of R and P. Hence for any P, *i.e.*, for any C and R,

$$f(A) = PRu(R^{-1}CR)R^{-1}P^{-1} = u(PCP^{-1}) = u(A).$$

Since the degree of $u(z)$ is less than that of the minimum polynomial of A, the matric polynomial $u(A)$ is unique.

Thus to each matrix A and to each single-valued function $f(z)$, Giorgi's definition yields a unique value $f(A)$ subject to the restriction already noted on the character of $f(z)$ at the characteristic roots of A.

2.6 *Schwerdtfeger*

H. Schwerdtfeger devoted about half of his doctoral dissertation (Bonn, 1935) to matric functions. He rediscovered Giorgi's definition for the case of distinct roots and showed that it was equivalent to the infinite series definition in the case of an entire function $f(z)$.

From power series considerations he was led in [**13**] to the following definition for the case of an $f(z)$ analytic at the repeated roots of the minimum polynomial of A. Letting $m(z)$ be the minimum polynomial, he made the partial fraction separation

$$\frac{1}{m(z)} = \sum_{j=1}^{t} \frac{h_j(z)}{(z - \lambda_j)^{s_j}} \tag{2.10}$$

where $\lambda_1, \cdots, \lambda_t$ are the distinct zeros of $m(z)$. He then defined the polynomials $g_j(z)$ by

$$g_j(z) = \frac{h_j(z)m(z)}{(z - \lambda_j)^{s_j}} \cdot \tag{2.11}$$

In terms of the corresponding matrices $G_j = g_j(A)$, the so-called Frobenius covariants of A, he then defined

$$f(A) = \sum_{j=1}^{t} G_j \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!} (A - \lambda_j I)^k. \tag{2.12}$$

This definition clearly has the same range of applicability as that of Buchheim, $f(z)$ being required to be defined at the characteristic roots of A and to be analytic at the repeated roots of the minimum polynomial. Schwerdtfeger did not relate his definition with any of the definitions given by earlier writers, but did show that the Cartan relation (2.6) was fulfilled by (2.12), and that in the case of distinct characteristic roots (2.12) was equivalent to the Lagrange inter-

polation formula for matrices.

It may be worth noting how (2.12) can be viewed as a generalization of a polynomial function. The matrices G_j, which are polynomials in A, have the following readily verifiable properties:

$$
\begin{aligned}
&\sum_{i=1}^{t} G_i = I \\
&G_j^2 = G_j, && (j = 1, \cdots, t), \\
&G_j(A - \lambda_j I)^k = 0, && \text{for } k \geqq s_j, \\
&G_i G_j = 0, && i \neq j.
\end{aligned} \tag{2.13}
$$

From (2.13), for integral $i \geqq 0$,

$$A^i = A^i \sum_{i=1}^{t} G_j = \sum_{j=1}^{t} A^i G_j = \sum_{j=1}^{t} (AG_j)^i.$$

Hence, if $p(z)$ is any polynomial,

$$p(A) = \sum_{j=1}^{t} p(AG_j). \tag{2.14}$$

Now if each $p(AG_j)$ is evaluated from the Taylor expansion of $p(z)$ about $z = \lambda_j$,

$$p(AG_j) = p(A)G_j = G_j \sum_{k=0}^{s_j-1} \frac{p^{(k)}(\lambda_j)}{k!} (A - \lambda_j I)^k. \tag{2.15}$$

This together with (2.14) yields (2.12) for the case of a polynomial $p(A)$, and since this also has meaning where $p(z)$ is non-algebraic it is therefore natural to view (2.12) as a definition of matric function.

For later work it is important to note that if $P^{-1}AP = C$ is in Jordan canonical form, then $P^{-1}G_jP$ is a matrix which has 1's in the diagonal positions corresponding to λ_j, and zeros elsewhere. For, from (2.11)

$$P^{-1}G_jP = P^{-1}h_j(A)P \prod_{i \neq j} (P^{-1}AP - \lambda_i I)^{s_i} = h_j(C) \prod_{i \neq j} (C - \lambda_i I)^{s_i}.$$

Now the right member is a direct sum of blocks all of which are zero except those corresponding to λ_j. From $\sum G_i = I$ follows $\sum P^{-1}G_iP = I$ and hence, each $P^{-1}G_jP$ is of the form stated.

In his thesis Schwerdtfeger used the characteristic function in (2.10), (2.11) and (2.12). The definition of $f(A)$ so obtained would be slightly less general than that of Buchheim since $f(z)$ is required to be analytic at the repeated roots of the *characteristic* polynomial of A.

2.7 *Cipolla*

M. Cipolla [5] in 1932 gave an extension of Giorgi's definition which, for a multiple valued function $v(z)$, yields additional meaningful values for $v(A)$ besides those provided by the Giorgi definition.

As in Giorgi's definition A is first reduced to a Jordan normal form.

$$C = P^{-1}AP = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_k.$$

A value of $v(C)$ is defined by

$$v(C) = v_1(C_1) \dotplus v_2(C_2) \dotplus \cdots \dotplus v_k(C_k)$$

where $v_i(z)$ is any single-valued branch of $v(z)$. The various possible values of $v(A)$ are then given by $v(A) = P^{-1}v(C)P$ where all values of $v(C)$, and all matrices P such that $P^{-1}AP = C$, are used.

If the same branch of $v(z)$ is used throughout the blocks corresponding to a given characteristic root λ_j, then this definition coincides with that of Giorgi, for a single-valued $f(z)$ may be composed by choosing branch $v_j(z)$ in a region enclosing λ_j but no other characteristic root. For such a function $f(z)$ the Giorgi and Cipolla definitions yield the same value, and as was shown earlier this value is expressible as a polynomial in A, and the value $f(A)$ is independent of the choice of P.

The same view of a multiple valued function can be taken in the definitions of Buchheim, Schwerdtfeger, Fantappiè, Cartan, and Richter.

If, however, different branches of $v(z)$ are chosen for two or more blocks C_i corresponding to a common characteristic root, values of $v(C)$ will be obtained which are not provided by Giorgi's definition. Furthermore, in this case, $v(C)$ need no longer be a polynomial in C, hence $v(A)$ need not be a polynomial in A, and the value $v(A)$ obtained from $v(C)$ will no longer be independent of the matrix P. Hence, in this case, the choice of the matrix P provides a further proliferation of the values yielded for $v(A)$.

If the reader is interested in applying Cipolla's definition to find the various square roots of the identity matrix of order 2, he will readily see the greater scope of Cipolla's definition.

2.8 *Richter*

Richter [12], in 1951, treated what he called the general continuous function, F_u, of a matrix, whose principal imposed properties were:

(1) The elements of $F_u(A)$ are continuous functions of the elements of A;

(2) $$F_u(P^{-1}AP) = P^{-1}F_u(A)P.$$

For a matric function corresponding to a scalar function $f(z)$ he resorted to a power series approach to define a functional value for a Jordan block, and ultimately defined

$$f(A) = \sum_{j=1}^{t} R_j \sum_{k=0}^{s_j-1} \frac{1}{k!} f^{(k)}(\lambda_j)(A - \lambda_j I)^k \tag{2.16}$$

where

$$R_j = w_j(A) \prod_{i \neq j} (A - \lambda_i I)^{s_i},$$

and the polynomials $w_j(z)$ are any polynomials satisfying

$$w_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i}\equiv 1, \qquad \mathrm{mod}\ (z-\lambda_j)^{s_j},$$

where s_j is the multiplicity of λ_j as a root of the minimum polynomial of A. The matrix $f(A)$ obtained does not depend on the choice of the $w_j(z)$.

The similarity of form of Richter's definition to those of Buchheim and Schwerdtfeger is evident. Richter, however, did not appear to have been aware of any of the earlier work in this field except that of Fantappiè.

It is clear that the sphere of applicability of Richter's definition is precisely that of Schwerdtfeger's.

3. The relationship of the definitions. To facilitate comparison of the definitions, some shorthand symbolism is desirable. Let D_i, $i=1, \cdots, 8$ represent symbolically the various definitions of matric function given in the preceding eight sub-sections. Let us define:

$$D_i = D_j$$

if the definitions D_i and D_j, (a) for each scalar function $f(z)$ are applicable for exactly the same set of matrices and (b) yield the same value, or set of values, for each $f(z)$ and admissible A; further define: $D_i \subset D_j$ if D_j is applicable whenever D_i is, but not conversely, and if for each $f(z)$ and admissible A for which both D_i and D_j are applicable, the values $f(A)$ are the same for both D_i and D_j; the relationship of the various definitions can then be stated in

THEOREM 1. $D_2\subset D_3=D_4\subset D_1=D_6=D_8=D_5\subset D_7$.

Proof.

In the discussion of the various definitions we have already pointed out the conditions under which each definition was applicable, and have thus established that aspect of Theorem 1, which is concerned with the relative breadths of applicability of the definitions.

It remains, therefore, to establish that the values ascribed to $f(A)$ are the same for any two definitions applicable to $f(z)$ and A. We have already observed this for D_5 and D_7. The balance of the theorem will then be proved, if we show $D_i\subseteq D_5$, $i\neq 5, 7$.

We prove first that $D_2\subset D_5$. Following essentially Hensel's proof of his theorem, we let, formally,

$$f(A) = \sum_{i=0}^{\infty}\frac{1}{i!}f^{(i)}(z_0)(A-z_0I)^i. \tag{3.1}$$

Consider the partial sums ω_k of this series; ω_k is a polynomial in A and hence, if P is a matrix transforming A to Jordan normal form $\omega_k(A)=P(\omega_k(P^{-1}AP))P^{-1}$ with $\omega_k(P^{-1}AP)=\omega_k(C_1)\dotplus\omega_k(C_2)\dotplus\cdots\dotplus\omega_k(C_t)$, where according to subsection 2.5, for $k\geqq \max$ of ν_j,

$$\omega_k(C_j) = \begin{pmatrix} \omega_k(\lambda_j) & \omega_k'(\lambda_j) & \cdots & \frac{1}{(\nu_j - 1)!}\omega_k^{(\nu_j-1)}(\lambda_j) \\ 0 & \omega_k(\lambda_j) & \cdots & \frac{1}{(\nu_j - 2)!}\omega_k^{(\nu_j-2)}(\lambda_j) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega_k(\lambda_j) \end{pmatrix}$$

where $\omega_k(z)$ is the corresponding partial sum of $f(z)$. Now it is clear that the right member of (3.1) converges if and only if the series $f(z)$ and each of its derivatives up to order $\nu_j - 1$ converge for $z = \lambda_j$, (which incidentally proves Hensel's theorem) and in this case $f(A) = \lim_{k\to\infty}\omega_k = P^{-1}G(P^{-1}AP)P$ where $G(P^{-1}AP)$ here means the value of "$f(C)$" according to Giorgi. Hence $D_2 \subset D_5$.

If the argument of the previous paragraph is carried through for the matrix $t(A - z_0 I)$, and t is subsequently replaced by 1, it is evident, since the characteristic roots of tA are $t\lambda_i$, that Ferrar's proposed extension of the power series definition is equivalent to Giorgi's definition, except for analyticity requirements.

Having previously shown that $D_3 = D_4$, we shall next prove $D_4 \subset D_5$. We first note that from the definition of the integral of a matrix, as the matrix of integrals of the elements, it is readily seen that a constant matric factor can be taken inside or outside the integral sign at will. Accordingly let $P^{-1}AP$ be in Jordan normal form. Then

$$\begin{aligned} \frac{1}{2\pi i}\int \frac{f(z)dz}{zI - A} &= \frac{1}{2\pi i} P^{-1}\cdot \left\{ \int \frac{f(z)dz}{P^{-1}(zI - A)P} \right\} P \\ &= \frac{1}{2\pi i} P^{-1} \left\{ \int \frac{f(z)dz}{zI - P^{-1}AP} \right\} P. \end{aligned}$$

Now $zI - P^{-1}AP$, considered as a matrix with elements in the field of all rational functions of z with complex coefficients, is non-singular and possesses an inverse. Further this inverse will be a direct sum of the inverses of the matrices $zI - C_i$, where C_i has the meaning of subsection 2.5, and I is the identity matrix of the same order as C_i. It is easily seen that

$$(zI - C_i)^{-1} = \begin{pmatrix} (z - \lambda_i)^{-1} & (z - \lambda_i)^{-2} & \cdots & (z - \lambda_i)^{-\nu_i} \\ 0 & (z - \lambda_i)^{-1} & \cdots & (z - \lambda_i)^{-(\nu_i - 1)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (z - \lambda_i)^{-1} \end{pmatrix}$$

and hence $(1/2\pi i)\int f(z)(zI - C_i)^{-1}dz$, taken around a set of admissible closed curves each enclosing a distinct characteristic root of A, gives precisely the $f(C_i)$ of Giorgi's definition, $(1/2\pi i)\int (z - \lambda_i)^{-k} f(z)dz$ being zero for each contour enclosing $\lambda_j \neq \lambda_i$, and equal to $(1/(k-1)!)f^{(k-1)}\lambda_i$ for the contour about λ_i.

We next show that D_6 equals D_1. D_6 can be written*

$$f(A) = \sum_{j=1}^{t} \prod_{i\neq j} (A - \lambda_i)^{s_i} \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!} h_j(A)(A - \lambda_j I)^k,$$

or, expressing the polynomial h_j, of degree s_j-1 or less, as $\sum_{m=0}^{s_j-1} b_m(A-\lambda_j I)^k$,

$$f(A) = \sum_{j=1}^{t} \prod_{i\neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{f^{(k)}(\lambda_j)}{k!} \sum_{m=0}^{s_j-1} b_m(A - \lambda_j I)^{k+m}. \tag{3.2}$$

Since $\prod_{j=1}^{t} (A-\lambda_j I)^{s_j}=0$, we need only consider those terms in the last sum for which $k+m<s_j$.

Now comparing (3.2) with Buchheim's definition, (2.3),

$$\sum_{j=1}^{t} \prod_{i\neq j} (A - \lambda_i I)^{s_i} \sum_{k=0}^{s_j-1} \frac{1}{k!} \frac{d^k}{dz^k} \left[\frac{f(z)}{\prod_{h\neq j} (z - z_h)}\right]_{z=\lambda_j} (A - \lambda_j I)^k,$$

the equality of the two formulas will be established if we show that the coefficients of $(A-\lambda_j I)^k$ in the second sums are the same in the two formulas: that is, we wish to show that

$$\frac{1}{k!} \frac{d^k}{dz^k}\left(\frac{f(z)}{\psi_j(z)}\right)\Bigg]_{z=\lambda_j} = \sum_{m=0}^{k} \frac{f^{(m)}(\lambda_j)}{m!} b_{k-m} \tag{3.3}$$

where

$$\psi_j(z) = \prod_{i\neq j} (z - \lambda_i)^{s_i}.$$

From (2.10),

$$\frac{1}{m(z)} = \sum_{i=1}^{t} \frac{h_i(z)}{(z - \lambda_i)^{s_i}}$$

follows

$$\frac{1}{\psi_j(z)} = \sum_{i=1}^{t} \frac{h_i(z)(z - \lambda_j)^{s_j}}{(z - \lambda_i)^{s_i}} = \phi_j(z)$$

from which

$$\phi_j^{(m)}(\lambda_j) = h^{(m)}(\lambda_j), \qquad \text{for } m < s_j.$$

Now

$$b_r = \frac{1}{r!} h^{(r)}(\lambda_j) = \frac{1}{r!} \phi^{(r)}(\lambda_j), \qquad \text{for } r < s_j.$$

Hence the right member of (3.3) can be written

* This proof was given by F. H. McGar, Jr. in an unpublished master's thesis, 1951.

$$\frac{1}{k!}\sum_{m=0}^{k}\frac{f^{(m)}(\lambda_j)}{m!}\cdot\frac{k!}{(k-m)!}\phi_j^{(k-m)}(\lambda_j)$$

which by Leibniz's formula can be written

$$\frac{1}{k!}\frac{d^k}{dz^k}\left[\left(\frac{f(z)}{\psi_j(z)}\right)\right]_{z=\lambda_j}$$

which is the left member of (3.3).

To show that $D_8 = D_7$ it is only necessary to observe that the definition of the $h_j(z)$ by

$$\frac{1}{m(z)} = \sum_{j=1}^{t}\frac{h_j(z)}{(z-\lambda_j)^{s_j}}$$

follows

$$\sum_{j=1}^{t} h_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i} = 1.$$

In other words the $h_j(z)$ satisfy the congruences

$$h_j(z)\prod_{i\neq j}(z-\lambda_i)^{s_i} \equiv 1, \qquad \text{mod } (z-\lambda_j)^{s_j},$$

and thus can serve as the polynomials $w_j(z)$ in Richter's definition. Richter's formula for $f(A)$ is thus seen to be identical with the Schwerdtfeger definition.

It remains to show that $D_6 = D_5$. According to D_6

$$f(A) = \sum_{j=1}^{t} G_j \sum_{k=0}^{s_j-1}\frac{f^{(k)}(\lambda_j)}{k!}(C-\lambda_j I)^k. \tag{3.4}$$

Then

$$P^{-1}f(A)P = \sum_{j=1}^{t} P^{-1}G_jP\sum_{k=0}^{s_j-1}\frac{f^{(k)}(\lambda_j)}{k!}(C-\lambda_j I)^k$$

where $C = C_1 \dotplus C_2 \dotplus \cdots \dotplus C_r$ is the Jordan normal form of A. Now from subsection 2.6, $P^{-1}G_jP$ has zeros everywhere except in the diagonal positions corresponding to λ_j, where it has 1's. Hence term number m of the right member of (3.4) is a direct sum of Giorgi submatrices

$$\begin{Bmatrix} f(\lambda_m) & f'(\lambda_m) & \cdots & \frac{1}{p!}f^{(p)}(\lambda_m) \\ 0 & f(\lambda_m) & \cdots & \frac{1}{(p-1)!}f^{(p-1)}(\lambda_m) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\lambda_m) \end{Bmatrix}$$

corresponding to the distinct root λ_m. Thus $f(A)$ as given by (3.4) is identical with $f(A)$ as given by Giorgi.

4. The fundamental laws of combination for matric functions. Among the principal writers on matric functions only Fantappiè, Cipolla and Schwerdtfeger concerned themselves with the fundamental question as to whether matric functions, as they defined them, obeyed the elementary rules of combination of scalar functions, although the utility of matric functions depends critically upon these rules. Fantappiè proved that *if* I–IV of section 1 are satisfied, and his additional condition V′ is fulfilled, then *necessarily* the matric function must be of the form he developed. He did not verify, however, that the formula he arrived at thereby necessarily must satisfy I–IV. Cipolla asserted without proof that his definition satisfied Fantappiè's requirements I–IV.

A direct application of Cipolla's definition to the functions k and z makes I and II obvious. The fulfillment of III is also clear, with the precaution that if in $h(z)=f(z)+g(z)$, $f(z)$ and/or $g(z)$ are multiple valued, then the determination used for $h(A)$ is dictated by the determinations used for $f(A)$ and/or $g(A)$. That is, (1) the choice of the branches of $h(z)$ relative to a given block of the Jordan normal form of A must be the sum of the branches of $f(z)$ and $g(z)$ used for that block; and (2) the same transforming matrix P must be used for each of $f(A)$, $g(A)$ and $h(A)$.

The fulfillment of IV is demonstrated as follows.* Let $h(z)=f(z)g(z)$. Then a value of $f(A)g(A)$ is, if the same matrix P is used in computing both $f(A)$ and $g(A)$,

$$\begin{aligned} f(A)g(A) &= P\,|\,f_1(C_1) \dotplus f_2(C_2) \dotplus \cdots \dotplus f_r(C_r)\,| \\ &\quad \cdot |\,g_1(C_1) \dotplus g_2(C_2) \dotplus \cdots \dotplus g_r(C_r)\,|\,P^{-1} \\ &= P\,|\,f_1(C_1)g_1(C_1) \dotplus f_2(C_2)g_2(C_2) \dotplus \cdots \dotplus f_r(C_r)g_r(C_r)\,|\,P^{-1}. \end{aligned}$$

Now a direct computation shows that $f_i(C_i)g_i(C_i)$ is again a triangular matrix with diagonal elements $f_i(\lambda_i)g_i(\lambda_i)$. The element in row r column s above the diagonal is, writing simply $f_i^{(j)}$ for $f_i^{(j)}(\lambda_i)$,

$$\begin{aligned} &\sum_{j=0}^{s-r-1} \frac{1}{j!(s-1-r-j)!} f_i^{(j)} g_i^{(s-1-r-j)} \\ &\qquad\qquad = \frac{1}{(s-r-1)!} \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} f_i^{(j)} g_i^{(s-1-r-j)}, \end{aligned}$$

which by Leibniz's formula is $(1/(s-r-1)!)(f_ig_i)^{(s-r-1)}$. This is exactly the element in row r column s of $h(C_i)$, for the choice $f_i(z)g_i(z)$ as branch of $h(z)$ for block C_i. Hence, if the same transforming matrix P is used in determining $h(A)$ for this set of branch choices, then $h(A)=f(A)g(A)$.

We now prove the important

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* This proof is given by Ferrar in [7]. A variation of it is found in [13].

THEOREM 2.* *Let A be a square matrix with the distinct characteristic roots $\lambda_1, \cdots, \lambda_t$ of multiplicities in the minimum polynomial of A $\mu_1, \cdots, \mu_t$, respectively. Let $g(z)$ be a function which is single valued at each λ_i and analytic at those λ_i whose $\mu_i > 1$. Let $f(z)$ be a function which is single valued at $z_i = g(\lambda_i)$, $i = 1, \cdots, t$ and is analytic at those z_i corresponding to $\mu_i > 1$. Let $h(z) = f(g(z))$. Then $h(A)$ is uniquely defined by Cipolla's (i.e., Giorgi's) definition and $h(A) = f(g(A))$.*

To compute $f(g(A))$ by Cipolla's definition, one begins by reducing $g(A)$ to Jordan normal form, recalling that under the assumption of single-valuedness the value of $f(g(A))$ will be independent of the matrix chosen to effect that reduction. First let P be a matrix reducing A to Jordan form. Then $P^{-1}f(g(A))P = f(P^{-1}g(A)P)$, where $P^{-1}g(A)P$ is a direct sum of blocks of the form

$$K_{ph} = \begin{pmatrix} g(\lambda_p) & g'(\lambda_p) & \cdots & \frac{1}{(\mu-1)!} g^{(\mu-1)}(\lambda_p) \\ 0 & g(\lambda_p) & \cdots & \frac{1}{(\mu-2)!} g^{(\mu-2)}(\lambda_p) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & g(\lambda_p) \end{pmatrix}$$

corresponding to the blocks C_{ph} of the Jordan form of A.

Now $P^{-1}g(A)P$ can be reduced to Jordan form by transforming by a matrix R which is a direct sum of matrices R_{ph} each of which transforms the corresponding block K_{ph} into Jordan form. We now form the functional value of each of the resulting Jordan blocks according to Cipolla. By (2.9) of subsection 2.5, for each of the Jordan blocks in $R_{ph}^{-1}K_{ph}R_{ph}$, and hence for their direct sum,

$$f(R_{ph}^{-1}K_{ph}R_{ph}) = \sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(R_{ph}^{-1}K_{ph}R_{ph} - g(\lambda_p)I)^r$$

holds, whence

$$R_{ph}^{-1}f(K_{ph})R_{ph} = R_{ph}^{-1}\left\{ \sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(K_{ph} - g(\lambda_p)I)^r \right\} R_{ph}.$$

Hence

$$f(K_{ph}) = \sum_{r=0}^{\mu-1} \frac{1}{r!} f^{(r)}(g(\lambda_p))(K_{ph} - g(\lambda_p)I)^r. \tag{4.1}$$

We now wish to show that $f(K_{ph})$ as given by (4.1) is the function $f(g(z))$ formed for the block C_{ph} of $P^{-1}AP$. The term $r=0$ in (4.1) gives $f(g(\lambda_p)$ as the

* A rather complicated proof of this theorem is given by Ferrar [7] for the case of convergent power series. In [13] Schwerdtfeger offers a proof of this theorem which appears to be based on one or more unwarranted assumptions or else to omit some crucial details.

diagonal elements of $f(K_{ph})$. We delete this term from the sum and write $K_{ph}-g(\lambda_p)I$ in terms of powers of the matrix

$$Q = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & & \\ \cdot & & \quad\ 1 \\ \cdot & & \\ 0 & 0 & 0 \cdots 0 \end{bmatrix}$$

which has 1's along the first super-diagonal and zeros elsewhere. Clearly Q^i is a matrix having 1's along the ith super-diagonal and zero's elsewhere. Hence, in terms of Q

$$K_{ph} = g(\lambda_p)I + g'(\lambda_p)Q + \frac{1}{2!}g''(\lambda_p)Q^2 + \cdots + \frac{1}{(\mu-1)!}g^{(\mu-1)}(\lambda_p)Q^{\mu-1}$$

and (4.1) can be written

$$(4.2)\ f(K_{ph}) - g(\lambda_p)I = \sum_{r=1}^{\mu-1} \frac{f^{(r)}(g(\lambda_p))}{r!}\left[g'(\lambda_p)Q + \cdots + \frac{g^{(\mu-1)}}{(\mu-1)!}(\lambda_p)Q^{\mu-1}\right]^r.$$

This shows that $f(K_{ph})$, being a polynomial in Q, will have all terms equal along a given super-diagonal.

Since all derivatives of f appearing are evaluated at $g(\lambda_p)$ and those of g at λ_p, we shall henceforth dispense with writing the arguments.

We now wish to show that the coefficient of Q^β, $\beta<\mu$, in (4.2) is equal to

$$\frac{d^\beta}{dz^\beta}[f(g(z))]_{z=\lambda_p}.$$

By the multinomial theorem the coefficient of Q^β in (4.2) is, since $Q^\mu=0$,

$$\sum_{r=1}^{\mu-1}\frac{f^{(r)}}{r!}\left\{\sum_{(s)}\frac{r!}{\alpha_1!\alpha_2!\cdots\alpha_\beta!}\prod_{i=1}^{\beta}\left[\frac{g^{(i)}}{i!}\right]^{\alpha_i}\right\},$$

where the inner sum is taken over all non-negative integral solutions (s) of the equations

$$(4.3)\qquad \sum_{i=1}^{\beta}\alpha_i = r, \qquad \sum_{i=1}^{\beta} i\alpha_i = \beta.$$

Now the coefficient of Q^β can be rewritten as

$$\frac{1}{\beta!}\sum_{r=1}^{\beta}f^{(r)}\left\{\sum_{(s)}\frac{\beta!}{\alpha_1!\alpha_2!\cdots\alpha_\beta!}\prod_{i=1}^{\beta}\left[\frac{g^{(i)}}{i!}\right]^{\alpha_i}\right\}.$$

By a theorem due to Faà di Bruno [16], this is precisely

$$\frac{1}{\beta!}\frac{d^\beta}{dz^\beta}[f(g(z))],$$

evaluated of course at $z=\lambda_p$.

This shows that the submatrix of $P^{-1}f(g(A))P$ corresponding to a Jordan block of $P^{-1}AP$ is exactly the same as the corresponding submatrix of $h(P^{-1}AP)$. Hence

$$h(A) = Ph(P^{-1}AP)P^{-1} = f(g(A)).$$

The preceding proof fails if either f or g is not single-valued, since the values of the matric function are then no longer independent of the transforming matrix, nor is the functional value of a K_{ph} expressible as a polynomial in A. The precise conditions under which $h(A)=f(g(A))$ for multiple-valued f and/or g, are not known.

5. Observations on Cipolla's definition.

5.1. The definition given by Cipolla is the most general which has been given for a matric function corresponding to a given scalar function. However, even Cipolla's definition does not give all of the values desired for a matric function in some cases, as has been pointed out by V. Amato [1], G. Andreoli [2] and Richter [12].

It can happen that a function $f(z)$ may not be analytic at a multiple root of the minimum polynomial of A, in which case Cipolla's definition fails, although there may be a perfectly "respectable" value for $f(A)$. Consider, for example, $f(z) = \pm\sqrt{z}$ for the argument

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that Cipolla's definition yields the sole value

$$f(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

although evidently

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is also a desired value to ascribe to $f(A)$.

This phenomenon will always occur, for instance, when $f(y)$ is the inverse function of a function $y=g(z)$, analytic at a repeated root of the minimum polynomial of A, for which $g'(\lambda)=0$, since then $f'(g(\lambda))$ is undefined.

Richter has indicated a method by which one may recover such desired values of an inverse function. Andreoli treated the general concept of matric functions from the standpoint of matric power series with matric coefficients.

He was not led to a definition of a matric function corresponding to a given scalar function different from that of Cipolla, but did show that there existed values of a matric function of the category just mentioned.

5.2 *Generalization of Frobenius's theorem.*

A well known and useful theorem in the algebra of matrices is that due to G. Frobenius: *Let A be any square matrix of order n with characteristic roots $\lambda_1, \cdots, \lambda_n$ and $f(z)=g(z)/h(z)$ be any rational function over the complex field, such that $h(A)$ is non-singular. Then the characteristic roots of the matrix $f(A)$ are precisely $f(\lambda_1), f(\lambda_2), \cdots, f(\lambda_n)$.*

From Cipolla's definition, it is obvious that Frobenius's theorem is valid for any function $f(z)$ and for any A to which the definition applies.

5.3 *Functional Values which are polynomials in A.*

In sections 2.5 and 2.6 it was shown that a single valued function $f(z)$, analytic at the repeated roots of the minimum polynomial of a matrix A, always yields a matric value $f(A)$ which is a polynomial in A. This has sometimes been misinterpreted as saying that any function of a matrix is simply a polynomial function. This is not the case, for the polynomial that arises for a given matrix A depends upon the argument matrix, and may be different for different A's. In other words for an indeterminate matrix X, $f(X)$ cannot be expressed as a polynomial in X with constant coefficients.

That $f(A)$ is a polynomial in A is somewhat analogous, for a scalar function of a complex variable considered as a generalization of a real function, to the fact that the value of a function $g(z)$ at an imaginary value $z=a$ can be expressed as a polynomial in a with real coefficients (indeed a polynomial of degree no greater than 1). This, of course, is far from saying that $g(z)$ is a polynomial in z.

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DIGITAL COMPUTERS

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In the field of digital computers great strides have been taken in the past decade. Since the present widespread availability of these machines is having a broad impact on applied mathematics, and since so many of our mathematics graduates are finding employment in computing laboratories attached to industrial and government establishments, it may be of interest to review the present status of this development.

During the war, the tremendous number of ballistic tables needed to describe the flight of the new missiles as they became available made the Army's Ordnance Department and the Navy's Bureau of Ordnance willing customers of the men who came forth with the idea of exploiting Babbage's old (1833) concept of a machine that would compute on the basis of orders given to it before it began its work—a machine that could remember. The Harvard Mark I, the first machine with this power, was developed by the IBM company and Professor Howard Aiken, and worked for the Bureau of Ordnance from the time it began operating in 1944 until after the war. And the Bell Telephone Laboratories Relay Computer gave round-the-clock service at Aberdeen. Toward the end of the war the ENIAC, the first electronic machine, was under development at the University of Pennsylvania for Aberdeen; and this machine, in spite of its limited capacity, achieved revolutionary speeds in the multiplication of two numbers. Its success pointed the way toward the vast changes that were to come.

Although all these machines were being developed by the military primarily to solve the ordinary differential equations of ballistics, they had many other more exciting uses; and there were a few men like John von Neumann, who realized the pressing need for much more powerful machines, capable of handling complex scientific problems, if our atomic efforts were to prosper; and a few others, like Norbert Wiener, who saw the character of the changes they might introduce into industry. This was in 1946. The ENIAC began successful operation in 1947.

At the beginning of 1955, the editors of *Management Methods* published a report on "What Management should know about Electronics for the Office" in which they said:

"The time has come for the businessman to drop the future tense as far as the Automatic Office is concerned. Managements who fail to grasp this new condition—and its implications—may find themselves hopelessly outdistanced by alert competitors before they have time to act.

Here are the facts. . . .

(Some misleading or controversial statements have been omitted)

"Fact I It will take a company not less than 24 months—perhaps longer —to prepare itself for automatic data processing equipment. This does *not* include lead-time for machine delivery. You can reasonably assume a three year target date for getting into electronic operation if you start from scratch to-day.

"Fact II Virtually any firm employing more than 100 clerical workers is ripe for some form of electronic data processing.

"Fact III The price of electronic data processing equipment is consistent with its ability to provide a proper rate of return. Example: a $1,000,000 computer system will pay for itself in three to four years if it replaces only 50 clerical employees.

"Fact IV Not all electronic systems cost $1,000,000. Lower priced, special purpose machines are already in actual use by private firms. New equipment is coming out fast.

"Fact V The fear of obsolescence of present-day equipment is no excuse for failure by management to act. 'ENIAC' . . . is 'obsolete' but still in profitable operation.

"Fact VI Human beings are going to be replaced in staggering numbers by electronic equipment. . . . *The effect of electronic equipment on our economic life is of the same magnitude as the effect of the H-bomb on our military strategy.*"

They go on:

"If businessmen have failed to recognize the actual advent of the Automatic Office, they are not wholly to be blamed. Ample evidence of a general nature has been available for several years, but only recently has specific proof been published. One big barometer could have been read four years ago when almost every one of the large business machine firms committed millions of dollars to electronic development. Companies like Burroughs, IBM, National Cash Register, Remington Rand, and Underwood—all with a tremendous stake in the status quo—made the irrevocable decision that their present products were 'obsolete.' . . . Dozens of the real giants like U. S. Steel and General Electric have completed their long preparation for electronic data processing and have equipment on order or delivered."

The business-machine companies mentioned in this quotation, and others, have all recognized the need of industrial personnel for training in the use of the new computers, and have instituted formal course work to provide this training. For example, Remington Rand has four courses. The first runs for two weeks, and is given about six times a year. It is designed for business executives and members of electronic evaluation committees, and is aimed at providing a general understanding of the techniques for applying general purpose electronic computers to business problems. There are two courses in programming, both running for six weeks, the elementary one given about six times a year, and the advanced one about three times a year. And, finally, there is a course given three times a year, lasting six weeks, designed for operation and maintenance personnel. National Cash Register has a four week basic course in programming, and a six week advanced course. Incidentally it may be worth mentioning that the National Cash Register machine is one of those costing less than a million dollars, and is a general purpose machine.

Probably the most important operational *Fact* among the quotations given above is the first one: It will take a company not less than 24 months—perhaps longer—to prepare itself for automatic data processing equipment. This fact is related to one of the most important characteristics of the great electronic "brains." They are capable of performing remarkable feats, with great speed and accuracy; but they must be told, in the greatest detail, exactly what they are to do. As Claude Shannon has put it, "a digital computer must be instructed in words of one microsyllable." And it is a fact that in a large business operation there is probably no one person who knows in detail and can write down in the most elementary form, all the steps that are taken in a complete business operation. It has proved to be worthwhile in several companies merely to analyze the operation itself, in an attempt to evaluate the desirability of introducing computers. For the analysis of the operation may well disclose undesirable complications and duplications, even though the decision may be reached that data-processing equipment is not needed.

What is the nature of these machines and what types of problems have they been solving?

Mechanical aids to computation go back at least as far as the abacus, but the distinctive thing about these modern machines is that they are automatically sequenced—they can be given instructions *before* they begin to operate, which will enable them to perform each tiny step that they must take in order to complete the problem. With modern electro-mechanical desk computers, as with the abacus, each computational step requires direct human intervention; but with the automatically sequenced machine, the human operator must think through, in fullest detail, the many steps that must be taken, and tell the machine about them on the instruction tape. There are advantages and hazards in this fact. With a desk calculator, the speed of the total operation is subject to human physiological limitations. With an automatically sequenced machine, the speed of computation is vastly increased, but the programming and coding

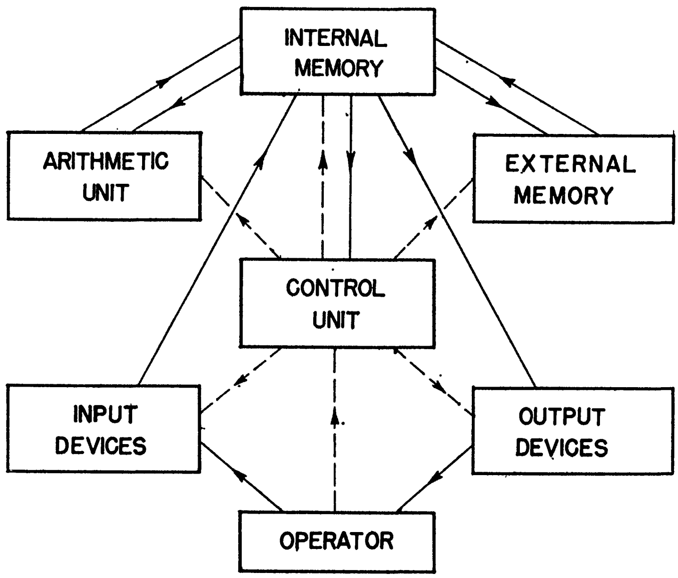
must take into account all the acts of memory and judgment that usually go into a desk calculation.

This means that a code (an instruction tape), once written, must be subjected to a period of "debugging," consisting of trial runs that show up some (hopefully all) of the omissions and mistakes that have been built into the instructions. In recent tests of an IBM machine and a Remington Rand machine on a weather forecasting problem, though there were other considerations that were decisive in selecting the best available machine for the problem, it was discovered that coding for one machine took about 6 man-weeks, while debugging the code took 15 hours of machine time; coding for the other machine took about 17 man-weeks, and debugging took about 50 machine hours. Since one machine hour on these two machines is roughly equivalent to four or five hundred man-weeks on a desk computer, you will see that the debugging operation is non-trivial. One device that has met, at least partially, the problems inherent in correct coding for high speed computers is the development of the library of sub-routines. For example, after a sine function has been coded and run successfully, whenever a sine function is to be computed as part of another problem, the coder splices in the previously tested sub-routine for this operation. This part of the new code then needs no further debugging.

If the disadvantages of extended debugging are great, the advantages of vastly increased speed are greater, so that the new machines have taken their places in our fast moving world. Original applications of these machines were to scientific problems which involved relatively little basic data, but required many thousands, and sometimes many millions of operations on these data. Since little information needed to be fed into the machines, and the answers were short, the original emphasis was on speed within the computing element, with little emphasis on speed in the input and output mechanisms. Business and management problems, on the other hand, ordinarily involve the performance of relatively simple mathematical operations on great masses of data. The answers, too, are sometimes very long. Thus effective utilization of the speed of electronic computers in the solution of problems of this type requires greatly increased speed in input and output mechanisms. In the recent past, there has been a good deal of emphasis on the solution of these problems, and some of the present input and output equipments have attained impressive speed. Many machines are equipped to type out results on an electrically operated typewriter at about 10 characters per second, but output printers operating 40 to 200 times as fast as this are beginning to find widespread use in business practice.

How are these machines put together and how do they operate? Though a full answer to this question would go far beyond the scope of this paper, some of the basic notions can be given. The parts of a computer are usually referred to as organs; and this designation reflects the similarity of their functions to the functions of certain organs of a human being. The block diagram of the machine may be given as:

BLOCK DIAGRAM OF AN AUTOMATIC DIGITAL COMPUTER



——— Transfer of Numbers or Instructions
– – – – Control of Operation

The operator puts problems on the computer through the input devices, causes it to commence operation through a signal to the control unit, and receives the results from the output devices. The input is typically punched paper tape, punched cards, magnetic tape or magnetic wire, although photographic film has also been tried. Whichever of these devices is used records the instructions and the data of the problem, and feeds them into the memory. The control interprets the instructions to the other parts of the machine and causes the desired arithmetical operations to be performed.

For most machines the preferred arithmetic is binary, as contrasted with our usual decimal system. This means that all numbers are represented by using only the two digits 0 and 1. Thus 1 1 0 1 represents

$$1\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2 + 1$$

The reason for this nearly universal use of some form of binary representation is that the typical equipment used in these machines has two states. Thus punched cards and tape are always either punched or not punched in each position, and magnetic tapes, wire and drums distinguish most reliably a pulse from a no-pulse. Likewise, electronic circuits are most easily built with two stable states.

Often the input and output are handled in decimal form, while internally the computer deals with numbers in binary representation, the conversion being made by the computer internally.

The operations that one of these "giant brains" can perform are essentially only the elementary operations of addition, subtraction, multiplication, and sometimes division, as well as one or two other orders like the comparison of two numbers to decide which is greater. This instruction makes it possible to instruct the computer to do one thing if, say, the first number is greater, and a different thing if the second number is greater. Addition is handled either by our usual method, from right to left, "carrying" the 1 when a sum gives ten or more (called in the machine serial addition), or by another method, parallel addition, which adds all the digits simultaneously, as in a desk calculator, reserving for later consideration the inclusion of the "carried" digits. Very fast machine methods have been devised for handling addition in this way.

Some of the characteristics of digital machines now in operation are:

1. They will perform multiplication at speeds sometimes as fast as 30 millionths of a second.
2. They have internal memories sometimes of several thousand words (either numbers or instructions), and external memories many times as large.
3. They are able to follow instructions in succession, and set up automatically the proper connections between machine parts.
4. If at any point in the calculation there are two or more alternative courses for the subsequent operation, they will select and perform the correct one according to specified conditions contained in their instructions.

Something more should be said about the memory devices that are in use, since the speed of computation is importantly affected by this organ, and the kind of problem a machine can handle depends significantly on the size and speed of the memory.

The earliest electronic machines (after the ENIAC) used mercury delay lines as the memory element. A little later magnetic drums were perfected; and then the research on electrostatic storage, which had been initiated very early, was finally successful and produced several varieties of vacuum tube storage. What do we mean by these terms?

A mercury delay line is a form of acoustic storage that consists of a number of tanks containing mercury and equipped with a quartz crystal at each end. Electrical impulses representing digital information are impressed on one of the crystals and are converted by it into acoustical energy in the mercury. The pulses then travel through the tank as sound waves, and are reconverted into electrical pulses by the crystal at the other end. After being corrected for loss of amplitude and definition, the pulses are fed back into the input of the tank. The information can then be recirculated continuously through the tank without loss of synchronism or definition, thus forming a memory device storing pulses the

number of which is determined by the length of the pulse and the length of time required for an acoustic disturbance to travel the length of the tank. The first computer with a mercury delay line memory to operate successfully was the EDSAC, designed and constructed at the Mathematical Laboratory of the University of Cambridge, England. It was completed in 1949. It was inspired by the EDVAC, a machine that was then under construction for Aberdeen at the University of Pennsylvania.

The characteristic multiplication speed of American machines using this type of memory is 2 or 3 thousandths of a second.

When these delay line machines began to operate they increased the multiplication speed by a factor of about one hundred over the speeds available with the older machines that had used telephone relays as the memory element (the multiplication speed of the ENIAC was comparable with that of the delay line machines, but the ENIAC'S limitation was its tiny memory—only 20 words).

Magnetic drum memories which were developed at about the same time as the delay line memories have proved the most reliable and the easiest to put into successful operation. They have been widely adopted by the manufacturers of the more moderately priced machines that are smaller in size and are intended for relatively small businesses. Most of these machines have attained a multiplication speed about one-tenth that of the delay line machines.

A magnetic drum is a cylinder four inches to a foot in diameter and two inches to three feet in length which is coated with a thin layer of ferromagnetic material. Magnetic heads serving, in turn, for both reading and writing, are placed along the drum, often with a clearance of one mil. These heads, between about ten and two hundred in number are generally mounted in groups, with about ten to the inch in the axial direction. The drum must be very accurately balanced so that, in rotating at a speed of perhaps 3600 rpm or faster, there is very little variation in the head-to-drum spacing.

The longest and most trying effort went into the development of the electrostatic tube memory, which makes use of the secondary-emission characteristic and high resistivity of the cathode ray tube phosphor. For a long time the hopes of American machine designers rested in a special tube that was being developed by the Radio Corporation of America, the Selectron tube. But F. C. Williams in Manchester, England, succeeded in building a memory using ordinary cathode ray tubes before the Selectron was put into operation. The Williams principles have been adapted to the design of the American machines, and this is the type of electrostatic tube machine most widely used in America. The speed of multiplication of these machines is about ten times that of the delay line machines.

Although the details of operation of this type of memory are complicated, the basic idea is simple enough. On a single cathode ray tube, a beam is made to store small positive or negative charges, resulting from secondary emission or a spray of secondary electrons, at the points of an array sometimes as dense as 32×32 on the screen of the tube. One cathode ray tube is used for each binary

digit of a word, the digits being recorded in corresponding positions on the faces of the several tubes.

These various types of memory are often used in combination. For example, one of the large commercially available machines has a fast internal memory of over 20,000 digits of electrostatic storage, an auxiliary magnetic drum storage of over 80,000 digits, and an external, magnetic tape memory of over 8,000,000 digits without changing tapes. A second large computer which is on the market has an internal memory of nearly 600,000 digits in magnetic drum storage and up to 150,000 digits of magnetic core storage. This magnetic core storage is the newest and most promising very fast memory element. It is made up of tiny magnetic rings strung together in a matrix. Within the next few years there are bound to be important machines built with this type of storage. The size and speed of this element make its use consistent with that of transistors, and this is undoubtedly a direction in which we may expect developments.

What is the story of the transition I have suggested from the immediate post-war years when the new machines were being developed by the Army and Navy primarily for the solution of ballistic problems to the present wide-spread interest in industry in the use of electronic computers for many facets of the problems of management? Clearly the present situation is a concomitant of large investments in the design and construction of high speed computers by some of the wealthiest companies operating in America today.

In 1946, and continuously until 1951 when the first UNIVAC operated successfully, the big machines were always believed to be just 18 months in the future. After World War II the concern to exploit the potentialities of digital computers for the benefit of federal departments not engaged extensively in military research was expressed first by the Census Bureau which secured the services of the Bureau of Standards as a contracting and monitoring agent in the purchase of a digital computer which was to be used in compiling the 1950 census. This was the UNIVAC that began operating in 1951. However, the staff of the Bureau of Standards, after repeated failures to secure delivery from its various contractors—and there were several other federal agencies for which the Bureau was serving as contracting agent—itself embarked on the construction, with a minimum of development, of an acoustic delay line computer, using the results that had been obtained by the other groups working in the field. This was the first successful delay line computer to operate successfully in America. It built upon the pioneering work in logical design of John von Neumann and his Institute for Advanced Study group as did virtually all other computers not only in the United States but in the world. It benefited by the successes and the difficulties of the EDVAC group at the University of Pennsylvania, which had been working for some time on a delay line computer for Aberdeen. It used much of the current know-how about input and output equipment, and it developed some very clever ideas all its own. But most important, it operated, at a time when many important activities in the United States were sorely in need of a

machine of its speed. This was the SEAC—the Standards Eastern Automatic Computer (as the name suggests, the West-coast branch of the Bureau was also in the business, and ultimately, but much later, had a successful machine in the SWAC—the Standards Western Automatic Computer). It was the SEAC that did the early automatic computations that were done for the Census Bureau, although much of the later work could wait for the UNIVAC (also a delay line computer). After the UNIVAC had been put into successful operation, one by one, other machines began to operate; and some machines that had been in operation, but on classified military problems, were adapted for the commercial market. Remington Rand and IBM embarked on extensive construction and sales programs; many of the small companies (a number of these had been established by imaginative young engineers who wished to enter the competitive market with machines of moderate price) were bought up by other large business-machine companies. Thus the automatically sequenced digital computer came of age as an element in industrial management.

Why has speed of multiplication been emphasized? Since the early machines were designed primarily for the solution of scientific problems, the essential time element to be considered was the time to make the computations that occurred within the machine; and by far the decisive one of these was multiplication. On one fast machine, for example, addition may take 36 microseconds, while multiplication takes 500. Thus a reasonable estimate of the speed of operation of a machine used primarily for scientific computations is given by the multiplication speed. For business uses, like inventory control, this observation is no longer true. Here, the number of items of input is enormous, compared with the computations performed on the items. Thus, input and output equipment becomes most important. We are now capitalizing on the great speeds that have been developed for the internal computing element in such esoteric applications as Russian translations and weather forecasting; and we are developing fast end-organs to take care of business needs.

In the early post war days, this whole development was financed almost exclusively by the government. More recently private capital has taken over responsibility for the specifically business uses, and has further developed the speed and versatility of the internal computing elements.

To give some idea of the scope of the present applications of automatically sequenced computers, a brief list is given of some of the types of scientific problems that have been and are being solved on large scale computers, and some of the categories of business applications that have been developed for these machines. Among scientific problems, solutions of the partial differential equations of hydrodynamics (whose solution in three dimensions involves millions of computations) play an important part; for these equations occur in weather prediction and in explosion theory as well as in various other aspects of wave problems. A sample of problems that have been attacked by these machines includes the computation of the so-called structure factors in crystallography (of great importance in metallurgy) analysis and design of optical systems by ray-

tracing techniques, analysis of emission and reflection of radiant energy from the surfaces of radiant heated enclosures, study of the vibration of aircraft wings, the computation of air flow past a delta wing or rocket fin, analysis of frictional gas flow in jet motors, study of shielding problems in the design of atomic reactors. One of the early IBM super-calculators spent a considerable portion of the time it devoted to scientific problems computing the positions of the five outer planets—Jupiter, Saturn, Neptune, Uranus, and Pluto—at 40-day intervals for a period of 407 years, from 1653 to 2060. The operation involved over 5,000,000 multiplications and divisions, and more than 7,000,000 separate additions and subtractions of large numbers. When this machine sold its time, the charge was $300 per hour. But it was available without charge for problems of general interest to science originating in university or similar research.

Among business problems, studies of inventory control and related questions have received considerable attention, and some machines have been designed specially to handle this problem. It involves simple operations, performed over and over again, and a very simple and inexpensive machine can make vast improvements in operation. General purpose machines have also been programmed for this problem as well as for sales analysis, billing, preparation of payroll on a completely automatic basis, taking account of the great variety of individual deductions and adjustments. At the Bureau of Standards, the SEAC has recently been used to determine the low bidder on government contracts. In general, this analysis is extremely complex. Features that raise complications are: simultaneous purchase for several depots, involving different freight costs from factories to depot and consequent price fluctuation within a given lot; price variation depending on size of purchase from a particular manufacturer, with minimum acceptable and maximum available quantity varying from one manufacturer to another; the necessity for determining the actual lowest bidder on the total transaction exactly and not approximately because of the requirements set by law. The mathematical analysis used in the solution of this problem was based on the method of linear programming developed in connection with the study of logistics problems by the Defense department. This is an illustration of mathematical methods that are emerging, and in fact must be developed if the new machines are to be exploited to anything like their full potential. A new kind of applied mathematics is coming of age.

General purpose computers are in operation in England, Canada, Belgium, Switzerland, Holland, Sweden, Australia, France, Germany, Norway, and Japan, as well as other countries about which we have less information. There are surely over seventy in operation in the United States, with the number constantly rising. The development goes on, with the expanded use of these machines providing one element of concern as the completely automatic factory and automatic office raise problems for labor and management alike.

A NOTE ON THE CLASSICAL GROUPS

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The purpose of this paper is to set forth some topological properties of and relations between certain classical groups. While much of the material included here may be known to a few, the main interest of this paper lies in the simplicity of the proofs of some important, though obscured, results. The groups considered are the full linear group of $n \times n$ matrices over the complex numbers, denoted by $GL(n, c)$, and some of its subgroups: $GL(n, r)$, the subset having real coefficients; $SL(n, c)$, the subset of $GL(n, r)$ with determinant unity; $U(n)$, the set of unitary matrices; $SU(n)$, the unimodular group; $O(n)$, the real orthogonal group; $SO(n)$, the rotation group; and $GL(n, r)^+$, the elements of $GL(n, r)$ having positive determinants. The symbolic notions and definitions of the groups mentioned above are the same as used in [1].

1. Decompositions as topological spaces. It has been shown in [1] that the underlying space of $GL(n, c)$ is the topological product of the space of the n-dimensional unitary group and that of the positive definite hermitian matrices. A second decomposition of this space gives $GL(n, c)$ as the topological product of the non-zero complex numbers and the space of $SL(n, c)$, the commutator subgroup of the algebraic group $GL(n, c)$ and a subgroup of the topological group $GL(n, c)$. Consider the matrix $A \in GL(n, c)$ and define the matrix $P = (p_{ij})$ such that $p_{11} = |A|$, $p_{ii} = 1$ for $i \neq 1$, and $p_{ij} = 0$, $i \neq j$. Also define $B = (b_{ij})$ by letting $b_{1j} = a_{1j}/|A|$ and $b_{ij} = a_{ij}$ for $i \neq 1$. Clearly $PB = A$ and $|B| = 1$. Since P and B are continuous functions of A, the decomposition is continuous. The intersection of all matrices of type P and all matrices of type B contains only the identity, and the decomposition is therefore unique. Since multiplication of matrices is continuous, the identification is a homeomorphism. The set of all matrices of type P is homeomorphic to the non-zero complex numbers.

Simple considerations of the above decompositions of $GL(n, c)$ will yield two decompositions of the underlying topological space of $GL(n, r)$. First, if $\sigma \in GL(n, r)$, then $\sigma = \tau\alpha$, where τ is unitary, α a positive definite hermitian matrix, and $\sigma = \bar{\sigma}$. Since the decomposition of σ, as an element of $GL(n, c)$, is unique; $\tau\alpha = \bar{\tau}\bar{\alpha}$ implies $\tau = \bar{\tau}$ and $\alpha = \bar{\alpha}$; therefore τ is orthogonal and α is positive definite symmetric. Thus, it is proved that the underlying topological space of $GL(n, r)$ is the topological product of that of $O(n)$ and the space of positive definite symmetric matrices. Secondly, let $A \in GL(n, r)$, then $A = PB$ where P and B are defined as above, and $B \in SL(n, c)$. Similarly, $A = \bar{A}$ and the uniqueness of the second decomposition in the preceding paragraph implies that P is real, $B \in SL(n, r)$, and that the underlying space of $GL(n, r)$ is the topological product of the non-zero reals and the underlying space of $SL(n, r)$. A third decomposition, due to the Gram-Schmidt orthogonalization process, gives $GL(n, r)$ as the product of the space of $O(n)$ and that of the triangular matrices with $a_{ii} > 0$.

It follows easily that the space of $GL(n, r)^+$, the component of the identity in $GL(n, r)$, can be decomposed into (1) the product of $SO(n)$, the n-dimensional rotation group, and the set of positive definite symmetric matrices (2) the product of the positive reals and $SL(n, r)$, and (3) the space of $SO(n)$ and the triangular matrices with $a_{ii}>0$.

Let τ be an element of $SL(n, r)$, then $\tau=\sigma\alpha$; σ is orthogonal and α is positive definite symmetric, since τ is also an element of $GL(n, r)$. Since the determinant of τ equals one, σ and α are of determinant one and the uniqueness of the decomposition in $GL(n, r)$ proves that $SL(n, r)$ is the topological product of the underlying space of $SO(n)$ and that of the positive definite symmetric matrices of determinant one. A similar argument shows that this space, $SL(n, r)$, is also the topological product of $SO(n)$ and the space of all triangular matrices having determinant one and main diagonal elements positive. That $SL(n, c)$ can be decomposed into the product of $SU(n)$ and the space of positive definite hermitian matrices of determinant one can be proved easily using the first decomposition of $GL(n, c)$. Moreover, $U(n)$ is shown to be the product of $SU(n)$ and the complex numbers of absolute value one in [1].

2. Connectivity. It is well known that many of the underlying spaces are connected [1]. But by proving $SU(n)$ is connected by first principles, the connectivity of the others comes easily through a consideration of their decompositions and an elementary property of projections. Any $S\in SU(n)$ can be written as UBU^{-1} with U unitary and B a diagonal matrix having $b_{jj}=e^{i\theta_j}$. Define B_t to be the diagonal matrix formed from B by letting $b_{tjj}=e^{it\theta_j}$ for $0\leqq t\leqq 1$. It is clear that $UB_tU^{-1}=f(t)$ is a curve from B to 1. *And $SU(n)$ is therefore connected.* Since $U(n)=T^1\times SU(n)$, and T^1 is homeomorphic to the unit circle, *$U(n)$ is connected.* The set of positive definite hermitian matrices is homeomorphic to a Euclidean space [1]. This fact together with the connectivity of $U(n)$ and the first decomposition of $GL(n, c)$ proves that *$GL(n, c)$ is connected.* Since the projection function of a product onto one of its factors is continuous, *$SL(n, c)$ is a connected set* (consider the second decomposition of $GL(n, c)$). *$GL(n, r)^+$ is a connected set,* because it is the component of 1 in $GL(n, r)$. $SO(n)$ is the image of this connected set under the projection function on the first decomposition of $GL(n, r)^+$, and *$SO(n)$ is therefore connected.* Similarly $SL(n, r)$, being the image of this function on the second decomposition of $GL(n, r)^+$, *is a connected set.*

3. Deformation retracts. A space X is a deformation retract of Y relative to X if there exists a homotopy f of Y into Y such that (1) $f(0, y)=y$, (2) $f(1, y)\subset X$ for all $y\in Y$, and (3) $f(t, x)=x$ for all $x\in X$.

The following considerations will show that $SO(n)$, $O(n)$ and $U(n)$ are deformation retracts of $GL(n, r)^+$, $GL(n, r)$, and $GL(n, c)$ respectively. The mapping

$$f(t, \tau) = \sigma(t1 + (1 - t)\alpha) \qquad 0 \leqq t \leqq 1$$

is clearly a homotopy of $GL(n, r)^+$, $GL(n, r)$ and $GL(n, c)$, where σ is an element of $SO(n)$, $O(n)$, and $U(n)$ respectively and α is a positive definite symmetric matrix if $\tau \in GL(n, r)^+$ and if $\tau \in GL(n, r)$, and a positive definite hermitian matrix if $\tau \in GL(n, c)$ (see first decompositions of these groups in section 1). Clearly $(t1+(1-t)\alpha)$ is symmetric (hermitian) according as α is symmetric (hermitian), since each set of matrices is closed with respect to scalar multiplication and addition. Moreover, if $\{\lambda_{ii}\}$ are eigenvalues of α, then $\{t+(1-t)\lambda_{ii}\}$ are eigenvalues of $t1+(1-t)\alpha$. Moreover for any t these eigenvalues are positive, if the λ_{ii} are positive. Finally if $\tau \in GL(n, r)^+$, $GL(n, r)$ and $GL(n, c)$, respectively, then $f(0, \tau)=\tau$, $f(1, \tau)=\sigma$, and $f(t, \sigma)=\sigma$ for all t.

Secondly, consider the mapping

$$f(t, A) = \frac{1}{(\sqrt[n]{|A|})^t} A, \qquad 0 \leqq t \leqq 1,$$

on each of the following sets: $GL(n, r)^+$, the set of positive definite symmetric matrices, the set of positive definite hermitian matrices, and the set of real triangular matrices $A=(a_{ij})$ with $a_{ii}>0$. It is clearly a homotopy of these sets. Moreover, using this homotopy one proves easily that $SL(n, r)$, the set of positive definite symmetric matrices of determinant one, the set of positive definite hermitian matrices of determinant one and the set of real triangular matrices $A=(a_{ij})$, $a_{ii}>0$ and $|A|=1$ are respectively deformation retracts of the above named sets.

Finally, let τ be an element of $SL(n, r)$, then by the second decomposition of this space, $\tau=\sigma\alpha$, where σ is an element of $SO(n)$ and $\alpha=(a_{ij})$, a triangular matrix having $a_{ii}>0$ and $|\alpha|=1$. Define $\alpha_t=(b_{ij})$, such that $b_{ii}=a_{ii}^t$, and $b_{ij}=ta_{ij}$ for $i\neq j$ and $0\leqq t\leqq 1$. The mapping

$$f(t, \tau) = \sigma\alpha_t$$

is clearly a homotopy of $SL(n, r)$. Moreover, it is easily seen that $f(0, \tau)=\sigma$, $f(1, \tau)=\tau$ and $f(t, \sigma)=\sigma$ for each t. Therefore $SO(n)$ is a deformation retract of $SL(n, r)$.

4. Homotopy types. It is well known that if Y_0 is a deformation retract of Y relative to Y_0, then Y_0 and Y are of the same homotopy type [3]. From the preceding considerations it follows that the following sets of spaces are of the same homotopy type:

(1) $U(n)$ and $GL(n, c)$
(2) $O(n)$ and $GL(n, r)$
(3) $SO(n)$, $GL(n, r)^+$ and $SL(n, r)$
(4) The set of positive definite symmetric (hermitian) matrices and the set of positive definite symmetric (hermitian) matrices of determinant one.
(5) The set of real triangular matrices with positive diagonal terms and its subset of matrices with determinant one.

It might be observed at this point that the set of positive definite symmetric matrices and the set of real triangular matrices with positive diagonal terms are homeomorphic, and hence are of the same homotopy type. For each of these sets is homeomorphic to $R^{n(n+1)/2}$. One notes that each is homeomorphic to $R^{n(n-1)/2} \times R^n(+)$, where $R^n(+)$ is that octant of R^n with all coordinates positive; this set is clearly homeomorphic to R^n.

Finally, since homeomorphic sets are of the same homotopy type and since this last relation is one of equivalence, the sets of symmetric matrices in (4) and the spaces in (5) are of the same homotopy type.

5. Further considerations of $SL(n, c)$, $U(n)$ and $SU(n)$. This paper does not establish $SU(n)$ as a deformation retract of $SL(n, c)$. But one observes that their n-th homotopy groups are isomorphic. The set of positive definite hermitian matrices of determinant one is homeomorphic to R^{n^2-1}. Therefore $SL(n, c) = R^{n^2-1} \times SU(n)$. It has been shown elsewhere [2], that if X_1, X_2, and X_3 are topological spaces and if $X_1 = X_2 \times X_3$, then $\pi_n(X_1) = \pi_n(X_2) \times \pi_n(X_3)$, where $\pi_n(X_i)$ are n-th homotopy groups of X_i. Also since any map into R^k is homotopic to the identity, $\pi_n(R^k) = I$ for all n and any k, and $\pi_m[SL(n, c)] = \pi_m[SU(n)]$ for all m. Since the m-th homotopy groups of the non-zero complex numbers and of the complex numbers of absolute value one contain only the identity for $m > 1$, it follows that $\pi_m[GL(n, c)] = \pi_m[SL(n, c)] = \pi_m[SU(n)] = \pi_m[U(n)]$, for all $m > 1$. Moreover, $\pi_m[GL(n, c)] = \pi_m[U(n)]$ for all m.

6. Further consideration of $GL(n, r)$, $O(n)$, $GL(n, r)^+$, $SL(n, r)$, $SO(n)$. The first two of these groups are not connected sets [1], but they are homogeneous spaces with $GL(n, r)^+$ and $SO(n)$ as components of their identity elements respectively. In [2, 114], one observes that $\pi_m[O(n)] = \pi_m(SO(n))$. Therefore these five groups have isomorphic m-th homotopy groups for $m \geqq 1$, since the first two groups are of the same homotopy type and the last three are of the same homotopy type. It is known that m-th homotopy groups are invariants of the same homotopy type [2].

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MATHEMATICAL NOTES

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AN INEQUALITY FOR POSITIVE DEFINITE MATRICES

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Making use of Hölder's inequality and an identity involving multiple integrals, R. Bellman [1] recently showed that, for any hermitian positive definite matrices A, B and any number α such that $0<\alpha<1$,

$$|\alpha A + (1-\alpha)B| \geqq |A|^{\alpha}|B|^{1-\alpha}, \tag{1}$$

where $|X|$ denotes the determinant of the matrix X. Our object is to present an alternative proof of (1) and to generalize this inequality.

We note, in the first place, that

$$\alpha\lambda + 1 - \alpha \geqq \lambda^{\alpha} \qquad (0<\alpha<1;\lambda>0), \tag{2}$$

with equality if and only if $\lambda=1$. This is proved by considering $\alpha\lambda+1-\alpha-\lambda^{\alpha}$ as a function of λ and using the first mean value theorem of the differential calculus.

Next, we recall that two hermitian positive definite matrices A and B (say of order n) can be reduced simultaneously to diagonal form. More precisely, there exists a non-singular matrix P such that

$$\bar{P}'AP = \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n), \qquad \bar{P}'BP = I,$$

where $\lambda_1, \cdots, \lambda_n$ are the (necessarily real) roots of the equation $|A-\lambda B|=0$. Moreover, since $\bar{P}'AP$ is again positive definite, it follows that $\lambda_1, \cdots, \lambda_n>0$. We now have

$$\alpha A + (1-\alpha)B = \bar{P}'^{-1}(\alpha\Lambda + (1-\alpha)I)P^{-1},$$

and therefore

$$\begin{aligned}|\alpha A + (1-\alpha)B| &= |\bar{P}'|^{-1}\cdot|P|^{-1}\cdot|\alpha\Lambda + (1-\alpha)I| = |B|\cdot|\alpha\Lambda+(1-\alpha)I| \\ &= |B|\cdot\prod_{r=1}^{n}(\alpha\lambda_r + 1 - \alpha).\end{aligned}$$

Hence, by (2),

$$|\alpha A + (1-\alpha)B| \geqq |B|\cdot\prod_{r=1}^{n}\lambda_r^{\alpha}, \tag{3}$$

and so

$$|\alpha A + (1-\alpha)B| \geqq |B| \cdot |\Lambda|^{\alpha} = |B| \cdot |\bar{P}'AP|^{\alpha} = |B|(|\bar{P}'| \cdot |P|)^{\alpha}|A|^{\alpha} = |A|^{\alpha}|B|^{1-\alpha}.$$

This establishes (1). Further, it is clear that there is equality in (1) if and only if there is equality in (3), and that is the case precisely if $\lambda_1 = \cdots = \lambda_n = 1$. Then $\bar{P}'AP = I$ and consequently $A = B$. Thus there is equality in (1) if and only if $A = B$.

It is now easy to deduce the following generalization:

THEOREM. *Let $\alpha_1, \cdots, \alpha_k$ be positive numbers such that $\alpha_1 + \cdots + \alpha_k = 1$, and let $A_1, \cdots, A_k$ be hermitian positive definite matrices. Then*

$$(4) \qquad |\alpha_1 A_1 + \cdots + \alpha_k A_k| \geqq |A_1|^{\alpha_1} \cdots |A_k|^{\alpha_k},$$

with equality if and only if

$$(5) \qquad A_1 = \cdots = A_k.$$

We prove this theorem by induction with respect to k. For $k=2$ it has been demonstrated above. Assume, then, that it holds for $k-1$, where $k \geqq 3$. Since

$$\alpha_1 A_1 + \cdots + \alpha_k A_k = (1-\alpha_k)\left(\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right) + \alpha_k A_k$$

and since $\alpha_1/(1-\alpha_k)A_1 + \cdots + \alpha_{k-1}/(1-\alpha_k)A_{k-1}$ is positive definite, it follows by the case $k=2$ that

$$(6) \quad |\alpha_1 A_1 + \cdots + \alpha_k A_k| \geqq \left|\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right|^{1-\alpha_k} |A_k|^{\alpha_k},$$

with equality if and only if

$$(7) \qquad \alpha_1 A_1 + \cdots + \alpha_{k-1}A_{k-1} = (1-\alpha_k)A_k.$$

Moreover,

$$\frac{\alpha_1}{1-\alpha_k} + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k} = 1,$$

and so, by the induction hypothesis,

$$(8) \quad \left|\frac{\alpha_1}{1-\alpha_k}A_1 + \cdots + \frac{\alpha_{k-1}}{1-\alpha_k}A_{k-1}\right| \geqq |A_1|^{\alpha_1/(1-\alpha_k)} \cdots |A_{k-1}|^{\alpha_{k-1}/(1-\alpha_k)},$$

with equality if and only if

$$(9) \qquad A_1 = \cdots = A_{k-1}.$$

From (6) and (8) we now at once infer (4). Also, there is equality in (4) if and

only if there are equalities in both (6) and (8), *i.e.*, if and only if both (7) and (9) are satisfied. But (7) and (9) are together equivalent to (5), and the proof is therefore complete.

It may be pointed out in conclusion that the theorem can also be derived from the generalized inequality of the arithmetic and geometric means and the relation (due to Minkowski)

$$|A_1 + \cdots + A_k|^{1/n} \geqq |A_1|^{1/n} + \cdots + |A_k|^{1/n},$$

valid for any hermitian positive definite matrices $A_1, \cdots, A_k$ of order n, with equality occurring if and only if every pair of A's is linearly dependent.

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CONVEX REGIONS IN PROJECTIVE N-SPACE

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We consider the following conditions on a set K in real projective n-space.

(1) Each point P of K is contained in an n-simplex region which is contained in K.

(2) Any two points P, Q of K are joined by a line segment $\overline{PQ}$ which is contained in K.

(3) Any two points P, Q of K are joined uniquely by a line segment which is contained in K.

(4) There exists an $(n-1)$-dimensional hyperplane of the projective n-space which does not meet the set K.

Condition (1) simply states that K is an open set; when $n=2$, a 2-simplex region is called a triangular region in [1]. A convex region is usually defined [1] as any set satisfying conditions (1), (2), and (4). We wish to prove here that conditions (1) and (3) also characterize the convex regions of a real projective n-space, $n \geqq 1$. Obviously (2) and (4) imply (3), and (3) implies (2). All that remains to be demonstrated is that (1) and (3) imply (4). For the case $n=1$, to deny (4) forces K to be the whole space which is impossible.

For $n=2$, we suppose that every line of the projective plane meets a set K satisfying conditions (1) and (3). Any line λ of the plane would then separate K into two disjoint sets K_1 and K_2 excluding the points of $K \cap \lambda$. K_1 is defined as the set of points which may be joined to some fixed point of K not on λ by segments in K which do not meet λ. K_2 is defined as the set of points of K which are not in K_1 or λ. Clearly K_1 and K_2 each separately satisfy conditions (1) and (4).

To verify that K_1 and K_2 each also satisfy (2) and (3) we first prove that any three non-collinear points P, Q, and R of K are vertices of a triangular region in K. If the segments in K determined by P, Q, and R together with the end-points bound a triangular region, the region is clearly in K. On the other hand, if we suppose the segments in K determined by P, Q, and R together

with the end-points do not bound a triangular region, then we take a point L of $\overline{QR}$ in K and consider the lines LQ' as Q' traverses segment $\overline{PQ}$ in K. We call R' the intersection of LQ' with PR. Let Q'' be the point Q' on segment $\overline{PQ}$ at which the segments $\overline{PQ'}$, $\overline{Q'R'}$, $\overline{R'P}$ in K together with the end-points first occur as not bounding a triangular region in K as Q' traverses $\overline{PQ}$. The points P, Q'', R'' are the vertices of a triangular region in K for which the segment $\overline{R''Q''}$ on its boundary contains a point not in K and the segment $\overline{Q''R''}$ not on its boundary lies entirely in K. Now the lines LQ' where Q' is on $\overline{PQ}$ between P and Q'' meet the triangular region $PQ''R''$ in segments $\overline{Q'R'}$ in K with segments $\overline{R'Q'}$ containing points not in K. Since each point of the segment $\overline{Q''R''}$ in K is contained in a triangular region in K and $\overline{Q''R''}$ with its end-points is compact so that it is covered by a finite collection of the triangular regions in K, we can find a line LQ' for which every point of the segment $\overline{R'Q'}$ as well as $\overline{Q'R'}$, Q', R' would be in K contrary to condition (3).

Now the axiom of Pasch clearly holds for the boundary of any triangular region so that any line not on a vertex meeting one side will meet exactly one of the other sides. When we apply this result to any triangle in K determined by the fixed point used to define K_1 and K_2 and any other two points of K, we see that (3) is true for both K_1 and K_2 and also that any segment in K joining a point of K_1 to a point of K_2 will meet λ.

Let S be any point on λ but not in K. We suppose every line of the pencil on S meets K. As T traverses some line μ not on S, the lines TS are all the lines of the pencil. TS can never meet both K_1 and K_2. There exists on μ a point $T_1 \neq T_0 = \lambda \cap \mu$ such that T_1 is the last T for which TS meets K_1 or is the first T for which TS meets K_2 as T traverses the entire line μ in the sense from K_2 through T_0 to K_1. If T_1S meets K_1 in a point V, then from condition (1) a whole interval of points T on μ about T_1 gives ST meeting K_1 contrary to T_1 being the last such T. A similar contradiction is obtained if T_1S meets K_2. Therefore our supposition concerning the pencil on S is false and (4) is true.

For $n > 2$, we proceed in a similar manner using λ as an $(n-1)$-dimensional hyperplane, S as an $(n-2)$-dimensional hyperplane in λ which we know exists not meeting K by assuming our proposition already proved true for $(n-1)$-dimensional real spaces. We take μ as any line which does not meet S, and to complete the proof we consider as before the pencil of $(n-1)$-dimensional hyperplanes ST as T traverses μ. Thus by induction (4) follows from (1) and (3) for all natural numbers n.

From the above result we may state that for $n > 1$ it is impossible to find in projective n-space two disjoint sets K and K' satisfying (1) and (3) with K' closure also satisfying (3) for which the union of K and the closure of K' is the whole space. Also any set satisfying (1) and (3) must be homeomorphic to an open convex set in euclidean n-space.

Reference

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A NOTE ON EULER TRANSFORMS

GEORGE BRAUER, University of Minnesota

If λ is a positive number and $\phi(t)$ is a non-constant function which is monotone non-decreasing for $0 \leqq t < \infty$, then the function

$$f(x) = \int_0^\infty (x+t)^{-\lambda} d\phi(t), \qquad 0 < x < \infty,$$

will be called the Euler λ-transform of ϕ (*cf.* [1]). The class of functions which can be represented as Euler λ-transforms will be denoted by E_λ. Hartman and Wintner [1] have shown that if λ and μ are two positive numbers such that $\lambda < \mu$, then $E_\lambda \subset E_\mu$. The following result is in the reverse direction.

THEOREM. *If $\lambda > 1$ and $f(x)$ can be written as the Euler λ-transform of a function $\phi(t)$ which is everywhere differentiable in* $(0, \infty)$, *such that its derivative $\phi'(t)$ satisfies the following conditions*

(i) *$\phi'(t)$ is monotone non-decreasing for* $0 < t < \infty$,

(ii) *$\phi'(t) = O(t^{\lambda-1})$ as $t \to \infty$,*

then $f(x)$ is in $E_{\lambda-1}$.

Proof. Since $f(x)$ is in E_λ, we have

$$f(x) = \int_0^\infty (x+t)^{-\lambda} d\phi(t) = \int_0^\infty (x+t)^{-\lambda}\phi'(t)dt.$$

If we integrate by parts, and make use of condition (ii) we obtain

$$f(x) = [1/(\lambda-1)]\left[x^{-\lambda+1}\phi'(0) + \int_0^\infty (x+t)^{-\lambda+1} d\phi'(t)\right].$$

Let $\epsilon(t)$ denote the function defined by

$$\epsilon(0) = 0, \qquad \epsilon(t) = 1 \qquad \text{for } t \neq 0,$$

and let

$$\theta(t) = [1/(\lambda-1)][\phi'(0)\epsilon(t) + \phi'(t)].$$

Then

$$x^{-\lambda+1} = \int_0^\infty (x+t)^{-\lambda+1} d\epsilon(t)$$

and

$$f(x) = \int_0^\infty (x+t)^{-\lambda+1} d\theta(t). \tag{1}$$

It remains to be shown that $\theta(t)$ is non-constant and non-decreasing. If

$\phi'(t)$ is identically equal to some constant a, then a cannot be negative or 0 since $\phi(t)$ is non-constant and non-decreasing. Thus $\alpha>0$ and $\theta(t)=[a/(\lambda-1)][\epsilon(t)+1]$ which is non-decreasing and not constant. If $\phi'(t)$ is not constant we have, for $t\neq 0$,

$$\begin{aligned}\theta(t) &= [\phi'(0) + \phi'(t)]/(\lambda - 1)\\ &= \theta(0) + \phi'(t)/(\lambda - 1)\end{aligned}$$

which is non-constant. The fact that $\phi'(t)$ is non-decreasing guarantees that $\theta(t)$ is non-decreasing. Thus $f(x)$ is the Euler $\lambda-1$ transform of $\theta(t)$.

Reference

1. Philip Hartman and Aurel Wintner, On Euler Transforms, Proc. Amer. Math. Soc., vol. 1, 1950, pp. 394–396.

THE LIMIT OF A CERTAIN INTEGRAL CONTAINING A PARAMETER*

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The purpose of this note is to establish the following estimate:

$$(1)\qquad \begin{cases} I_n \equiv \dfrac{2}{\pi n}\displaystyle\int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{2\sin\theta}\cot\theta - \frac{(2n+1)\cos(2n+1)\theta}{2\sin\theta}\right|\sin 2\theta d\theta \\ \quad = \dfrac{8}{\pi^2} + O\left(\dfrac{\log n}{n}\right), \qquad (n\to\infty). \end{cases}$$

This integral, with $\theta=\frac{1}{2}\phi$, is found in Rabson [3] where it occurs as $1/n$ times the n-th Lebesgue constant for the convergence of Fourier series on the quaternions of norm one. He shows [3, (5.5)] that

$$(2)\qquad \frac{4}{\pi} + \epsilon \geqq I_n \geqq \frac{2^{1/2}}{\pi} - \epsilon, \qquad n \geqq N_\epsilon,$$

for each $\epsilon>0$, all that is needed for his purposes.

The derivation of (1) is along different lines from Rabson's proof of (2) and is based on (3) and (4) below:

$$(3)\qquad \int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{\sin\theta}\right|\cos^2\theta d\theta = O(\log n);$$

$$(4)\qquad \int_0^{\pi/2}\cos\theta\,|\cos(2n+1)\theta|\,d\theta = \frac{2}{\pi} + O\left(\frac{1}{n}\right).$$

Equation (3) follows from the well-known estimate of the Lebesgue constants arising in the convergence theory of Fourier series, since $0\leqq\cos^2\theta\leqq 1$. As a matter of fact, a much more precise result is obtainable for the integral

* This note was written while the author was the recipient of a Research Corporation grant.

in (3) simply by replacing $\cos^2\theta$ by $1-\sin^2\theta$, separating into two integrals, using the known asymptotic formulas for the Lebesgue constants and the resulting analog of (4). But this is not needed here.

Equation (4) is a special case of a result which, except for the remainder term, is due to Fejér [1]. The remainder term and a somewhat more general result are found in [2, (2.6)].

Now replace $\sin 2\theta$ by $2\sin\theta\cos\theta$, getting

$$\begin{aligned} I_n &= \frac{2}{\pi n}\int_0^{\pi/2}\left|\frac{\sin(2n+1)\theta}{\sin\theta}\cos^2\theta - (2n+1)\cos\theta\cos(2n+1)\theta\right| d\theta \\ &= \frac{2(2n+1)}{\pi n}\int_0^{\pi/2}\cos\theta\,|\cos(2n+1)\theta|\,d\theta + O\left(\frac{\log n}{n}\right) \\ &= \frac{8}{\pi^2} + O\left(\frac{1}{n}\right) + O\left(\frac{\log n}{n}\right) = \frac{8}{\pi^2} + O\left(\frac{\log n}{n}\right). \end{aligned}$$

Remark. Rabson needs (2) only when n is an integer. The proof of (1), however, is valid for arbitrary, not necessarily integer-valued, growth of n.

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3. G. Rabson, *Summability of Fourier series on the quaternions of norm one*, Trans. Amer. Math. Soc., vol. 75, 1953, pp. 287–303.

A SHORT PROOF OF A CLASSICAL THEOREM IN THE THEORY OF FOURIER INTEGRALS

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We give here a short self-contained proof of an important theorem of Pringsheim concerning the Fourier integral formula (Math. Ann. vol. 68, 1910 and vol. 71, 1912). We have been unable to find such a proof in the literature.

The theorem in question is the following:

Let f be a complex-valued function of bounded variation on the line $-\infty<t<\infty$, and suppose that $f(t)\to 0$ as $|t|\to\infty$. Then

$$\text{(1)}\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixy}dy\int_{-\infty}^{\infty}f(t)e^{iyt}dt = \frac{f(x+0)+f(x-0)}{2},\qquad -\infty<x<\infty,$$

where the inner integral is convergent except possibly at $y=0$, and the outer integral is to be interpreted as

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$$\lim_{\epsilon\to 0, R\to\infty}\left(\int_{-R}^{-\epsilon}+\int_{\epsilon}^{R}\right).$$

Our proof makes frequent use of the following lemma which has a certain interest of its own:

LEMMA. *Let f be of bounded variation on the interval $a \leqq t \leqq b$, and let g be continuous on the same interval. Then*

(a).
$$\left|\int_a^b fgdt\right| \leqq V_f(a, b) \sup_{a\leqq \xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right| + \inf_{a\leqq x\leqq b}\left|f(x)\int_a^b gdt\right|$$
$$\leqq \left\{V_f(a, b) + \inf_{a\leqq t\leqq b}|f(t)|\right\} \sup_{a\leqq \xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|,$$

where $V_f(x, y)$ denotes the total variation of f on the interval $x \leqq t \leqq y$.

(b). *If f is real and vanishes or changes sign in the interval $a \leqq t \leqq b$, then the inequality* (1) *reduces to*

$$\left|\int_a^b fgdt\right| \leqq V_f(a, b) \sup_{a\leqq \xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|.$$

(The result (b) is not needed here and is given only for the sake of completeness).

The proof of (a) is quite simple. Suppose that f vanishes at one of the end-points of the interval $a \leqq t \leqq b$, say at $t = a$. Integrating by parts, we find that

$$\left|\int_a^b fgdt\right| = \left|\int_a^b\left\{\int_t^b g(u)du\right\}df(t)\right| \leqq V_f(a, b) \sup_{a\leqq \xi,\eta\leqq b}\left|\int_\xi^\eta gdt\right|.$$

Since $V_f(x, y)$ is an additive interval function, this inequality remains true if f vanishes at *any* point of the interval $a \leqq t \leqq b$. The general result follows from the obvious inequality

$$\left|\int_a^b fgdt\right| \leqq \left|f(c)\int_a^b gdt\right| + \left|\int_a^b \{f(t) - f(c)\}g(t)dt\right|.$$

In proving (b), we can, because of (a), restrict ourselves to the case where $f(t)$ is bounded away from zero but changes sign. There is then a finite subinterval at whose end-points $f(t)$ has opposite signs. Bisecting this interval, we get two new intervals of which one and only one has the same property. Continuing in the usual way, we arrive at a nested set of closed intervals whose lengths tend to zero and at whose end-points $f(t)$ has opposite signs. There is a point ξ common to all these intervals. If ξ is a right-hand or left-hand end-point, respectively, of all but a finite number of these intervals, then $f(\xi)$ has sign opposite to $f(\xi-0)$ or $f(\xi+0)$. Otherwise, $f(\xi-0)$ and $f(\xi+0)$ have opposite signs. In either

case, the required result follows from (a) upon redefining f to be zero at ξ. Indeed, this does not alter the integral of fg, and the total variation of the redefined f does not exceed that of f.

The following rather well known properties of

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt$$

will be needed in the sequel:

$$\text{(P1).}\qquad \int_0^\infty \frac{\sin \alpha t}{t}\,dt = \pi/2 \qquad \text{for} \quad \alpha > 0;$$

$$\begin{matrix}\text{(P2).}\\ \text{(P3).}\end{matrix}\qquad \left|\int_\xi^\eta \frac{\sin \alpha t}{t}\,dt\right| \begin{cases} \leqq \mathrm{Si}\,(\pi) & \text{for} \quad \xi\eta \geqq 0,\ \alpha > 0; \\ \leqq 2/(\alpha\xi) & \text{for} \quad 0 < \xi < \eta = \infty,\ \alpha > 0. \end{cases}$$

Note that the property (P2) is a consequence of the known inequalities $0 \leqq \mathrm{Si}\,(x) \leqq \mathrm{Si}\,(\pi)$ for $x \geqq 0$, the oddness of $\mathrm{Si}\,(x)$, and the fact that the integral in (P2) is equal to $\mathrm{Si}\,(\alpha\eta) - \mathrm{Si}\,(\alpha\xi)$.

Now to the proof of the theorem.

First off, the inner integral in (1) converges uniformly for $|y| \geqq \epsilon > 0$ as can be seen by applying the lemma to

$$\int_S^T f(t) e^{iyt}\,dt$$

as $S, T \to \infty\ (\to -\infty)$ and noting that

$$\left|\int_\xi^\eta e^{iyt}\,dt\right| \leqq 2/|y|$$

for $y \neq 0$ and $S \leqq \xi,\ \eta \leqq T$. Consequently,

$$\begin{aligned} J_{\epsilon,R}(t) &\equiv \frac{1}{2\pi}\left(\int_{-R}^{-\epsilon} + \int_\epsilon^R\right) e^{-ixy}\,dy \int_{-\infty}^\infty f(t) e^{iyt}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f(t)\,dt \left(\int_{-R}^{-\epsilon} + \int_\epsilon^R\right) e^{iy(t-x)}\,dy \\ &= I_R(x) - I_\epsilon(x), \end{aligned}$$

where

$$I_\alpha(x) = \frac{1}{\pi}\int_{-\infty}^\infty f(x+t)\,\frac{\sin \alpha t}{t}\,dt.$$

If we split $I_\epsilon(x)$ into integrals over the intervals $(-\infty, -T]$, $[-T, T]$, and $[T, \infty)$, apply the lemma and (P2) to the first and third of these and use the inequality $|\sin \epsilon t| \leqq \epsilon |t|$ in the remaining one, we see that $I_\epsilon(x) \to 0$ as $\epsilon \to 0$.

We introduce the notation

$$h(t) \equiv h_x(t) \begin{cases} = 0, & t = 0, \\ = f(x+t) + f(x-t) - \{f(x+0) + f(x-0)\}, & t \neq 0. \end{cases}$$

In view of (P1), we then find that

$$\pi\left\{I_R(x) - \frac{f(x+0)+f(x-0)}{2}\right\} = \int_0^\infty h(t)\frac{\sin Rt}{t}\,dt = \int_0^\delta + \int_\delta^\infty .$$

By the lemma and (P2), $\left|\int_0^\delta\right| \leqq \mathrm{Si}\,(\pi)V_h(0, \delta)$. Since $h(t)$ and, therefore, $V_h(0, t)$ are continuous at $t=0$, it follows that $\left|\int_0^\delta\right|$ can be made arbitrarily small by taking δ small enough. On the other hand, the lemma and (P3) give $\left|\int_\delta^\infty\right| \leqq 2V_h(\delta, \infty)/(R\delta)$ which, for fixed δ, tends to zero as $R\to\infty$.

Remark. It is clear that the above argument applies equally well to show that $J_{\epsilon,R}(x)\to f(x)$ uniformly in any proper subinterval of an interval of continuity of f.

CLASSROOM NOTES

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THE IRRATIONALITY OF $\sqrt{2}$

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The following proof has some novelty in that it does not use the usual assumption that the fraction n/m which is to represent $\sqrt{2}$ is such that n and m have no common factors.

Suppose $\sqrt{2}=(n/m)$, where n and m are positive integers. Then $n>m$, and there is an integer $p>0$ such that $n=m+p$, and $2m^2=m^2+2pm+p^2$. This implies $m>p$. Consequently, for some integer $a>0$, $m=p+a$, $n=2p+a$ and $2(p+a)^2=(2p+a)^2$. The last equality implies $a^2=2p^2$ so that the entire process may be repeated indefinitely giving $n>m>a>p>\cdots$, but since every non-null set of positive integers has a smallest element, this is a contradiction and $\sqrt{2}$ is irrational.

ON DISSIMILAR SOLUTIONS OF A DIFFERENTIAL EQUATION

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A common complaint of undergraduate students in Differential Equations classes is that the major problem is "to put their solutions in the form of the textbook's answer." This problem of reconciling two dissimilar solutions can frequently lead to quite laborious algebraic manipulations. An interesting example of a similar difficulty is provided by a consideration of two apparently conflicting treatments of Van der Pol's equation [1], [2].

By a suitable choice of scale factors Van der Pol's equation may be written in the convenient form:

$$\ddot{y} + y = \mu(1 - y^2)\dot{y}. \tag{1}$$

The problem is to find an approximation to the periodic solution of this equation which reduces to a cosine solution of the linear equation for $\mu = 0$. (This solution is a stable limit cycle and is quite important in non-linear mechanics.) Both authors used a perturbation method which makes use of the small parameter μ. If the solution presented by [1] is corrected for two algebraic errors, it can be written in the form:

$$\begin{aligned} y &= 2\cos\omega t + \frac{\mu}{4}(3\sin\omega t - \sin 3\omega t) + \cdots, \\ \omega &= 1 - \frac{\mu^2}{16} + \cdots. \end{aligned} \tag{2}$$

By a simple transformation McLachlan's solution may be written in a similar form:

$$\begin{aligned} y &= 2\cos\omega t + \frac{\mu}{4}(-\sin\omega t - \sin 3\omega t) + \cdots, \\ \omega &= 1 - \frac{\mu^2}{16} + \cdots. \end{aligned} \tag{3}$$

An examination of these solutions shows that both are periodic and reduce to $2\cos t$ for $\mu = 0$. We see that if squares and higher powers of μ are neglected both have the same value of $y(0)$, but differ in their value of $\dot{y}(0)$. This conflict seems to be irreducible. The solution is, as it happens all too frequently, quite simple. It is clear that a shift in the origin of the time axis is allowable. We, therefore, shift this origin in (2) a small amount by replacing ωt by $\omega t + a\mu$. If squares and higher powers of μ are neglected, we have

$$y = 2\cos\omega t + \frac{\mu}{4}[(3 - 8a)\sin\omega t - \sin 3\omega t]. \tag{4}$$

Hence we see that the choice $a = \frac{1}{2}$ reduces (2) to (3) and that the coefficient of

sin ωt in the *linear* approximation is completely arbitrary. Thus we see, as was pointed out by the referee, there actually is a family of approximations which approach the solution $y=2\cos t$ as μ approaches zero.

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INTEGRALS OF INVERSE FUNCTIONS

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If $y=f(x)$ and its inverse $x=g(y)$ are single-valued and continuously differentiable, the area under the curve $y=f(x)$ and between the ordinates $x=a$ and $x=b$ may be calculated by using the inverse function. In particular

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy.$$

This leads to an interesting formula for indefinite integrals

$$\int^x f(x)dx = xf(x) - \int^{f(x)} g(y)dy.$$

For the undergraduate student, this formula has two uses. First, it gives a method of formal integration for the logarithmic function, the inverse circular and hyperbolic functions, and many others. For example,

$$\begin{aligned}\int \ln x dx &= x\ln x - \int^{\ln x} e^y dy\\ &= x\ln x - x.\end{aligned}$$

Second, it will sometimes determine whether an integral is an elementary function or not. For example, $\int(\ln x)^{1/2}dx$ is not elementary, since $\int e^{y^2}dy$ is not.

The method of deriving this formula suggests a comparison between the "ring" and "shell" methods of finding a volume of revolution, and the two methods of calculating the first moment of area. These lead to the result

$$\int^x f^2(x)dx = xf^2(x) - 2\int^{f(x)} yg(y)dy.$$

This appears to be less useful, but should suggest to the student the two methods of calculating moments of inertia, which lead to the result

$$\int^x f^3(x)dx = xf^3(x) - 3\int^{f(x)} y^2g(y)dy.$$

This suggests the generalization

$$\int^{x} f^{n}(x)dx = xf^{n}(x) - n\int^{f(x)} y^{n-1}g(y)dy.$$

This relation is indeed valid and provides at the undergraduate level an interesting problem in differentiation.

The form of these results suggests that they are closely connected to the formulas obtained by integration by parts. In fact, the last relation may be quickly obtained by making the substitution $x=g(y)$ in the left hand side, then integrating by parts taking $u=y^n$ and $dv=g'(y)dy$.

THE FUNDAMENTAL THEOREM OF THE CALCULUS

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Both proof and content of the fundamental theorem of the integral calculus are foreshadowed by the

FUNDAMENTAL THEOREM OF THE SUM CALCULUS. *If $F(n)$ is a function of the integral variable n having $f(n)$ as difference, then*

$$\sum_{n=a}^{b} f(n) = F(b+1) - F(a). \tag{1}$$

The proof is immediate; for the sum in (1) is

$$\begin{aligned}\sum_{n=a}^{b} f(n) &= F(a+1) - F(a) + F(a+2) - F(a+1) + \cdots + F(b+1) - F(b) \\ &= F(b+1) - F(a).\end{aligned}$$

This theorem is readily and effectively illustrated by forming a short table of *anti-differences*. Thus to verify the brief table

$f(n)$	$F(n) = \Delta^{-1}f(n)$
r^n	$\dfrac{r^n}{r-1}\ (r \neq 1)$
$n^{(k)}$	$\dfrac{n^{(k+1)}}{k+1}$
$\cos n\alpha$	$\dfrac{\sin(n-\frac{1}{2}\alpha)}{2\sin\frac{1}{2}\alpha}$

we need only show that $\Delta F(n) = f(n)$. We can now compute sums such as

$$\Sigma ar^n, \qquad \Sigma n^2 = \Sigma(n^{(2)} + n^{(1)}), \qquad \Sigma \cos n\alpha,$$

between integral limits by use of (1).

A direct analogue of (1) is the

FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS. *If $f(x)$ is integrable in the interval (a, b) and $F(x)$ is a function having $f(x)$ as derivative, then*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{2}$$

The proof again depends upon "telescopic" cancellation. Divide (a, b) into subintervals δ_i by the intermediate points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and let $\delta = \max \delta_i$; then

$$\int_a^b f(x)dx = \lim_{\delta\to 0} \sum_{i=1}^{n} f(t_i)\delta_i, \qquad x_{i-1} \leqq t_i \leqq x_i. \tag{3}$$

Since $F(x)$ is differentiable it is also continuous in (a, b); and from the mean value theorem

$$F(x_i) - F(x_{i-1}) = f(\xi_i)\delta_i, \qquad x_{i-1} \leqq \xi_i \leqq x_i. \tag{4}$$

Since the points t_i in (3) are at our disposal we may take $t_i = \xi_i$. Then the sum in (3) becomes

$$F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1}),$$

that is, $F(b) - F(a)$. Since this is true for any subdivision of (a, b), it must hold as $\delta \to 0$.

The analogy between the theorems is enhanced by writing them in the form:

$$\sum_{n=a}^{b} f(n)\,\Delta n = \Delta^{-1}f(n)\Big|_a^{b+1}; \qquad \int_a^b f(x)dx = D^{-1}f(x)\Big|_a^b.$$

The requirement in the fundamental theorem that $f(x)$ be integrable is essential, for the derivative of a function, $f(x) = F'(x)$, is not necessarily integrable. For example the function

$$F(x) = x^2 \sin\frac{1}{x^2}\ (x \neq 0), \qquad F(0) = 0$$

has the derivative

$$F'(x) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}\ (x \neq 0), \qquad F'(0) = 0;$$

but since $F'(x)$ is unbounded in any neighborhood of the origin, $F'(x)$ is not integrable over $0 \leqq x \leqq 1$.

(2) Given four mutually tangent spheres of slightly different radii $r_1 < r_2 < r_3 < r_4$, find the radius R of the largest fifth sphere which will fit in the space which is more or less bounded by the four given spheres.

SOLUTIONS

Bounds on a Solution of a Special Riccati Equation

E 1113 [1954, 259 and 714]. *Proposed by Peter Treuenfels, Ballistic Research Laboratory, Aberdeen Proving Ground*

Let $y(x)$ denote that solution of the differential equation $dy/dx = x^2 + y^2$ which passes through the origin. Show that $y(1) < 23/53$.

II. *Note by Aaron Herschfeld, Polytechnic Institute of Brooklyn.* The value $y(1) = 0.350168+$ given in the discussion of this problem is incorrect. From the 1948 *Tables of Bessel Functions of Fractional Order*, vol. 1, put out by the National Bureau of Standards, one finds

$$\begin{aligned} y(1) &= J_{3/4}(0.5)/J_{-1/4}(0.5) \\ &= 0.37110,55199/1.05959,95935 \\ &= 0.35023,18443. \end{aligned}$$

It is perhaps also interesting to note that the simple solution

$$y = xJ_{3/4}(x^2/2)/J_{-1/4}(x^2/2)$$

has its first infinity at the least positive zero of $J_{-1/4}(x^2/2)$, which is easily found from the *NBS* tables. Since $J_{-1/4}(u)$ vanishes between $u = 2.00$ and $u = 2.01$, a little inverse interpolation gives the zero at $u = 2.00629,97$, or $x = 2.00314,73$, where $y(x) = \infty$.

Polygonal Path Covering a Square Lattice

E 1123 [1954, 423; 1955, 124]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N \times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

II. *Addendum by John Selfridge, U.C.L.A.* We answer the question at the end of the problem in the negative.

Let there be R rows and S columns which have none of the given segments lying on them. The $R \times S$ point lattice formed by these has $2R+2S-4$ boundary points if R and S are each greater than 1. Each oblique line covers at most 2 of these boundary points. Thus in the polygonal covering there are at least $R+S-2$ oblique segments, $N-R$ horizontal segments, and $N-S$ vertical segments, or at least $2N-2$ segments in all.

If R or S is 0 or 1 there are at least $2N-1$ segments.

Inradius of a Mixtilinear Triangle

E 1141 [1954, 711]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Find the radius of the circle inscribed in the mixtilinear triangle formed by the two legs of a given right triangle ABC and the semi-circumference described externally upon the hypotenuse AB.

I. *Solution by the Proposer.* Let ω be the center and ρ the radius of the circle inscribed in the mixtilinear triangle. The semi-circumference and its reflection in the line AB is the circumcircle of triangle ABC. Extend CA and CB their own lengths to A' and B' and draw $A'B'$. The circumcircle of triangle ABC is the nine-point circle of triangle $A'B'C$. By the converse of Feuerbach's theorem, the circle (ω) is the incircle of triangle $A'B'C$, since it is tangent to sides $A'C$, $B'C$ and to the nine-point circle internally. Consequently $\rho=2(s-c)=a+b-c$.

II. *Solution by M. A. Kirchberg, Hopkins, Mich.* The centers of the inscribed circle and the given circle being at (ρ, ρ) and $(a/2, b/2)$, respectively, on a coordinate system defined by the legs of the given triangle, where ρ is the unknown radius, we may simply equate the distance between centers to the difference in radii and solve, finding $\rho=a+b-c$.

Also solved by J. W. Baldwin, G. B. Charlesworth, P. A. Clement, Hüseyin Demir, I. A. Dodes, A. L. Epstein, R. L. Helmbold, A. R. Hyde, M. S. Klamkin, Viktors Linis, D. C. B. Marsh, Beckham Martin, R. K. Morley, T. F. Mulcrone, S. Parameswaran, C. F. Pinzka, P. W. A. Raine, G. B. Robison, David Rothman, Sister M. Stephanie, J. A. Tierney, Roy Westwick, and Roscoe Woods.

Morley studied the locus of ω when A and B are fixed and C varies.

Semi-vertical Angle of a Right Circular Cone

E 1142 [1954, 711]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α, 2β, 2γ, with each other.

Solution by Leon Bankoff, Los Angeles, Calif. The sides of the base of the triangular pyramid determined by the three generating lines are $a=2y\sin\alpha$, $b=2y\sin\beta$, $c=2y\sin\gamma$, where y is the slant height of the cone. The radius of the base of the cone is given by

$$R = abc/4\sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter of the base of the pyramid. Since the semi-vertical angle ϕ is equal to arc sin R/y, we obtain

$$\phi = \arcsin \frac{2\sin\alpha\sin\beta\sin\gamma}{\sqrt{(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)^2-2(\sin^4\alpha+\sin^4\beta+\sin^4\gamma)}}\cdot$$

Also solved by A. Buckley, G. B. Charlesworth, H. K. Crowder, Hüseyin

Demir, A. L. Epstein, R. L. Helmbold, A. S. Hendler, Raymond Huck, A. R. Hyde, Viktors Linis, D. C. B. Marsh, R. K. Morley, C. S. Ogilvy, S. Parameswaran, Walter Penney, Azriel Rosenfeld, D. C. Russell, Chih-yi Wang, Roy Westwick, R. H. Wilson, Jr., Roscoe Woods, and the proposer.

Demir gave the equivalent answer

$$\sin^2 \phi = -\frac{16 \sin^2 \alpha \sin^2 \beta \sin^2 \gamma}{\begin{vmatrix} 0 & \sin \alpha & \sin \beta & \sin \gamma \\ \sin \alpha & 0 & \sin \gamma & \sin \beta \\ \sin \beta & \sin \gamma & 0 & \sin \alpha \\ \sin \gamma & \sin \beta & \sin \alpha & 0 \end{vmatrix}}.$$

Helmbold considered the analogous problem in an n-dimensional Euclidean vector space.

Concerning the Coefficients of a Polynomial

E 1143 [1954, 711]. *Proposed by C. D. Olds, San Jose State College, Calif.*

If the roots of the equation

$$a_0 x^n - n a_1 x^{n-1} + \tfrac{1}{2} n(n-1) a_2 x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive and distinct, prove that $a_r a_{n-r} > a_0 a_n$, $r = 1, 2, \cdots, n-1$.

Solution by E. P. Starke, Rutgers University. With no additional difficulty we can prove more, namely, $a_p a_q > a_r a_s$ for all p, q, r, s with $p+q=r+s$, $|p-q| < |r-s|$. Since the given equation has n distinct positive roots, so also has the reciprocal equation

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i x^i = 0.$$

Furthermore, by the law of the mean, each of the equations obtained by differentiation has $n-1$ distinct positive roots,

$$\sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} a_i x^{n-i-1} = 0, \qquad \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} a_{n-i} x^{n-i-1} = 0.$$

Since the proposed conclusion is obviously true for $n=2$, we proceed by induction, assuming $a_p a_q > a_r a_s$ is true whenever $p+q=r+s$, $|p-q| < |r-s| \leqq n-1$. Hence

$$a_k a_{n-k} > \cdots > a_2 a_{n-2} > a_1 a_{n-1},$$

where $k = [n/2]$. Further, $a_1^2 > a_0 a_2$ and $a_2 a_{n-1} > a_1 a_n$ imply, since the a's are positive, that

$$a_1 a_{n-1} > a_0 a_n,$$

which completes the proof.

The example $16x^3-64x^2+81x-34=0$ shows that the converse does not hold.

Also solved by I. A. Dodes, H. M. Feldman, N. J. Fine, A. J. Goldman, M. S. Klamkin, Viktors Linis, S. Parameswaran, Azriel Rosenfeld, Georgia Smith, and O. E. Stanaitis. Late solutions by A. E. Danese, C. F. Pinzka, and Albert Wolinsky.

A Logarithmic Inequality

E 1144 [1954, 711]. *Proposed by A. S. Hendler, Rensselaer Polytechnic Institute, N. Y.*

For what positive values of a is $\log_a b<b$ for all positive b?

Solution by F. R. Olson, University of Buffalo. If $a<1$, then $b=a$ would imply $1<a$. It follows that $a>1$. Consequently the inequality is equivalent to $b<a^b$ or $b^{1/b}<a$. By the usual derivative process the maximum of $b^{1/b}$ is found to occur when $b=e$. Hence $\log_a b<b$ for $a>e^{1/e}$.

Also solved by R. H. Ayres, Jr., P. T. Bateman, B. H. Bissinger, Julian Braun, G. B. Charlesworth, H. K. Crowder, Hüseyin Demir, I. A. Dodes, N. J. Fine, D. S. Greenstein, R. L. Helmbold, A. R. Hyde, M. S. Klamkin, Viktors Linis, D. C. B. Marsh, Morris Morduchow, T. F. Mulcrone, C. S. Ogilvy, M. W. Oliphant, S. Parameswaran, Walter Penney, L. L. Pennisi and N. C. Scholomiti (jointly), Azriel Rosenfeld and L. I. Sherry (jointly), D. C. Russell, C. M. Sandwick, Sr., O. E. Stanaitis, D. R. Sudborough, T. H. Sumner, Roy Westwick, R. H. Wilson, Jr., and the proposer. Late solution by C. F. Pinzka.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4639 [1955, 371]. *Correction.* For $k\geqq j$, read $k\neq j$.

4643. *Proposed by Norman Anning, Alhambra, California*

If r is a prime of the form $4k+1$ or is the product of such primes, show that the number of points with integral coordinates on the sphere

$$x^2+y^2+z^2=r^2$$

is $6r$.

4644. *Proposed by Azriel Rosenfeld, Columbia University*

Prove that in no field K can the additive group K^+ be isomorphic to the multiplicative group K^*.

4645. *Proposed by Leonard Carlitz, Duke University*

Let $0 \leqq r \leqq p-1$, where p is prime, and let $a_0, a_1, \cdots, a_r$ be integers not divisible by p. Define the determinant of order $r+1$

$$\Delta_r = \left| a_i^{p^j} \right| \qquad (i, j = 0, 1, \cdots, r).$$

Show that

$$\Delta_r \equiv 0 \pmod{p^{r(r+1)(r+2)/6}}.$$

4646. *Proposed by Oliver Gross, the Rand Corporation*

Let $F_1(x) = G_1(x) = x$ for x real, and for $n > 1$ define F_n, G_n recursively by

$$F_n(x_1, \cdots, x_n) = \max\,(x_1, G_{n-1}(x_2, \cdots, x_n)),$$
$$G_n(x_1, \cdots, x_n) = \min\,(x_1, F_{n-1}(x_2, \cdots, x_n)).$$

Are the following relations true:

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 F_n(x_1, \cdots, x_n)dx_1 \cdots dx_n = \frac{\pi\sqrt{3}}{9},$$
$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 G_n(x_1, \cdots, x_n)dx_1 \cdots dx_n = 1 - \frac{\pi\sqrt{3}}{9}?$$

4647. *Proposed by James Munkres, Los Alamos Scientific Laboratory*

A permutation of the integers $1, \cdots, n$ is called an n-chain; two n-chains are disjoint if any two integers which are adjacent in one chain are not adjacent in the other. (The first integer is considered adjacent to the last for this purpose.) Does there exist, for each n, a collection containing $[(n-1)/2]$ mutually disjoint n-chains?

SOLUTIONS

4553 [1955, 259–260]. *Correction.*

Change the number in the last line from 1.12 to $6^{1/2} - 2^{1/2} = 1.035$.

Asymptotic Summation Formulas

4582 [1954, 199]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Show that

$$\text{(a)} \qquad \sum_{n=1}^{p} \frac{S_{1,n}^2}{n} \sim \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3}(\gamma + \log p)^3,$$

(b) $$\sum_{n=1}^{p} \frac{S_{2,n}}{n} \sim \frac{\pi^2}{6}(\gamma + \log p) - \sum_{n=1}^{\infty} \frac{1}{n^3},$$

where γ is Euler's constant and $S_{m,n} = \sum_{r=1}^{n} r^{-m}$.

Solution by Leonard Carlitz, Duke University. We shall show that

$$A = \sum_{n=1}^{k} \frac{1}{n} S_{2,n} = \frac{\pi^2}{6}(\gamma + \log k) - \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{\log k}{k}\right), \tag{1}$$

$$B = \sum_{n=1}^{k} \frac{1}{n} S_{1,n}^2 = \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{3}(\gamma + \log k)^3 + O\left(\frac{\log k}{k}\right), \tag{2}$$

where $S_{m,k} = \sum_{n=1}^{k} n^{-m}$.

We recall that

$$S_{1,k} = \log k + \gamma + O\left(\frac{1}{k}\right), \qquad S_{2,k} = \frac{\pi^2}{6} + O\left(\frac{1}{k}\right). \tag{3}$$

We have

$$S_{1,k}S_{2,k} = A + C, \tag{4}$$

where

$$C = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=n+1}^{k} \frac{1}{r^2} = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{n} = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \frac{1}{r-n}.$$

Thus

$$2C = \sum_{r=2}^{k} \frac{1}{r^2} \sum_{n=1}^{r-1} \left(\frac{1}{n} + \frac{1}{r-n}\right) = \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n(r-n)},$$

$$\begin{aligned}
2C + \sum_{r=2}^{k} \frac{1}{r} \sum_{n=1}^{r-1} \frac{1}{n^2} &= \sum_{r=2}^{k} \sum_{n=1}^{r-1} \frac{1}{n^2(r-n)} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=n+1}^{k} \frac{1}{r-n} = \sum_{n=1}^{k-1} \frac{1}{n^2} \sum_{r=1}^{k-n} \frac{1}{r} \\
&= \sum_{r=1}^{k-1} \frac{1}{r} \sum_{n=1}^{k-r} \frac{1}{n^2} = \sum_{r=1}^{k-1} \frac{1}{r} \left\{\frac{\pi^2}{6} + O\left(\frac{1}{k-r}\right)\right\} \\
&= \frac{\pi^2}{6} S_{1,k-1} + O\left(\sum_{r=1}^{k-1} \frac{1}{r(k-r)}\right).
\end{aligned}$$

But

$$\sum_{r=1}^{k-1} \frac{1}{r(k-r)} = \frac{1}{k} \sum_{r=1}^{k-1} \left(\frac{1}{r} + \frac{1}{k-r}\right) = \frac{2}{k} S_{1,k-1} = O\left(\frac{\log k}{k}\right),$$

so that

$$2C + A = \frac{\pi^2}{6} S_{1,k} + S_{3,k} + O\left(\frac{\log k}{k}\right).$$

Thus (4) becomes

$$2S_{1,k}S_{2,k} = 2A + 2C = A + \frac{\pi^2}{6}S_{1,k} + S_{3,k} + O\left(\frac{\log k}{k}\right).$$

Using (3) and

$$S_{3,k} = \sum_{n=1}^{\infty} \frac{1}{n^3} + O\left(\frac{1}{k^2}\right),$$

we get

$$2\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\}\left\{\frac{\pi^2}{6} + O\left(\frac{1}{k}\right)\right\}$$
$$= A + \frac{\pi^2}{6}\left\{\log k + \gamma + O\left(\frac{1}{k}\right)\right\} + \sum_{n=1}^{\infty}\frac{1}{n^3} + O\left(\frac{\log k}{k}\right),$$

which reduces to (1).

In the next place

$$S_{1,k}^3 = B + DS_{1,k} + E, \tag{5}$$

where

$$D = \sum_{n=1}^{k} \frac{1}{n} \sum_{s=n+1}^{k} \frac{1}{s}, \qquad E = \sum_{n=1}^{k} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} \sum_{s=n+1}^{k} \frac{1}{s}.$$

Now

$$S_{1,k}^2 = 2D + S_{2,k}. \tag{6}$$

As for E, we have

$$E = \sum_{s=2}^{k} \frac{1}{s} \sum_{n=1}^{s-1} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{n=1}^{s} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{r} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r}$$
$$= \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r} \sum_{n=r}^{s} \frac{1}{n} - \sum_{s=1}^{k} \frac{1}{s^2} \sum_{r=1}^{s} \frac{1}{r},$$

so that

$$\begin{aligned} 2E &= \sum_{s=1}^{k} \frac{1}{s}\left(\sum_{r=1}^{s} \frac{1}{r}\right)^2 + \sum_{s=1}^{k} \frac{1}{s} \sum_{r=1}^{s} \frac{1}{r^2} - 2\sum_{r=1}^{k} \frac{1}{r} \sum_{s=r}^{k} \frac{1}{s^2} \\ &= B - 2S_{1,k}S_{2,k} + 3A - 2S_{3,k}. \end{aligned} \tag{7}$$

Thus using (5), (6), (7) we get

$$S_{1,k}^3 = \frac{3}{2} B + \frac{1}{2} (S_{1,k}^2 - S_{2,k})S_{1,k} - S_{1,k}S_{2,k} + \frac{3}{2} A - S_{3,k},$$

which reduces to

$$\begin{aligned} S_{1,k}^3 &= 3B - 3S_{2,k}S_{1,k} + 3A - 2S_{3,k} \\ &= 3B - 3(S_{2,k}S_{1,k} - A - S_{3,k}) + 5S_{3,k}. \end{aligned}$$

Now using (1) and (3) we immediately get (2).

Also solved by the Proposer.

A Set of Transcendental Numbers

4583 [1954, 263]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For $a_n = 1$, $n = 1, 2, \cdots$,

$$S = \sum_{r=1}^{\infty} [r!]^{-a_r}$$

is transcendental. Find other non-decreasing sequences $\{a_n\}$ such that S is transcendental.

Solution by the Proposer. Let

$$\sum_{r=1}^{n} [r!]^{-a_r}$$

be an approximation to S. If S is algebraic, then by the theorem of Liouville on algebraic numbers

$$\sum_{r=n+1}^{\infty} [r!]^{-a_r} \geqq [n!]^{-ma_n} \tag{1}$$

must hold for $m > 2$ and n sufficiently large. If we take $a_{j+1} > 1 + a_j$, (1) implies

$$\frac{[(n+1)!]^{-a_{n+1}}}{1 - \dfrac{1}{(n+1)!}} \geqq [n!]^{-ma_n}$$

which, in turn, implies that

$$2[n!]^{ma_n} \geqq [(n+1)!]^{a_{n+1}}. \tag{2}$$

But (2) will not hold if we choose a_n of sufficiently high order in n. In fact, since $2[n!]^m < [(n+1)!]^{n+1}$ for $n > m$, we have

$$2^{n!}[n!]^{mn!} < [(n+1)!]^{(n+1)!}$$

which contradicts (2) with $a_n = n!$ and hence also contradicts (1). Thus S is transcendental for $a_n = n!$ and also for any a_n of equivalent or higher order, such as $(n!)^p$, n^n, *etc.*

Approximations for π and for e

4584 [1954, 263]. *Proposed by W. F. Cheney, Jr., University of Connecticut*

The equation $15x^2-78x+97=0$ has a root $3.14160+$, and the equation $27x^2-55x-50=0$ has a root $2.71829+$. It is required to find quadratic equations with integer coefficients, each less than 100 in absolute value, which have roots closest to π and e, respectively.

Solution by Walter Penney, Washington, D. C. If π is a root of the equation $ax^2+bx+c=0$, we must have $a\pi^2+b\pi+c=0$. Since a can be taken positive without any loss of generality, we seek integers $a(0<a<100)$ and $b(-100<b<100)$ such that $9.8696044010\cdots a+3.1415926535\cdots b$ is as close as possible to an integer $c(-100<c<100)$; a must be less than 42, since $42\pi^2-100\pi$ is greater than 100. If a is 41, b must be 99, 98 or 97. If a is 40, $94\leqq b\leqq 99$, *etc.* When a value of c close to an integer is found, $7\pi(=21.9911485745\cdots)$ can be added to or subtracted from this value to see whether the result is improved. In this way we find the following pairs of values of (a, b) which give near-integral values of c: $(36, -94)$, $(28, -66)$, $(27, -74)$, $(26, -82)$, $(19, -39)$, $(15, -78)$, $(11, -4)$. Testing each in turn, we find the best result is $19x^2-39x-65=0$ with the root 3.14158805.

Similarly we find $35x^2-83x-33=0$ has a root 2.71828582.

Also solved by J. P. Ballantine, Bart Park, and the Proposer.

A Series Involving the Partition Function

4585 [1954, 263]. *Proposed by H. F. Sandham, Institute for Advanced Studies, Dublin, Ireland*

Denoting the number of partitions of n by $p(n)$, prove

$$\sum_{n=1}^{\infty} \frac{np(n)}{\cosh \pi\sqrt{(2n-\frac{1}{4})}} = \frac{1}{4\pi}.$$

Solution by Chih-yi Wang, University of Minnesota. This solution was suggested by the Proposer's paper *Two series of partitions* (THIS MONTHLY, v. 61 (1954), pp. 104–106), where we have the relations

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \sum_{n=0}^{\infty} p(n)q^n, \tag{1}$$

$$\int_0^1 q^{\beta+1/8}\{(1-q)(1-q^2)(1-q^3)\cdots\}^3 \frac{dq}{q} = \frac{2\pi}{\cosh \pi\sqrt{2\beta}}, \tag{2}$$

$(\beta > -1/8)$. By putting $\beta = n - 1/8$ in (2), we get

$$\int_0^1 \{(1-q)(1-q^2)(1-q^3)\cdots\}^3 d(q^n) = \frac{2\pi n}{\cosh \pi\sqrt{(2n-\frac{1}{4})}}.$$

Multiplying across by $p(n)$ and summing with respect to n from 1 to ∞, we have, by aid of (1),

$$\begin{aligned} 2\pi \sum_{n=1}^{\infty} \frac{np(n)}{\cosh \pi\sqrt{(2n-\frac{1}{4})}} \\ &= \int_0^1 \{(1-q)(1-q^2)(1-q^3)\cdots\}^3 d\,\frac{1}{(1-q)(1-q^2)\cdots} \\ &= -\frac{1}{2}\big[\{(1-q)(1-q^2)\cdots\}^2\big]_0^1 = \frac{1}{2}. \end{aligned}$$

Hence the stated result follows.

Also solved by the Proposer.

Behavior of Improper Integrals

4587 [1954, 264]. *Proposed by A. E. Livingston, University of Washington, and Harry Pollard, Cornell University*

Show that

$$-(2n+1)\pi \int_{(2n+1)\pi}^{\infty} x^{-1} \sin x dx \uparrow 1.$$

(A similar statement holds for

$$2n\pi \int_{2n\pi}^{\infty} x^{-1} \sin x dx.)$$

Solution by Chih-yi Wang, University of Minnesota. Let the given integral be denoted by $I(n)$. Using the formula for integration by parts, we obtain

$$I(n) = 1 + (2n+1)\pi \int_{(2n+1)\pi}^{\infty} x^{-2} \cos x dx.$$

Now consider $I(n)$ as a continuous function of the positive real variable n. We get, by differentiating with respect to n,

$$\begin{aligned} I'(n) &= \frac{2}{2n+1} + 2\pi \int_{(2n+1)\pi}^{\infty} x^{-2} \cos x dx \\ &\geqq \frac{2}{2n+1} - 2\pi \int_{(2n+1)\pi}^{\infty} x^{-2} dx = 0, \end{aligned}$$

which implies that $I(n)\uparrow$. The proof will be complete if we integrate by parts three times, then

$$I(n) = 1 - 2(2n + 1)^{-2}\pi^{-2} + O(n^{-2}),$$

which approaches 1 as $n \to \infty$.

The second proposed statement is established by the same procedure.

Also solved by D. L. Greenstein, M. S. Klamkin, M. R. Spiegel, and the Proposers.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Theory of Games and Statistical Decisions. By David Blackwell and M. A. Girshick. New York, John Wiley and Sons, Inc., 1954. ix+355 pages. $7.50.

As the first textbook to appear on Wald's theory of statistical decision functions, Blackwell and Girshick's book ought to attract the interest of the general mathematician without a specific interest in research in mathematical statistics. It is Wald's achievement to have placed the problems of mathematical statistics within the frame of a precise axiomatic mathematical theory and through this transformation to have made significant practical advances in the field through the development of sequential analysis and sequential decision theory. This is as straightforward an illustration of the power of the abstract mathematical method as anyone could wish.

The two basic principles underlying Wald's work are first, that a statistical procedure should be tested by its consequences under various circumstances as measured by the expected risk function, and second, that for a procedure with a variable number of steps, part of the risk consists in the cost of the steps. From the first of these principles, it follows that statistical decision theory can be set in the framework of the Von Neumann theory of the normal form of a zero-sum two-person game. It follows from the absence of a specific conscious opponent in the statistical "game" that the objects of interest and consequently the mathematical problems may be different in the two theories, but both fall under the aegis of the theory of convex sets and functions in linear spaces. Chapters 1 and 2 give a very clear and complete presentation of the theory of two-person zero-sum games. The formal description of statistical and sequential games is presented in Chapter 3, while Chapter 4 on principles of choice indicates why the study of the minimax strategy solutions of these games is inadequate from the point of view of the statistical theory. The formal theory of more

general classes of optimal strategies of significance from this viewpoint, particularly the admissible and Bayes strategies, is developed in Chapter 5. The remaining seven chapters apply these general concepts to the determination of the Bayes strategies for statistical games under various special assumptions, and in particular for sequential games in Chapters 9 and 10.

The amount of mathematical knowledge explicitly presupposed by the book is fairly limited as far as elementary analysis and linear algebra are concerned. What is presupposed, however, is not so much knowledge as familiarity with the whole process of abstract reasoning in the framework of elementary set theory. The great clarity of the presentation is achieved by taking full advantage of the notation and conventions of abstract mathematics as commonly practiced at the present time. If the first two chapters on game theory are compared to the related sections of McKinsey's book, one is struck both by the absence of discursive material in the present book and by its enormously greater mathematical sophistication. It is curious that one could read this whole book on statistics without knowing any but the most elementary facts about probability theory and without once meeting Stirling's formula. On a different tack, it might be mentioned that the reviewer was puzzled by the fact that, in the midst of extremely long formal definitions and proofs, the authors chose to use the property of compactness of sets in Euclidian spaces without ever defining the term and in the process made a number of petty but inelegant mis-statements. The book is singularly free of printer's errors and the only mathematical mistake the reviewer could find was on page 154, where a qualitative discussion correct for discrete distributions is incorrectly extrapolated to continuous distributions.

Perhaps the pressure of books as important as this one will aid in the modernization of the undergraduate mathematical curriculum in American colleges to the extent that it may be possible in the future to use this book on a wide scale, as the authors' preface states they intend it, as a textbook for first-year graduate students.

F. E. BROWDER
Fayetteville, North Carolina

Differential and Integral Calculus. By C. E. Love and E. D. Rainville. Fifth Edition. The Macmillan Company, New York, 1954. vi+526 pages. $5.75.

Calculus. By G. E. F. Sherwood and A. E. Taylor. Third Edition. New York, Prentice-Hall, Inc., 1954. vii+579 pages. $7.65.

Calculus, an Introduction to Analysis, and A Tool for the Sciences. By G. M. Merriman. Henry Holt and Company, New York, 1954. viii+625 pages. $6.50.

These three texts are alike in many ways: each is as long as an historical novel, as rich as a well-made fruitcake, and as carefully organized as a mountain-climbing expedition. Starting with some work on limits, each of the books develops differentiation of algebraic functions, mentioning the process of anti-differentiation fairly early. Other topics considered (not in this order in all the

books) are differentials, the mean-value theorem, the definite integral, the elementary transcendental functions, technique of integration, partial differentiation, applications to problems of geometry and physics in two and three dimensions, series, approximate integration, and differential equations. And lots mo'. There is an abundance of excellent problems in each book; some of the problems are, of course, the old stand-bys ("A Norman window consists . . . ") but many have interesting novel features.

The texts differ, one from another, with respect to rigor of treatment, formality of tone, degree of explicit motivation in the introduction of topics, and "reconditeness," to borrow a word from Whitehead's essay, *The Mathematical Curriculum.* One way to convey a feeling for the flavor of the books is to present some quotations from each, with relatively little comment. The limitations of such a method are obvious; nevertheless this reviewer feels that the following quotations present a faithful picture of the spirit of the books.

I. Limits

1. *L. and R., p. 9.* "Let $f(x)$ be a function of x and let a be constant. If there is a number L such that, *in order to make the value of $f(x)$ as close to L as may be desired, it is sufficient to choose x close enough to a, but different from a,* then we say that the limit of $f(x)$, as x approaches a, is L." This statement is followed by a page and a half of elaboration. P. 495 (appendix), "Definition of a limit. We say that $\lim_{x\to a} f(x)=L$, if for every positive number ϵ (arbitrarily small), there exists a number δ such that, in order to make $|f(x)-L|<\epsilon$, it is sufficient that x satisfy $|x-a|<\delta$, $x\neq a$."

2. *S. and T.* After two pages of discussion of derivatives, involving an intuitive notion of a limit, we find, on p. 22, "As yet we have not given a formal or precise definition of what this means . . . We are considering some particular function, and we ask the question: Does $f(x)$ approach some limit as x approaches x_0? To answer this, we inspect the table, looking at all the values of x near x_0. *Perhaps the inspection will show us that there is a certain number A with the property that $f(x)$ differs from A by very little when the difference between x and x_0 is small, and that the difference $f(x)-A$ can be brought down to any desired smallness, and maintained that small or even smaller, simply by requiring x to be near enough to x_0. If so, this is exactly what we mean by saying that $f(x)\to A$ as $x\to x_0$.*" This statement is followed by a page and a half of elaboration, which includes the following "brief formal statement of the definition of a limit": "Definition. We write $\lim_{x\to x_0} f(x)=A$ and say that $f(x)\to A$ as $x\to x_0$, if corresponding to each positive number ϵ there is some positive number δ such that $|f(x)-A|<\epsilon$ whenever $0<|x-x_0|<\delta$."

3. *M.* After four pages of discussion of the instantaneous velocity of a vertically thrown ball, we have, p. 15, the following: "Definition: If a variable v moves on its range $\boldsymbol{R}$ so that all values assumed by v after a certain one lie arbitrarily close to a constant l (not necessarily on $\boldsymbol{R}$), then l is the *limit* of v: $l=\lim v$." After two pages of elaboration, and one page dealing with functions,

we come to the following: "Definition. If (and only if) for arbitrary measure $\epsilon > 0$ a corresponding dependent measure $\delta > 0$ can be exhibited such that for all x on the interval $0 < |x-a| < \delta$ the functional values $f(x)$ satisfy $0 < |f(x) - L| < \epsilon$, then the number L is the *limit* (*value*) of $f(x)$ as $x \to a$."

It seems clear that S. and T. win this round.

II. Plane Areas

1. *L. and R., pp. 112–120.* "Calculus grew out of the attempts . . . of mathematicians to solve two major problems . . . The second problem was to obtain the area bounded by a curve $y=f(x)$, the x-axis, and two ordinates $x=a$ and $x=b$.

"This second problem is solved by a judicious extension of the elementary concept of the area of a rectangle as the product of its base and its altitude.

"In Fig. 57, let the interval $a \leqq x \leqq b$ be divided into n parts in any manner . . .

"It is reasonable that, if the maximum width of the rectangles shown be taken sufficiently small, and the number of rectangles correspondingly large, then the sum of the areas of the rectangles will approximate, as closely as desired, a quantity which agrees with our intuitive concept . . . of the required area.

"Therefore, we proceed to lay down as our definition of the area A . . . the following:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k) \Delta x_k,$$

in which, since the widths Δx_k of the rectangles approach zero, the number of them, n, must approach infinity.

"At once we are confronted with the question of whether the limit . . . exists, and, if it does exist, with the problem of determining how to compute that limit . . . "

The foregoing exposition is followed by a statement of the Fundamental Theorem of the Integral Calculus, and then:

"Proof of the Fundamental Theorem will be omitted . . . Here we content ourselves with the discussion in the next section, a treatment intended to make the Fundamental Theorem plausible, not to prove it."

Next come five pages concerned with an intuitive discussion of the Fundamental Theorem, and with the computation of plane areas.

With the authors' premises about the appropriate degree of depth and rigor, it is hard to imagine a better exposition than the foregoing.

2. *S. and T., pp. 28–31.* "The problem of calculating areas bounded all or in part by curved lines is a fundamental problem of geometry. Let us consider the problem for the area OAB shown in Fig. 7 where the bounding curve is $y=x^2$. This problem was solved by Archimedes in the third century B.C.

"Area is measured by comparing the given figure with a standard figure, the unit area, which is a square with sides one unit long. The formula for the area

of a rectangle is deduced from such a comparison. In dealing with curved figures, we may attempt to choose a set of rectangles which do not overlap and which nearly cover the figure. An approximation to the area of the figure is then furnished by the total area of the rectangles. This is the basis for our procedure."

After a page devoted to algebraic development, we find:

"Hence we can see that S_n approaches $\frac{1}{3}c^3$ as a limiting value. We conclude that the area between the curve $y=x^2$ and the x-axis, from $x=0$ to $x=c$, is $S=\frac{1}{3}c^3$." (Note that area has not yet been defined.)

On the following page, the problem of area is considered for an arbitrary function which is continuous and never negative on the interval $a\leqq x\leqq b$: " . . . We then consider the sequence $S_1, S_2, S_3, \cdots, S_n, \cdots$. It always turns out that, as n is made larger and larger, the numbers S_n approach a limiting value S. This number S is, *by definition*, taken to be the area of the figure . . . "

Pp. 63–66. "Finding areas by antidifferentiation. . . . "

Pp. 246–258. These pages are devoted to a discussion of lower and upper sums, to re-definitions of the area under a curve and of the area between two curves, and to proofs of the mean-value theorem for integrals and of the Fundamental Theorem.

3. *M., pp. 206–232.* Starting with the problem of finding the work done in emptying a tank, the author is led to the geometrical problem of finding "a numerical value for an area. But the 'area problem' is a new problem. . . . To extend the notion of area we must lead off from known ground. In terms of the area unit of the plane content of a 1×1 square, the reader knows . . . This is the extent of his basic information. It is not fair to include here the familiar area of a circle . . . Any extension, then, of the concept of area and calculation of area must of mathematical necessity be based on the fundamental area of rectangle. In a larger sense it can be pointed out that we actually need to *define* what is meant by the area enclosed by one or more *curved* arcs. Such assignment of meaning in terms of *square* units presents difficulties. The present goal is the invention of a suitable extended definition of area; the result will also furnish a calculus of its numerical value."

Then follow three pages of discussion of "Area and Perimeter of a Circle in Greek Geometry." Next comes the statement: "As in so many extensions of old notions toward new fields of conquest, we first review an old problem in which we know the answer, but seek a generalizable method of obtaining it." Four pages are devoted to the calculation of sums of areas of rectangles inscribed in $y=2x$ and in $y=12x-x^2$, and of the limits of these sums. "Yet, from the intuitive geometry of the situation, any member of $\{S_n\}$ *encloses* the area OAB sought, while any member of $\{s_n\}$ *is enclosed by* this area. Thus OAB is caught between $\{S_n\}$ and $\{s_n\}$, which sequences *both* approach 144; how, then, can we mistrust the appropriation of this common limit $S=144$ as the definition of area and area-number?"

The definite integral over a closed range of a single-valued function is next defined, the existence of the definite integral for continuous functions is stated,

the area bounded by the graph of a continuous non-negative function over $a \leqq x \leqq b$ and the lines $y=0$, $x=a$, $x=b$, is defined, and the Fundamental Theorem is stated and proved, using the mean-value theorem and the existence of the definite integral for continuous functions.

III. The Area of a Surface of Revolution

1. *L. and R., pp. 305–306.* " . . . By elementary geometry, the surface area of this conical frustum is *the circumference of the middle section multiplied by the slant height*, or $2\pi(y_i+\frac{1}{2}\Delta y_i)\Delta s'_i$. Discarding higher order infinitesimals as usual, . . . "

2. *S. and T., pp. 321–327.* " . . . We can write this sum as two separate sums . . . Each of these is a sum to which the formula of Bliss . . . applies . . . " Proof of the formula of Bliss is omitted.

3. *M., pp. 274–279.* " . . . Bliss' Theorem is easy to understand and to apply; it is modernly finding its way into the literature of teaching elementary calculus. We shall accept the theorem without proof, but we attempt to make it plausible at the end of this section. . . . "

Each of these books has so many good features that the reviewer is tempted to suggest a liaison of L, R, S, T, and M to produce one text incorporating the advantages of all. Mindful, however, of Bernard Shaw's retort to Isadora Duncan, he resists the temptation and accepts these books for what they are: excellent additions to the library of American texts.

R. A. Rosenbaum
Wesleyan University

Trigonometry. By Roy Dubisch. New York, The Ronald Press, 1955. xiv+283 +Index+tables. $5.00.

Trigonometry. By E. P. Vance. Cambridge, Massachusetts, Addison-Wesley Publishing Company, 1954. viii+140+Index+tables. $3.00.

For years the teacher of mathematics has been inundated by elementary textbooks which, in mathematical content and presentation, might well have been printed from the same century-old plates. It is a pleasure to report that neither of the books under review falls into this category. Both evidence sincere attempts to modernize the teaching of trigonometry and to present the subject as part of the main stream of mathematics.

The Vance book is a compact (124 pages of text) volume which the instructor would have to amplify in class. Professor Vance presents trigonometry just as a mathematician thinks of the subject, his emphasis clearly upon the mathematical aspects. The unity and clarity thus achieved are remarkable. There is some question, however, in regard to its use as a text. Will the student be as enthusiastic over the Vance book as is his teacher?

After a first chapter on Numbers and Coordinate Systems, Vance proceeds to circular functions. His very insistence upon this term seems to leave open the

possibility of other functions related to conic sections. The third chapter, Functions Involving More Than One Angle, contains the first completely satisfactory derivation of composite angle formulas to be found in an elementary text. Chapter 4, Solution of Triangles, is very neatly done, properly beginning with the law of sines and only then specializing to right triangles. Chapter 5, Inverse Functions and Graphs, and Chapter 6, Identities and Equations, are both good but not unusual. Chapter 7, Complex Numbers, begins with a discussion of the algebra of number pairs and continues to exploit the quite general approach. The last chapter, Applications of Circular Functions to Periodic Phenomena, is wonderful, even going so far as to talk about harmonic analysis. It should whet the appetite of even a mildly interested student. Finally, it should be noted that Vance does not include the study of logarithms in his book except in an appendix.

In contrast to the brevity of the Vance book, which would place interesting demands upon the instructor, Professor Dubisch has written such an extensive and lucid treatment that an instructor would be almost superfluous. His book might well be labelled as "self-teaching." Still the presentation has much the same spirit as does the Vance book.

The introductory chapter contains a nice explanation of "function," reminiscent of that in Begle's recent calculus text. The second chapter deals with the arc-length function $ALF(ALF(x)$ is the point on the unit circle with coordinates $(\cos x, \sin x))$. This appears to be an artificiality but it leads to a very neat formulation. In Chapters 3 through 7, Dubisch takes up trigonometric functions, tables and graphs, inverse functions, identities and equations, all in terms of ALF. Then in Chapter 8, the functions are re-defined in terms of angles. This approach introduces radian measurement very naturally. Chapter 9 repeats the study of identities and equations, now in terms of angles. Solution of Triangles is the title of Chapter 10 and the material here is quite standard, as is the eleventh chapter on complex numbers. The final two chapters take up logarithms and are well done but unspectacular. (Why does the purely algebraic study of logarithms traditionally appear in a trigonometry course?) Briefly then, Dubisch starts out with a new and mathematically interesting approach. He follows his new line very well until he gets to the traditional material of trigonometry whereupon the treatment becomes quite standard. This is not to be construed entirely as a derogatory comment since a sound foundation is actually the bulk of the work in any construction.

The format of the Dubisch book is good but cannot quite compare with the excellent standards associated with Addison-Wesley.

Both authors and both publishers deserve praise for leaving the old and beaten track and heading toward the modern. It is to be hoped that others will be stimulated by these two books to write much-needed modern texts for other elementary courses.

J. G. Hocking
Michigan State College

Mathematics in Western Culture. By Morris Kline. New York, Oxford University Press, 1953. 12+484 pages. $7.50.

The object of this book is to show that mathematics has been an important influence in the development of our Western culture. This might be relatively easy if the author were writing for mathematicians; but the foreword by Professor Courant implies that the book is addressed to the group of intelligent people who do not have a background of mathematical knowledge, and this makes the undertaking a formidable one. In his introductory chapter the author says, "The subject is not a series of techniques. These are indeed the least important aspect, and they fall as far short of representing mathematics as color mixing does of painting. The techniques are mathematics stripped of motivation, reasoning, beauty, and significance." He then proceeds to present mathematics stripped of techniques. The non-mathematical reader, without encountering any technique beyond a little elementary algebra and some simple graphs, may possibly get some notion of the cultural significance of the conception and growth of various mathematical ideas.

After the introduction one finds three chapters on Egyptian, Babylonian, and Greek mathematics, and it is pointed out that Euclid and the Greek philosophers represent the beginnings of abstract thinking and deductive reasoning. We are told that "the contribution of the Greeks that did most to determine the character of present-day civilization was their mathematics." Certainly Euclid's contribution to logical thinking could hardly be overestimated; but in emphasizing this point it was not necessary for the author to imply, somewhat uncritically, that Euclid deduced all his geometric theorems by flawless reasoning from just ten well chosen axioms.

Two chapters are devoted to the Alexandrian era, in which are found the beginnings of trigonometry, with practical applications to navigation, surveying, and such astronomical measurements as the radius of the earth and the distance to the moon. Then, under the heading "Interlude," a single chapter disposes of the nine or ten centuries of the medieval period in which there was no progress in mathematics and little or none in the development of Western culture.

The next six chapters describe the rapid mathematical advances of the two centuries beginning with the Renaissance. It is pointed out that the work of Copernicus, Kepler, and Galileo in astronomy did much to free scientific thought from the tyranny of the church; that the mathematical theory of perspective of Leonardo da Vinci and Dürer brought about important changes in the art of painting; and that Descartes' mathematical approach to the search for truth had a profound influence on religious and philosophical thinking.

There is a chapter on Newton's discovery of the law of gravitation, showing that it wiped out any distinction between celestial and terrestrial mechanics; and the author then devotes four chapters to the work of Leibniz and Newton on the calculus, and a description of the Newtonian influence on science, religion, literature, and aesthetics. In discussing the work of Leibniz and Newton the

author is distinctly more critical than he was in the case of Euclid's *Elements.* He says:

"No one can read the details of their writings on the calculus without being amazed by the variety of ways in which they stabbed at, around, and about the correct version of the limit concept without actually striking it. Several times they changed their approaches and contradicted their earlier statements. Neither man succeeded in doing more with the limit concept proper than confusing himself, his contemporaries, and even his successors."

Two chapters discuss the mathematical treatment of sound waves, of electromagnetic and light waves, and the effect of this work on music and the transmission of music and other sound.

The mathematical approach to the social sciences, especially through the application of the theories of statistics and probability, is discussed in three chapters. The author uses the device of a rather amusing Platonic dialogue between "Mr. Determinist" and "Mr. Probability of High Degree" to point out the conflict between the philosophy of determinism and the statistical point of view. The final chapters discuss the infinite and related paradoxes, non-Euclidean geometries and the belief in absolute truth, Einstein's relativity and the question of absolute time and space, and mathematics as itself a form of creative art.

Many mathematicians are skeptical with regard to books about mathematics written by mathematicians for the non-mathematical reader. The book under review makes a strong case in support of its thesis, and will be of very considerable interest *to mathematicians.* The reviewer would be interested, however, to read an appraisal of the book written by one of the non-mathematical laymen to whom it is supposedly addressed.

W. B. CARVER
Ithaca, New York

NEW BOOKS RECEIVED

American Men of Science, I, The Physical Sciences. New York, R. R. Bowker Company, Feb. 1955, $20.00.

An Introduction to the Theory of Numbers. By G. H. Hardy and E. M. Wright. Oxford, Clarendon Press, England, 1954. xvi+419 pages. 42s. Net.

Functional Mathematics, Grade 8. By W. A. Gager, D. H. Johnson, C. N. Shuster, R. Madden, F. W. Kokomoor. New York, Charles Scribner's Sons, 1955. viii+373 pages. $2.24.

Memorial des Sciences Mathematiques. Published by L'Academie des Sciences de Paris, Henri Villat, Director. Fascicule CXXIX, *Integration des equations aux derivees partielles du second ordre par la methode de Drach.* By M. Georges Heilbronn. Paris, Gauthier-Villars, 1955. 96 pages.

Automatic Feedback Control System Synthesis. By John G. Truxal. New York, McGraw-Hill Book Company, 1955. xiii+675 pages. $12.50.

Philosophy and Analysis. Edited by Margaret MacDonald. New York, Philosophical Library, 1954. 296 pages. $7.50.

Higher Transcendental Functions, Volume 3. Sponsored by California Institute of Technology, The Bateman Project Staff, Editor, A. Erdélyi. New York, McGraw-Hill Book Company, 1955.

One Hundred Mathematical Curiosities. By William R. Ransom. Portland, Maine, Box 1075, J. Weston Walch, Publisher, 1955. 212 pages.

Table of Salvo Kill Probabilities for Square Targets. By U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series 44. Washington, 25, D. C., 1955. ix+33 pages, 30 cents.

A Short Table for Bessel Functions of Integer Orders and Large Arguments. By L. Fox. Published for the Royal Society at the University Press, Cambridge, 1954. 27 pages. 6s.6d. net.

Royal Society Mathematical Tables, 3, Table of Binomial Coefficients. Edited by J. C. P. Miller. New York, Cambridge University Press, American Branch, 1954. 162 pages. $5.50.

Selected Papers in Statistics and Probability. By Abraham Wald. Edited by the Institute of Mathematical Statistics. New York, McGraw-Hill Book Company, 1955. 702 pages. $8.00.

Differential and Integral Calculus, 2nd Edition. By H. M. Bacon. New York, McGraw-Hill Book Company, 1955. 547 pages. $6.00.

Proceedings of the First Conference on Training Personnel for the Computing Machine Field, (Held at Wayne University, Detroit, Mich., June 22 and 23, 1954). Edited by Arvid W. Jacobson. Detroit, Wayne University Press, 1955. 104 pages. $5.00.

Partial Differential Equations. By I. G. Petrovsky. New York, Interscience Publishers, Inc., 1954. 245 pages. $5.75.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York, New York University Press (Distributed by Interscience Publishers, New York), 1954. 154 pages. $5.00.

The Elements of Probability Theory and Some of its Applications. By Harald Cramer. New York, John Wiley and Sons, Inc., 1955. 281 pages. $7.00.

An Introduction to Deductive Logic. By Hugues Leblanc. New York, John Wiley and Sons, Company, 1955. 244 pages. $4.75.

Advanced Mathematics for Engineers, 3rd Ed. Revised by F. H. Miller. New York, John Wiley and Sons, Inc., 1955. 548 pages. $6.50.

Modern Trigonometry. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1955. 209 pages.

Science and the Human Imagination, Aspects of the History and Logic of Physical Science. By Mary B. Hesse. New York, Philosophical Library, 1955. 171 pages. $3.75.

Introduction to College Mathematics, Revised Edition. By M. A. Hill, Jr., and J. Burton Linker. New York, Henry Holt and Company, 1955. 428 pages +99 pages of tables. $5.25.

Plane Trigonometry. By C. R. Wylie, Jr., New York, McGraw-Hill Book Company, 1955. 381 pages. $4.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

TOPOLOGY NOTES BY A. W. TUCKER

Microfilms of Professor Tucker's notes on Elementary Combinatorial Topology, given at Haverford College under the Philips Fund, and also the notes of several lectures given at the University of Washington are available at *University Microfilms*, 313 N. First Street, Ann Arbor, Mich., at a total cost of $2.70. Requests should be sent directly to *University Microfilms*.

PERSONAL ITEMS

Professor F. E. Johnston of George Washington University was the official representative of the Association at the dedication of the Glenn L. Martin Institute of Technology at the University of Maryland on March 25, 1955.

Professor A. W. McGaughey of Bradley University represented the Association at the inauguration of President Harold P. Rodes of Bradley University on May 6, 1955.

Dr. A. Helen Tappan, who has retired from her position as Head of the Department of Mathematics of Western College for Women, was awarded the honorary degree of Doctor of Humane Letters by the College in June, 1954. Previously, she was cited by the Alumnae Association as one of its ten outstanding alumnae.

Franklin and Marshall College announces the following: Associate Professor D. W. Western has been promoted to a professorship; Assistant Professors V. H. Haag and J. R. Holzinger have been promoted to associate professorships.

Mr. D. L. Arenson, formerly a technical engineer for Cook Research Laboratory, Aerophysics Section, Skokie, Illinois, has a position as Manager of Ex-Cel Development Laboratory, Chicago, Illinois.

Associate Professor H. A. Bender of the University of Rhode Island has been promoted to a professorship.

Associate Professor A. B. Brown of Queens College has been promoted to a professorship.

Mr. J. S. Chesna, previously a physicist for Eastman Kodak Company, Rochester, New York, is President of the Infrared Electro-Optical Bolometer Company, Rochester, New York.

Associate Professor D. E. Coffey of Lawrence Institute of Technology has been appointed to an instructorship at Northwest Missouri State College.

Associate Professor Elsie T. Church of Northwestern State College has been promoted to a professorship.

Dr. W. J. Coles, formerly a Fulbright Scholar at King's College, Cambridge,

England, has been appointed to an instructorship at the University of Wisconsin.

Associate Professor P. R. Culwell, chairman of the Department of Mathematics, Trinity University, has been appointed to an instructorship at San Antonio College.

Mr. S. H. Dalrymple, formerly a student at the University of Texas, is now a research assistant at Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico.

Assistant Professor A. E. Danese of Western Reserve University has been appointed to an instructorship at the University of Tennessee.

Dr. T. C. Doyle, previously a mathematician at the Naval Research Laboratory, Washington, D. C., is now a member of the staff of Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Dr. J. W. Forman of the University of Kansas has a position as an applied science representative for I.B.M. Corporation, Kansas City, Missouri.

Mr. E. P. Graney, recently a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California, has a digital computer programming and consulting service in Ann Arbor, Michigan.

Dr. T. R. Horton, applied science representative for I.B.M. Corporation, Atlanta, Georgia, has been transferred to Asheville, North Carolina.

Mr. W. R. Hydeman, formerly a mathematician with Engineering Research Associates, Arlington, Virginia, has accepted a position as sales engineer, Electronic Computer Department, Remington Rand, Inc., New York, New York.

Mr. A. R. Kneer, formerly a student at Gonzaga University, has a position as a mathematician in the Department of Commerce, Weather Bureau, Washington, D. C.

Mr. David Loev, previously an assistant development engineer at Burroughs Adding Machine Company, Philadelphia, Pennsylvania, is now an assistant scientist at the Weizmann Institute of Science, Rehovot, Israel.

Miss Anna E. Many, counselor to women at Newcomb Memorial College, Tulane University, has retired.

Mr. G. E. Meike, formerly a teaching fellow at the University of Detroit, has been promoted to an instructorship.

Mr. John Neufeld has been appointed to an instructorship at Detroit Institute of Technology.

Dr. C. V. Newsom, Associate Commissioner for Higher and Professional Education, State University of New York, has been named Executive Vice Chancellor of New York University, effective July 15, 1955.

Mr. R. D. Oeder, previously a mathematician at the Radiation Laboratory, University of California, Livermore Site, has accepted a position as an applied science representative, I.B.M. Corporation, Seattle, Washington.

Mr. J. H. Oppenheim, formerly a computer at the University of Chicago, is now a graduate student at the University of Illinois.

Mr. J. A. Painter, previously a graduate assistant at the University of Pitts-

burgh, has accepted a position as technical engineer at I.B.M. Corporation, Endicott, New York.

Dr. M. A. Shader of Stanford University is now an applied science representative with I.B.M. Corporation, San Francisco, California.

Sister M. Bibiana, formerly an instructor at the College of St. Teresa, Winona, Minnesota, is now at Cotter High School, Winona, Minnesota.

Professor Emeritus J. B. Coleman of the University of South Carolina died in Vienna, Austria, on February 21, 1955. He was a charter member of the Association.

Professor L. S. Johnston of the University of Detroit died on February 18, 1955. He was a member of the Association for thirty-eight years. The Johnston family is establishing the Leon S. Johnston Prize in Mathematics at the University of Detroit. Friends of Professor Johnston who may wish to make a contribution to the fund may write to the Chairman of the Department of Mathematics of the University.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

Official Reports and Communications

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 118 persons have been elected to membership by the Board of Governors on applications duly certified.

BRIAN ABRAHAMSON, S.M. (Chicago) Lecturer, University of Cape Town, South Africa.

R. H. ACKERSON, A.M. (Columbia U.) Teaching fellow, Alabama Polytechnic Institute.

H. G. APOSTLE, Ph.D. (Harvard) Chairman, Department of Mathematics, Grinnell College.

LT. COL. J. W. AULT, M.A. (Ohio S.U.) Asso. Professor, U. S. Air Force Academy, Denver, Colo.

MRS. JOAN L. AUSTRA, A.B. (Montclair S.T.C.) Teacher, Florence Avenue School, Irvington, N. J.

C. H. BAILEY, M.A. (W. Va. U.) Asso. Professor, Concord College.

R. G. BARTLE, Ph.D. (Chicago) Instr., Yale University.

R. V. BENSON, A. M. (Southern Calif.) Instr., Los Angeles City College.

C. E. BOULWARE, Ed.D. (Columbia U.) Asso. Professor, North Carolina College.

A. H. BROWN, Ph.D. (Chicago) Instr., Rice Institute.

WINNIE E. BROWN, B.S. (Tennessee P.I.) Instr., Tennessee Polytechnic Institute.

P. L. BUTZER, Ph.D. (Toronto) Asst. Professor, McGill University.

H. C. CARTER, Ph.D. (Missouri) Professor, Mary Washington College.

E. C. CASALE, M.A. (Temple) Instr., Temple University.

LOUIS CHILD, M.A. (North Carolina) New Mexico College of Agriculture and Mechanic Arts.

VIRGINIA CLOVER, A.B. (Augustana C.) Fellow, University of Arizona.

WHITFIELD COBB, A.M. (North Carolina) Asso. Professor, Guilford College.

LAURENCE COOK, M.S. in Ed. (Drake) Teacher, Junior High School, Eau Claire, Wisc.

GRACE V. CRENSHAW, M.A. (Virginia) Asso. Professor, Danville Branch, Virginia Poly-

technic Institute; Head of Department of Mathematics, Averett College.

ARNO CRONHEIM, Ph.D. (Illinois) Instr., Brandeis University.

D. A. DEFELICE, M.S. (Pittsburgh) Instr., Duquesne University.

MARCEL DELCOURTE, C.E. (Louvain) General Manager, Les Assurances du Boerenbond Belge, Louvain, Belgium.

J. M. DORAN, M.A. (George Peabody) Instr., Tennessee Polytechnic Institute.

CARL DUBOVY, B.S. (C.C.N.Y.) Grad. Student, City College of the City of New York.

HELEN M. DUNN, B.S. (Washington) Teacher, West Seattle High School, Seattle, Wash.

M. R. ELIZER, M.A. (Georgia) Asst. Professor, Emory Junior College.

M. P. EPSTEIN, Ph.D. (Columbia U.) Instr., University of California, Berkeley.

CORDELL EVANS, M.Litt. (Pittsburgh) Grad. Asst., University of Pittsburgh.

JACQUELINE P. EVANS, Ph.D. (Radcliffe) Instr., Smith College.

R. F. FARRELL, B.S. (Canisius) Dynamics Analyst, Bell Aircraft Corp., Niagara Falls, N. Y.

BEVERLY R. FERNER, A.M. (Wisconsin) Instr., Ohio University.

EMMA W. GARNETT, M.A. (George Peabody) Instr., Tennessee Polytechnic Institute.

L. L. GAVURIN, S.M. (Brown) Brooklyn, New York.

P. R. GERWITZ, B.A. (Buffalo) Grad. Student, University of Buffalo.

HERBERT GINSBERG, B.A. (St. Lawrence) Numerical Analyst, General Electric Co., Evendale, O.

O. H. GOEHRING, M.A. (Columbia U.) Mathematician, Cornell Aeronautical Lab., Buffalo, N. Y.

BROTHER JAMES FRANCIS GRAY, M.S. (Northwestern) Grad. Student, University of Notre Dame.

R. P. GROBE, B.S. (Montana S.C.) Grad. Student, Montana State College.

CORNELIUS GROENEWOUD, M.A. (Michigan S.C.) Dynamics Engineer, Bell Aircraft Corp., Buffalo, N. Y.

C. J. HANKS, Ed.D (Arkansas) Instr., California State Polytechnic College.

H. L. HARTER, Ph.D. (Purdue) Math. Statistician, Wright-Patterson Air Force Base.

K. M. HERSTEIN, B.S. (Columbia U.) President, Herstein Laboratories, New York, N. Y.

J. H. HODGES, Ph.D. (Duke) Instr., Duke University.

A. V. HOUGHTON III, M.S. (Illinois) Asst. Professor, Bradley University.

J. L. HOWELL, Ph.D. (Yale) Instr., University of Delaware.

C. W. HUNTER, M.A. (Boston C.) 1st Lieutenant, U.S.M.C., Camp Lejeune, N. C.

PATRICIA A. INMAN, B.A. (Reed) Grad. Student and Research Asst., University of Oregon.

W. H. JONES, M.S. (Chicago) Mathematician, Department of Defense, Washington, D. C.

J. D. KAPLAN, Student, Rutgers University.

CAROLYN E. KAPPEL, Student, Carleton College.

J. E. KELLEY, M.S. (Marquette) Instr., Marquette University.

H. P. KERFOOT, Ph.D. (Southern Calif.) Lecturer, University of Southern California.

M. Y. KITAMURA, Student, University of Hawaii.

ANN E. KLEIN, M.A. (Wichita) Instr., University of Wichita.

D. L. KLIPPENSTEIN, M.S. (Iowa S.C.) Instr., Bethel College.

LOUISE M. KNIFLEY, M.A. (Kentucky) Instr., University of Tennessee, Martin Branch.

R. R. KORFHAGE, B.S.E. (Michigan) Grad. Student, University of Michigan.

E. C. KOVACS, B.S. (Washington & Jefferson) Asst. Professor, University of Pittsburgh.

G. R. KUHN, B.S. (St. Joseph's C.) Teacher, Academy of the Sacred Heart, St. Louis, Mo.

M. A. LAFRAMBOISE, M.A., M.S. (Michigan) Instr., University of Detroit.

R. W. LANZKRON, M.S. (Wisconsin) Fellow, University of Wisconsin.

D. M. LEVY, B.A. (William & Mary) Teacher, Chuckatuck High School, Virginia.

D. R. LEWIS, M.A. (Minnesota) Asst. Professor, State Teachers College, Mankato, Minn.

Cpl. R. L. LIBOFF, B.S. (Brooklyn) Physicist, Army Chemical Center, Md.

SHEN LIN, M.A. (Ohio S.U.) Asst. Instr., Ohio State University.

B. W. LINDGREN, Ph.D. (Minnesota) Instr., University of Minnesota.

MICHAEL LIONE, M.S. (N.Y.U.) Instr., Newark College of Engineering.
R. J. LOCKHART, B. A. (Western Ontario) Lecturer, University of Manitoba.
CAPT. W. D. MARSLAND, JR., M.A. (N.Y.S. College for Teachers at Albany) Instr., U. S. Air Force Academy, Denver, Colo.
EARL MAXIE, M.S. (Tuskegee) Teacher, Grambling College.
C. W. MCARTHUR, Ph.D. (Tulane) Asst. Professor, Alabama Polytechnic Institute.
L. F. MCAULEY, Ph.D. (North Carolina) Instr., University of Maryland.
GRANVILLE MCCORMICK, M.A. (Oregon) Teaching Asst., University of Washington.
NEILL MCSHANE, Student, University of Virginia.
V. J. MONACELLA, B.S. (Gannon) Mathematician, David Taylor Model Basin, Washington, D. C.
J. G. MOSER, Student, Rose Polytechnic Institute.
H. L. NEWMAN, M.S. in E.E. (Buffalo) Sylvania Electric Products, Buffalo, N. Y.
E. D. NICHOLS, M.A. (Illinois) Instr., University of Illinois.
D. A. PAGE, A.M. (Illinois) Instr., University of Illinois.
R. W. PAUL, JR., Student, California Institute of Technology.
R. S. PIERCE, Ph.D. (Calif. I.T.) Research Asst., Harvard University.
A. G. POORMAN, B.A. (Ashland) Grad. Asst., Iowa State College.
FRANK PROPP, M.A. in Ed. (C.C.N.Y.) Teacher, Morris High School, New York, N. Y.
T. D. RINEY, Ph.D. (Purdue) Member, Technical Staff, Bell Telephone Labs., Allentown, Pa.
T. P. ROBINSON, M.S. (Oklahoma A. & M.) Grad. Student, Oklahoma Agricultural and Mechanical College.
J. N. ROGERS, Student, Butler University.
JACK ROSEMAN, B.A. (Boston U.) Grad. Student, University of Massachusetts.
F. W. SCHNEIDER, Columbus, Ohio.
M. A. SHADER, Ph.D. (Syracuse) Applied Science Representative, I.B.M. Corp., San Francisco, Calif.
E. T. SHEFFIELD, M.A. (Minnesota) Physicist, U. S. Naval Radiological Defense Lab., San Francisco, Calif.
F. C. SHERBURNE, JR., B.S. (Toledo) Grad. Asst., Michigan State College.
G. J. SIMMONS, Research Asst., Sandia Corp., Albuquerque, N. M.
SISTER ANITA SIBILIA, A.B. (Montclair S.T.C.) Teacher, St. Joseph's High School, Hammonton, N. J.
SISTER M. IGNATIA, M.S. (Catholic) Head, Department of Mathematics, Marygrove College.
SISTER M. THEODORA, M.S. (Marquette) Instr., Ursuline College.
SISTER MARY ANDREA, Ph.D. (Catholic) Head, Department of Mathematics, St. Mary College, Xavier, Kansas.
SISTER MARY DE SALES, Student, Georgetown University.
H. L. SLONECKER, JR., M.A. (George Peabody) Grad. Student, Vanderbilt University.
J. L. SMITH, B.A. (Louisville) Grad. Asst., University of Pittsburgh.
MAJ. N. H. SMITH, M.S. (S.U. of Iowa) Instr., U. S. Naval Academy.
J. R. STOCK, Dr.Sc. (Swiss Federal I.T.) Mathematician, National Carbon Research Labs., Cleveland, O.
R. T. STUBBS, B.S. (Georgia I.T.) Savannah, Georgia.
E. J. THOMAS, M.A. (Ohio S.U.) Asst. Professor, Mississippi Southern College.
HOWARD TINSLEY, B.S. (Tennessee P.I.) Instr., Tennessee Polytechnic Institute.
R. N. WALTER, Ed.D. (Columbia U.) Teacher, Manhattan High School of Aviation Trades.
R. M. WARTEN, Student, Brooklyn College.
E. C. WATTERS, Ph.D. (Maryland) Asst. Professor, U. S. Naval Academy.
D. F. WEHN, Student, Brooklyn College.
A. M. WHITE, M.A. (U.C.L.A.) Instr., University of Santa Clara.
MRS. MARIE S. WILCOX, M.A. (Indiana) Head, Department of Mathematics, Thomas Carr Howe High School, Indianapolis, Indiana.
J. C. WILSON, M.S. (S.U. of Iowa) Instr., Central College, Pella, Iowa.
MRS. ANNA L. WILSTED, B.S. (Edinboro S.T.C.) Teacher, Townville Consolidated School, Pa.
SAMUEL WOLF, M.S. (Case I.T.) Engineer, Westinghouse Electric Corp., Baltimore, Md.

MRS. DOROTHY WOLFE, M.A. (Wayne) Asst., University of Pennsylvania.
F. B. WRIGHT, JR., Ph.D. (Chicago) Instr., Tulane University.
R. E. WYLLYS, B.A. (Arizona S.C.) Mathematician, National Security Agency, Washington, D. C.
HIDEHIKO YAMABE, Dr.Sc. (Osaka U.) Asst. Professor, University of Minnesota.
W. A. YERKES, Student, Lebanon Valley College.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-second annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, with Mississippi Southern College the host institution, on February 18 and 19, 1955. Professor J. W. McClimans, Chairman of the Section, presided at the evening session and Professor P. K. Rees presided at the Saturday morning session.

There were 92 persons registered, including the following 45 members of the Association.

Ruth E. Biggers, Ermon H. Bryant, N. A. Childress, W. H. Cleveland, M. P. Dossey, W. L. Duren, Jr., Elizabeth Freas, L. M. Garrison, M. E. Gillis, A. L. Gilmore, Jr., G. W. Hiller, H. G. Jacob, Jr., John Jones, Jr., H. T. Karnes, Margaret M. LaSalle, Z. L. Loflin, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, T. F. Mulcrone, J. D. Munn, S. B. Murray, Arthur Ollivier, Judith M. Pillow, P. K. Rees, L. A. Rife, W. M. Sanders, W. C. Sangren, H. F. Schroeder, Fariebee B. Self, S. W. Shelton, Jr., P. K. Smith, S. M. Spencer, Jr., R. A. Stokes, V. B. Temple, W. B. Temple, E. J. Thomas, Earl Thomas, W. E. Timon, Jr., B. B. Townsend, P. M. Tullier, Jr., B. O. Van Hook, Dale Woods.

The following officers were elected for the coming year: Chairman, Professor T. L. Reynolds, Millsaps College; Louisiana Vice-Chairman, Mr. P. L. Ford, McNeese State College; Mississippi Vice-Chairman, Professor John Jones, Jr., Mississippi Southern College; Secretary-Treasurer, Professor Z. L. Loflin, Southwestern Louisiana Institute.

The invited speaker for the meeting was Dr. W. C. Sangren, Oak Ridge National Laboratory. His address on Friday evening was entitled "The ORACLE, and Other Computers, as a Scientific Tool" and at the Saturday morning session, "Numerical Calculations Associated with the Membrane Problem."

The following papers were presented:

1. *On the triangular form of certain matrices*, by Professor John Jones, Jr., Mississippi Southern College.

I. Schur has shown that if A is a matrix with elements in the field of complex numbers there exists a unitary matrix U such that U^*AU is triangular with U^* the conjugate transpose of U. The purpose of this note is to obtain the triangular form of a class of matrices which have elements that are functions of a real variable x defined on an interval $a \leqq x \leqq b$.

2. *Stieltjes integral representation of convex functions*, by Professor W. B. Temple, Louisiana Polytechnic Institute.

The main result of this paper is contained in the following theorem: Let $f(x)$ be continuous and convex of order k, $(k \geqq 2)$ in $a<x<b$, $a<0$ and $b>1$. Then there exists a non-decreasing function $\mu(x)$ and a polynomial of degree not more than $k-1$ determined by the conditions $f(1)=g(1)$,

$f(0)=g(0), f'(0)=g'(0), \cdots, f^{k-2}(0)=g^{k-2}(0)$, such that

$$f(x) = g(x) - \int_0^1 G_k(x, t)d\mu(t)$$

where

$$G_k(x, t) = \frac{1}{(k-1)!}\left[(1-t)^{k-1}x^{k-1} - (x-t)^{k-1}\right], \qquad 0 \leqq t \leqq x < 1,$$

$$G_k(x, t) = \frac{1}{(k-1)!}(1-t)^{k-1}x^{k-1}, \qquad 1 \geqq t \geqq x > 0.$$

3. *On the rise of a liquid in a rotating container*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

This short paper deals with the equations concerning the behavior of a liquid under rotation in several types of containers. First, the well known fact is derived that a fluid rotating about an axis will assume the form of a paraboloid of revolution. Then for several types of containers the rise of the liquid in the container is shown to be a well-determined function of the angular velocity of the container.

4. *Roses not clovers*, by Professor V. B. Temple, Louisiana College.

Leaves (petals) of roses overlap, leaves of clovers do not. Hence, the curves commonly called roses should be clovers. These curves are defined by writers only by equations of the form $\rho=a \sin k\phi$ and $\rho=a \cos k\phi$. We have defined them geometrically and their general equations are

$$\rho = \frac{a}{n}(n \mp 2) \sin\left[\left(\frac{n}{n \mp 2\lambda}\right)\frac{\pi}{2} \mp \phi\right], \qquad n > 0, \lambda \neq 0.$$

The upper and lower signs represent hypo- and epi-cyclo-roses corresponding to the hypo- and epicycloids. See this MONTHLY, vol. 61, 1954, p. 265, and vol. 59, 1952, p. 67. If λ is chosen so that $n/(n-2\lambda)=K$, an integer, these equations reduce to the simple forms above, according as K is even or odd. In this case we have the clovers. If K is not an integer we have the roses.

5. *An exponential Diophantine equation possessing integral solutions*, by Professor W. M. Sanders, Mississippi Southern College.

The exponential Diophantine equation $zx+y=y^x$ for $0<x<z$ and $0<y<z$ is considered. A solution for $x=2$ is obtained.

6. *The vibrations of a circular membrane*, by Professor Wallace Hebert, Louisiana Polytechnic Institute, introduced by the Secretary.

This paper contains a solution of the basic equation of motion of the vibrating membrane with suitable boundary values and initial conditions given. With certain assumptions regarding symmetry the solution can be obtained.

The basic equation of motion of the membrane is given in the form:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where ∇^2 is the Laplacian operator. Since we wish to consider a membrane of circular form, such as a drum, the equation can conveniently be written in polar co-ordinates, which then becomes

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + k^2v = 0.$$

The solution of the above equation can be shown to be

$$v = AJ_0(kr) + BY_0(kr)$$

where J_0 and Y_0 are the Bessel functions of the zeroth order and of the first and second kind.

7. *Vector duality*, by Professor Benjamin Ernest Mitchell, University of Mississippi.

The vector equation $\lambda\rho = c$ says that c is the power of the common initial point 0 of λ and ρ with respect to a circle constructed on AB as a diameter where A and B are the termini of λ and ρ respectively—and vice versa, according to Clifford. This circle is orthogonal to the circle center O and radius $\sqrt{c}$. If we pin down λ the terminus of ρ will trace a line, the polar of A with respect to circle $(0, \sqrt{c})$. We therefore call λ a line vector (not a line-bound vector). If we pin down ρ and release λ there results a pencil of lines through the terminus of ρ, hence the duality. We therefore call ρ a point vector. The relationship is extended to reciprocation or polar duality (Graustein). The principle may be extended to higher dimensions with the proper choice of dual elements.

8. *On some groups associated with finite geometries*, by Professors John Jones, Jr., W. M. Sanders, and L. A. Rife, Mississippi Southern College.

L. A. Rife established the existence of a finite projective geometry of 57 points in a thesis in 1947, *On Some Finite Geometries*, University of Nebraska (unpublished thesis). Marshall Hall, Jr., has established the uniqueness of the projective plane with 57 points. Using the postulates of O. Veblen and W. H. Bussey to construct finite projective geometries, permutation groups associated with their finite projective geometries are considered.

9. *An affine definition of pi*, by Professor W. L. Duren, Jr., Tulane University.

The customary definition of π is formulated in the euclidean geometry of the plane. Despite the non-invariance of arc length in the plane represented as the Cartesian product of affine line and affine line, the number π is defined naturally in this geometry by means of an ellipse. In fact it is defined also by means of an invariant measure.

10. *A note on the Poncelet line*, by Reverend T. F. Mulcrone, St. Charles College.

This paper is concerned with the form assumed by various theorems involving the Simson or pedal line of a point relative to a triangle when extension is made to the Poncelet line.

11. *The ORACLE, and other computers, as a scientific tool*, by Dr. W. C. Sangren, Oak Ridge National Laboratory.

The last ten years have seen the viewpoint towards computations change radically because of the advent of high speed machines. This should be expected in any field where an improvement of high order, say 10^4 or more, has suddenly taken place. The improvement is in the length of time needed to perform certain mathematical operations. It should be emphasized that the need for mathematics and mathematicians is greatly increased because of the increasing use of machines. The ORACLE and the Mathematical Panel at Oak Ridge are examples of a high speed digital machine and a group which use such machines.

12. *Numerical calculations associated with the membrane problem*, by Dr. W. C. Sangren, Oak Ridge National Laboratory.

One of the problems that arises in nuclear reactor theory is the finding of the fundamental eigenvalue and eigenfunction associated with the classical membrane problem with a fixed boundary. The membrane problem may be characterized either through a differential equation system or by a variational problem. There exist basically two mathematical techniques, analytical and

numerical, for obtaining the desired value and function associated with the given geometry. Since the analytical techniques have been fairly well exploited, the numerical techniques show the greatest future promise. A number of programs or codes have been prepared and run on high speed digital machines, primarily the ORACLE, with the objective of obtaining the desired value and function for odd-shaped membranes. Among the peculiar shaped regions being investigated are L-shapes, T-shapes, crosses and square doughnuts.

Z. L. LOFLIN, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-fifth regular meeting of the Southern California Section of the Mathematical Association of America was held at Santa Monica City College on March 12, 1955. Professor E. C. Rex, Chairman of the Section, presided.

The attendance was 105, including the following 83 members of the Association:

L. J. Adams, Frances Campbell Amemiya, A. R. Amir-Moez, Norman Anning, T. M. Apostol, Leon Bankoff, J. L. Barnes, Mabel S. Barnes, Lulu Bechtolsheim, May M. Beenken, Clifford Bell, J. S. Bendat, R. V. Benson, L. T. Black, H. F. Bohnenblust, R. E. Bruce, Jessie R. Campbell, L. M. Coffin, E. P. Coleman, P. H. Daus, R. A. Dean, C. R. DePrima, R. P. Dilworth, G. M. Eddington, L. R. Ford, Jr., G. E. Forsythe, Ruth M. Frisch, W. V. Gamzon, W. H. Glenn, Jr., B. K. Gold, Jr., Edison Greer, Nathaniel Grossman, C. J. A. Halberg, Jr., W. L. Hart, V. C. Harris, A. R. Harvey, H. L. Hendricks, R. B. Herrera, R. E. Horton, D. H. Hyers, C. G. Jaeger, P. B. Johnson, G. R. Kaelin, Rosella Kanarik, P. J. Kelly, G. J. Kleinhesselink, L. C. Lay, Fred Marer, G. F. McEwen, E. E. Moots, F. R. Morris, A. B. Neale, J. B. Nelson, L. J. Paige, R. W. Paul, Jr., D. J. Peterson, R. R. Phelps, H. R. Pyle, F. C. Reed, E. C. Rex, J. M. Robb, E. M. Scheuer, Sister M. Madeleine Rose, Abe Sklar, Samuel Skolnik, N. B. Smith, T. H. Southard, Maria Weber Steinberg, A. C. Sugar, T. E. Sydnor, P. Y. Tani, Elmer Tolsted, C. B. Tompkins, C. W. Trigg, S. E. Urner, F. A. Valentine, R. J. Walker, Morgan Ward, P. A. White, R. L. White, A. L. Whiteman, B. R. Wicker, A. D. Wirshup.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor S. E. Urner, Los Angeles State College; Vice-Chairman, Professor H. F. Bohnenblust, California Institute of Technology; Program Committee: Chairman, Professor Elmer Tolsted, Pomona College, Mr. R. B. Herrera, Los Angeles City College, Professor D. V. Steed, University of Southern California, Mr. T. E. Sydnor, Pasadena City College.

The following program was presented:

1. *A role for complex numbers in analytic geometry*, by Professor V. C. Harris, San Diego State College.

By using one plane as both a real plane and a complex plane, the student can graph $y=f(x)$ as a complex function y of a real variable x. If $f(x)$ represents functions of the type usually considered in analytic geometry, little if any additional technique needs to be developed. Alternative methods for this purpose are too difficult or less informative. The result is intended to be a better understanding of the place of complex numbers in mathematics.

2. *Group-varieties*, by Professor Iacopo Barsotti, University of Pittsburgh and University of Southern California, introduced by Professor A. L. Whiteman.

A group-variety is an irreducible algebraic subvariety G of a projective space over a field k, such that there exists a proper subvariety F of G, and a rational law of composition $(P, Q)\rightarrow PQ$ which turns $G-F$ into a group. If F is empty, G is called an abelian variety (necessarily commuta-

tive). If G is commutative but not abelian, it contains a maximal rational group-subvariety V, and G/V is abelian; V is a projective space, and the law of composition on V, if k has characteristic zero, is obtained by addition of certain cartesian co-ordinates (as in vector spaces), and multiplication of others. If G is not commutative, and C is its center, G/C is a Vessiot variety (representative of an algebraic group of matrices); moreover, G contains a maximal Vessiot variety V; V is invariant, and G/V is abelian.

3. *The experimental mathematics program at Haverford College*, by Professor P. B. Johnson, Occidental College.

The mathematics program can not be divorced from the entire curriculum at Haverford, where emphasis is on thinking and doing significant things rather than transmitting the culture. The freshman program gives major attention to basic ideas which have proved to be the seeds of modern mathematical thinking. The usual topics are then covered rapidly. A traditional calculus course follows, but this will be revised. A distinguished mathematician is brought in to teach a special topic in modern mathematics from an undergraduate point of view. The rest of the program is traditional. Many elements of the program can be successful elsewhere.

4. *Soap films and minimal surfaces*, by Professor C. R. DePrima, California Institute of Technology.

For more than a decade, R. Courant has employed soap film experiments as a heuristic guide to the study of the general Plateau-Douglas problems and related minimal surface problems. In this lecture the general phenomena of non-uniqueness, discontinuous dependence and instability associated with these minimal surface problems were demonstrated with soap film experiments.

5. *Report on the Los Angeles Mathematics Newsletter*, by Mr. Samuel Skolnik, Los Angeles City College.

The Los Angeles Mathematics Newsletter was initiated at Los Angeles City College in the Fall of 1953. This Newsletter is devoted to the interests of high school students of mathematics. Each senior high school in Los Angeles gets a sufficient number of copies to give to each student in the advanced mathematics classes (beyond plane geometry). During 1953–54 two issues were published. Three issues will be published during the current academic year. The readers of this abstract are invited to send in articles and problems for publication in the Newsletter. The articles should be 100 to 500 words in length on any topic suitable to a good high school mathematics major. Of course, the subject matter need not be new or original.

6. *The minimal cut theorem*, by Dr. L. R. Ford, Jr., RAND Corporation.

In the theory of flows through networks, a cut is a set of arcs whose removal disconnects the source from the sink. The value of a cut is the sum of the capacities of its arcs. The minimal cut theorem, stating that the maximal flow value is equal to the minimal cut value, is proved. Menger's theorem (König, *Theorie der Graphen*, Chelsea, N. Y., 1950, p. 244) is derived as an immediate corollary. A computational procedure for certain types of planar graphs is discussed and illustrated. This paper is a report of the work of the speaker and D. R. Fulkerson.

7. *Change ringing*, by Professor R. J. Walker, Cornell University.

The traditional English method of ringing permutations on a set of church bells leads to several problems in group theory. For example: Let $a_1, \cdots, a_n$ be a set of elements of a finite group G of order n. Is it possible to obtain each element b_i of G by the recursion process $b_{i+1}=a_{j_i}b_i$, $i=1, \cdots, n-1$, $b_1=I$? How small can n be chosen, and how does one determine the sequence j_i? These questions also arise in the problem of generating finite groups in automatic calculating machines.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.
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THE METHOD OF ARCHIMEDES*

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The works of Archimedes have come down to us in two streams of tradition, one of them continuous, the other broken by a gap of a thousand years between the tenth century and the year 1906, when the discovery of a manuscript in Constantinople brought to light an important work called the *Method*, on the subject of integration.

Newton and his contemporaries in the seventeenth century were much puzzled by one aspect of the integrations to be found in the continuous tradition. In the books on the *Sphere and Cylinder*, for example, it is clear that the somewhat complicated method employed there for finding the volume of a sphere represents merely a rigorous proof of the correctness of the result and gives no indication how Archimedes was led to it originally. The discovery of 1906 removes the veil, at least to some extent.

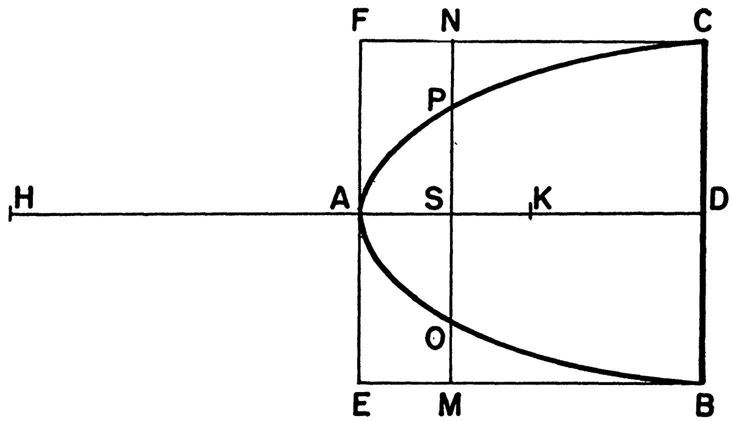


FIG. 1

The newly discovered *Method* consists of imagining the desired volume as cut up into a very large number of thin parallel slices or discs, which are then suspended at one end of an imaginary lever in such a way that they are in equilibrium with a solid whose volume and center of gravity are known. Thus, in Proposition 4 of the *Method*, Archimedes shows that the volume of a paraboloid of revolution is one-half of the volume of the circumscribing cylinder by slicing the two solids (see Figure 1 which represents a plane section through their common axis AD) at right angles to AD. For let us take HAD to be the bar of a balance with $HA=AD$ and with the fulcrum at A, and imagine the circle PO to be removed from the paraboloid and suspended at H. Since $AD/AS=DB^2/SO^2$ in the parabola BAC, we have

* An address to the Mathematical Association of America at the 1953 Summer Meeting in Kingston, Ontario, Canada.

$$\frac{HA}{AS} = \frac{AD}{AS} = \frac{MS^2}{SO^2} = \frac{\text{(circle in cylinder)}}{\text{(circle in paraboloid)}},$$

so that, by the law of the lever, the circle in the cylinder, remaining where it is, is in equilibrium with the circle from the paraboloid resting in its new position. If we deal in the same way with all the circles making up the paraboloid, we find that the cylinder, resting where it is with its center of gravity at the midpoint K of AD, is in equilibrium about A with the paraboloid placed with its center of gravity at H. Since $HA = AD = 2AK$, the volume of the paraboloid is therefore one-half of that of the cylinder, as desired.

Many accounts of the *Method* have been given since its discovery in 1906; for example, by T. L. Heath in his *Supplement to the Works of Archimedes*, Cambridge, 1912. In all of them, as in the original work of Archimedes himself, we are invited to *imagine* the lever and the objects suspended from it. But if we construct an *actual* lever and *actual* discs, the various figures, which may be spheres, cones, *etc.*, see below, will be observed to balance, slice by slice, as successive slices are added. The whole procedure then becomes a picturesque and effective illustration of the concept of an integral as the limit of a sum.

To find the volume of a sphere, a problem which Archimedes considered so important that he asked to have the result engraved on his tombstone, a cone and a sphere are together weighed against a cylinder (see Figure 2 and the accompanying sketch). Here the circle NM, resting where it is in the large cylinder $GLEF$ is in equilibrium about A with two circles placed at H, the one circle PO being taken from the given sphere and the other RQ from the cone FAE. For we have

$$OS^2 + QS^2 = OS^2 + AS^2 = AO^2 = CA \cdot AS = MS \cdot SQ$$

and therefore

$$\frac{HA}{AS} = \frac{MS}{SQ} = \frac{MS^2}{MS \cdot SQ} = \frac{MS^2}{OS^2 + QS^2}.$$

Thus, by the law of the lever as before,

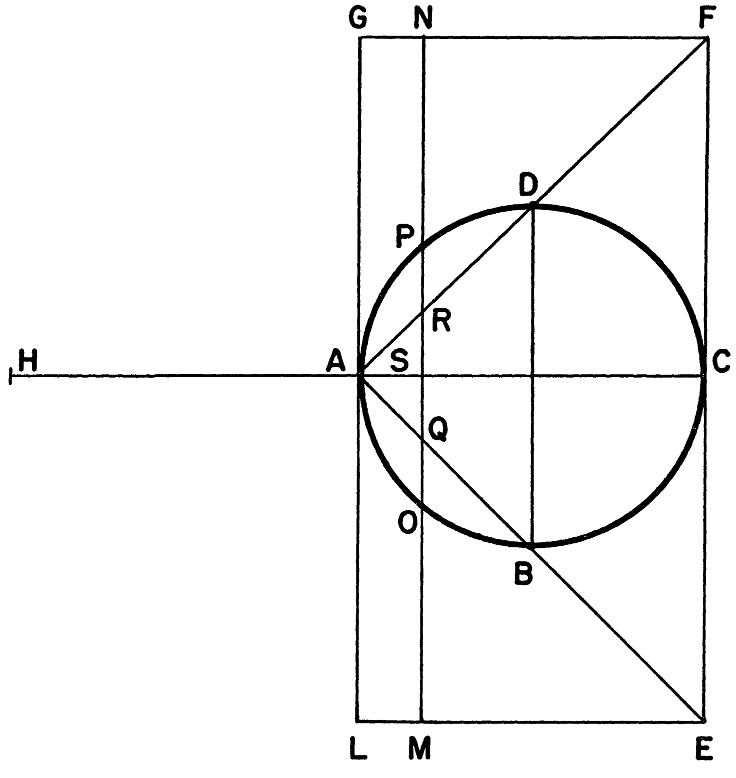
one-half of cylinder equals cone plus sphere

from which, since the cone is one-third of the cylinder,

sphere equals one-sixth cylinder.

Thus the cylinder circumscribed about the sphere, being one-quarter as great as the large cylinder $GLEF$, is three-halves as great as the sphere, which is the result stated on the tombstone of Archimedes.

If squares are substituted for the circles of cross-section in these figures, the argument remains unchanged and we have the solution of another famous problem (Proposition 15 in the *Method*), namely to find the volume common to



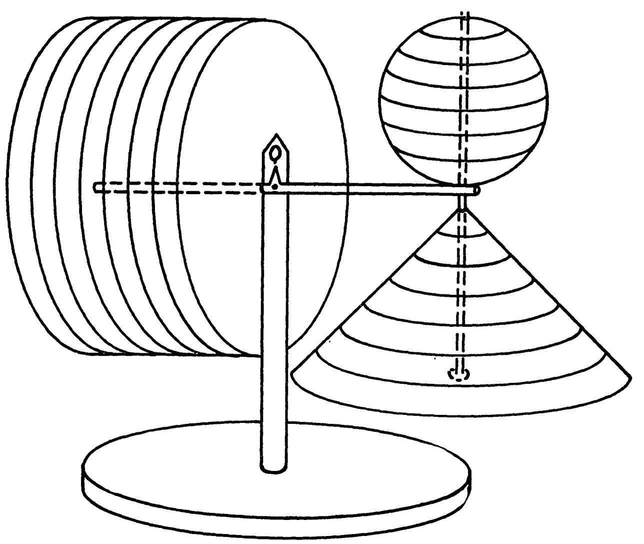


FIG. 2

two right circular cylinders intersecting at right angles.

The actual models were constructed by D. A. Eberle of the Psychology Workshop at Purdue University. The various slices were cut from a piece of white pine 1/2″ thick and 7″ wide. Thus the cylinder *GLEF* is composed of seven slices, each with a diameter of 7″. The seven slices for the cone, being first cut as stepwise increasing cylindrical discs with easily calculated radii, were placed all together on a mandrel passing through a 3/16″ hole through their centers and were then shaped down on a lathe, a procedure found to be especially necessary for the square cross-sections in the problem of the intersecting cylinders. The lever itself is a piece of steel 9″ by 1/2″ by 1/32″, placed so that its 1/2″ face is vertical. In each disc a thin slit was cut with a fine hacksaw from edge to center so that the disc could be slipped onto the lever.

SIMPLE DEVICES FOR EFFICIENCY IN THE ELEMENTARY THEORY OF EQUATIONS

FLORA DINKINES, University of Illinois, Chicago

1. Introduction. The instructor who attempts to teach his students to find the "best" method for solving each problem frequently finds that the process of producing several solutions, from which the "best" is to be chosen, tends to stimulate ingenuity and produce a habit of efficiency in most students. Not infrequently, it is necessary to consider whether the author has indicated the "best" procedure in the directions given, or whether the ultimate aim—such as finding all of the roots of a given equation—can be more readily attained by disregarding the author's directions involving some specific method—such as testing for rational roots. In such cases shall we disregard the directions and obtain the results more efficiently; shall we consider the problem satisfactorily worked only when we have followed the author's directions; or shall we work the problem efficiently and then follow the author's instructions? The answer probably depends upon how frequently such directions are given, and whether or not there are enough problems remaining to teach the students the desired standard techniques.

We shall consider here the "best" methods for working some of the college algebra problems ordinarily contained in the chapter on theory of equations, and the frequency with which these methods can be applied to problems in certain text books.*

Before entering upon the main points of discussion the author would like to say that no special method was used in selecting the books referred to. Those at hand were merely used to decide whether or not the devices to be given here

* These books are listed at the end of this article and referred to by number.

could be applied frequently enough to make their consideration worthwhile.

It is obvious that a special device, even though it is a great time saver, should be very easy to teach, or should be frequently applicable, in order to merit its being taught in preference to, or in addition to, the standard or more general procedures. In view of this the author believes that two devices here presented should be considered as standard techniques in any treatment, however brief, of the chapter on theory of equations presented in the usual college algebra course.

2. The two main devices. Throughout this paper we shall confine ourselves to polynomial equations with real coefficients. Usually the coefficients will be integers. Such a polynomial we shall represent in the usual manner by

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.$$

The first device arises from the observation that if $\sqrt{a}$ is a root of such a polynomial equation with integral coefficients it must be a root of the polynomial consisting of the even powered terms only, and also a root of the polynomial consisting of the odd powered terms only. This follows from the fact that the substitution of $\sqrt{a}$ into a term produces a multiple of $\sqrt{a}$ if the term is of odd power and an integer if it is of even power. This suggests grouping the odd powers and the even powers and looking for common factors. This simple method frequently makes the equation at hand appear rather transparent, and it is interesting to find that it frequently reveals factors other than those of the type x^2-a, where a is not a perfect square.

The following examples indicate the usefulness of this type of factoring. (We shall consider the problem completed when we have reduced it to the solution of one or more quadratic equations.)

Example 1. [7], p. 221, no. 13. The instructions are to solve the equation given that i and $-1+3i$ are roots. We ignore the instructions and regroup the equation into odd and even powers.

$$\begin{aligned} x^6 + 2x^5 + 10x^4 - x^2 - 2x - 10 &= x^6 + 10x^4 - x^2 - 10 + 2x(x^4 - 1) \\ &= (x^2 + 10)(x^4 - 1) + 2x(x^4 - 1) \\ &= (x^4 - 1)(x^2 + 2x + 10) = 0, \\ x &= \pm 1, \quad \pm i, \quad -1 \pm 3i. \end{aligned}$$

Note that this equation can be completely solved without using the information given or any information on rational roots.

Example 2. [7], p. 200, no. 17. The instructions are to find all rational roots.

$$9x^4 - 30x^3 + 16x^2 + 30x - 25 = 0.$$

$$\begin{aligned} 9x^4 + 16x^2 - 25 - 30x^3 + 30x &= (9x^2 + 25)(x^2 - 1) - 30x(x^2 - 1) \\ &= (9x^2 - 30x + 25)(x^2 - 1) = 0. \\ x &= 5/3, \quad 5/3, \quad \pm 1. \end{aligned}$$

All roots can be found without using the test for rational roots.

Example 3. [2], p. 376, no. 13. The instructions are to find all roots. As usual, it is assumed that the student must use the test for rational roots to reduce this to a quadratic.

$$\begin{aligned}15y^4 + 4y^3 + 56y^2 + 16y - 16 &= 15y^4 + 56y^2 - 16 + 4y^3 + 16y\\ &= (y^2 + 4)(15y^2 - 4) + 4y(y^2 + 4)\\ &= (15y^2 + 4y - 4)(y^2 + 4) = 0.\end{aligned}$$

Examples 4–7. [6], p. 247, nos. 1, 4, 6, 7. Here the instructions are to approximate a specific root to the nearest hundredth. Each of these is readily factored into the product of two quadratics, by this first device, and *all* roots can readily be obtained.

Example 8. [5], p. 262, no. 8. The instructions are to find the rational roots. There are none. This method of factoring quickly reveals that there are none, and also produces *all* of the roots. No synthetic division is necessary.

$$x^5 + 5x^3 - 7x^2 - 35 = 0.$$

$$\begin{aligned}x^3(x^2 + 5) - 7(x^2 + 5) &= (x^3 - 7)(x^2 + 5)\\ &= (x - \sqrt[3]{7})(x^2 + \sqrt[3]{7}x + \sqrt[3]{49})(x^2 + 5)\end{aligned}$$

Example 9. [5], p. 269, no. 8. The instructions are to compute, correct to three decimal places, the larger of the two irrational roots of this equation. $x^4-4x^3+5x^2-16x+4=0$. But why compute when we can factor the equation into two quadratics and thus obtain the exact answers?

$$\begin{aligned}x^4 + 5x^2 + 4 - 4x^3 - 16x &= (x^2 + 4)(x^2 + 1) - 4x(x^2 + 4)\\ &= (x^2 + 4)(x^2 - 4x + 1) = 0.\end{aligned}$$

Also, why compute for [5], p. 276, nos. 21, 22?

Example 10. [1], p. 324, no. 18. Here the instructions are to find all rational roots. If the depressed equation is a quadratic, find all roots.

$$x^5 - 10x^4 + 32x^3 - 38x^2 + 31x - 28 = 0.$$

One readily finds that 4 is the only rational root. This is the answer given by the author. However, since the depressed equation is $x^4-6x^3+8x^2-6x+7=0$, which factors by this first device into $(x^2+1)(x^2-6x+7)=0$, *all* roots are readily obtained.

Example 11. [1], p. 336, Illustrative Example. Here the author explains how to apply Ferrari's solution of the quartic to the equation

$$x^4 - 4x^3 + 6x^2 - 4x + 5 = 0,$$

which factors by this first device into $(x^2+1)(x^2-4x+5)=0$. Example 10 on this page also factors readily. However, it should be admitted, in all fairness to the author, that it is frequently desirable to teach the solution of the quartic by

means of a simple example, but is it not imperative that we not mislead the student into thinking that there is no simpler method of working this specific problem?

Examples 12–14. [4], p. 242, Examples 1 and 2; p. 244, Example 1.

Here the author has chosen equations which could be factored by this first device to use in explaining methods for finding rational roots, or all of the roots.

$$x^4 - 6x^3 + 3x^2 + 24x - 28 = (x^2 - 4)(x^2 - 6x + 7).$$
$$3x^3 + 2x^2 - 3x - 2 = (3x + 2)(x^2 - 1).$$
$$64x^3 - 16x^2 + 12x - 3 = (4x - 1)(16x^2 + 3).$$

Example 15. As a last example we point out that an equation like

$$x^6 - x^5 + 4x^4 - 2x^3 - 31x^2 + 35x - 70 = 0,$$

which has no rational roots, is readily solved by this method. Regrouping and factoring the odd powered terms, we have $x^6+4x^4-31x^2-70-x(x^4+2x^2-35)$. In factoring the even powered terms we merely test to see if x^4+2x^2-35 is a factor. No division is necessary since we observe that no odd powers can be obtained in the multiplication of the factors; therefore the second factor must be a binomial, which is obviously x^2+2. After checking the multiplication we have

$$\begin{aligned}&(x^2 + 2)(x^4 + 2x^2 - 35) - x(x^2 + 2x^2 - 35)\\ &\qquad = (x^2 - x + 2)(x^4 + 2x^2 - 35) = (x^2 - x + 2)(x^2 + 7)(x^2 - 5) = 0.\end{aligned}$$

$$x = \pm i\sqrt{7}, \qquad x = \pm\sqrt{5}, \qquad x = \frac{-1 \pm i\sqrt{7}}{2}\cdot$$

With a little effort the author was able to find the following problems in the listed text books for which this method will give a complete solution or will aid in the solution. Although these books have not been thoroughly checked, this should indicate that there are many problems for which we are teaching unnecessarily difficult methods of solution, or stated differently, there are many for which we are not teaching the most efficient methods of solution.

Problems to which the first device can be applied.

[1], p. 324, nos. 16, 18; p. 336, Example 1, nos. 7, 10.

[2], p. 376, nos. 13, 15.

[3], p. 46, no. 2; p. 51, nos 1, 2; p. 52, nos. 1, 4.

[4], p. 243, no. 7; p. 242, Examples 1, 2; p. 244, Example 1, no. 15; p. 248, no. 4; p. 257, nos. 8, 9, 10.

[5], p. 246, no. 20; p. 252, no. 39; p. 254, Example 2; p. 258, nos. 25, 26, 28; p. 262, nos. 1, 6, 8, 12, 13, 15, 16, 17, 19, 20, 21, 22; p. 269, no. 8; p. 276, nos. 21, 22.

[6], p. 221, nos. 31, 33, 34; p. 223, nos. 5, 9, 10; p. 226, no. 29; p. 229, nos. 11, 12, 13, 14, 15, 27; p. 234, nos. 4, 7, 11, 13, 15, 19, 20, 21, 22, 24, 27, 33, 34; p. 239, nos. 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16; p. 244, nos.

27, 28; p. 247, nos. 1, 4, 6, 7; p. 255, nos. 17, 18, 19, 20; p. 260, no. 2.
[7], p. 200, no. 17; p. 221, no. 13.
[8], p. 63, no. 1(g).

The second device is merely a way of factoring a quartic equation with integral coefficients into the product of two quadratic factors with integral coefficients. Assume that $ax^4+bx^3+cx^2+dx+e=0$ can be factored into

$$(mx^2 + kx + r)(nx^2 + tx + s) = 0.$$

Then

$$ax^4 + bx^3 + cx^2 + dx + e$$
$$= mnx^4 + (mt + nk)x^3 + (tk + nr + ms)x^2 + (rt + sk)x + rs.$$

If now, we choose two integers m and n whose product is a, and two integers r and s whose product is e, we can use these values in the linear equations obtained by equating the coefficients of x^3 and x. Thus we seek integers t and k satisfying

$$(1) \qquad \begin{cases} mt + nk = b \\ rt + sk = d. \end{cases}$$

If integral values can be found which also check in the equation

$$(2) \qquad tk + nr + ms = c,$$

obtained by equating the coefficients of the second degree terms, then the quartic has the factors mx^2+kx+r and nx^2+tx+s, as desired.

This method is especially useful when $a=1$, for then the equations reduce to

$$(3) \qquad \begin{cases} t + k = b \\ rt + sk = d \end{cases}$$

and

$$(4) \qquad tk + r + s = c,$$

and one can try all possible integral factorizations of e.

Some examples of the use of this second device are given here.

Examples 1–6. [1], p. 331, nos. 7, 8, 9, 10, 16, 18.

For nos. 7 and 8 the instructions are to find specified roots correct to three decimal places. For no. 7, $x^4+2x^3-8x^2-6x-1=0$, the roots indicated are between 2 and 3, and between -3 and -4.

We ignore the author's directions and choose $r=-1$, $s=1$. Then

$$\begin{cases} t + k = 2 \\ -t + k = -6, \end{cases}$$

and we obtain $k=-2$, $t=4$. These values check $tk+r+s=-8$, and the quadratic factors are x^2-2x-1 and x^2+4x+1. From these we can get the answers in

exact form, or their decimal approximations, if these are needed.

For no. 8, $x^4-6x^2-12x-8=0$, the roots indicated are between 3 and 4, and between -1 and -2. Since $b=0$, (3) becomes

$$\begin{cases} t + k = 0 \\ rt + sk = -12, \end{cases}$$

from which we obtain $t=-k$ and $k=12/(r-s)$. From (4) we have that $k^2=r+s+6$, for which it is easy to see that $r=-4$, and $s=2$ are integral solutions. Then the sign of k is given by $k=12/(r-s)$, and the factors are x^2-2x-4 and x^2+2x+2. (The same type of solution readily produces the factors of [6], p. 239, no. 10: $x^4+17x^2-30x-144=0$).

The instructions for nos. 16 and 18 are to find all real roots exactly or correctly to two decimal places.

No. 16, $2x^4-5x^3-33x^2-4x+40=0$, can be broken into the factors $2x^2+3x-5$ and x^2-4x-8, by this second device, and the four real roots can be obtained without using any method of approximation.

No. 18, $2x^5+x^4-16x^3+27x^2-14x-6=0$, cannot be easily factored. We use the usual procedure for rational roots to locate the root 3/2. The depressed equation $x^4+2x^3-5x^2+6x+2=0$ readily factors into x^2-2x+2 and x^2+4x+1 by the second device. Thus again the longer methods of approximation have been avoided.

Example 7. [5], p. 276, no. 26 can be readily factored by this method. We shall not repeat the author's lengthy instructions.

$$x^4 - 9x^3 + 14x^2 + 13x + 23 = (x^2 - 10x + 23)(x^2 + x + 1) = 0.$$

Examples 8–17. [7], p. 229, nos. 1–10. Here the instructions are to solve the quartic equations by the method outlined in the test. This is Ferrari's solution of the quartic, though the author doesn't say that. Every one of these problems can be solved by this second device and some of them very easily, such as number 2: $x^4+6x^3-8x^2-22x-105=(x^2+4x-21)(x^2+2x+5)=0$.

Example 18. [5], p. 281, no. 18. By means of this device the resulting equation $27x^4-45x^3-15x^2+24x+16=0$ can be factored into

$$(9x^2 - 24x + 16)(3x^2 + 3x + 1) = 0.$$

The frequency with which the second device can be applied is indicated by the following set of problems. Again the author wishes to indicate that the supply has not been exhausted.

Problems to which the second device can be applied.

[1], p. 321, nos. 13, 14, 16; p. 323, nos. 3, 4, 5, 6, 11, 14; p. 331, nos. 7, 8, 9, 10, 15, 16, 18.

[2], p. 376, nos. 8, 9; p. 378, Example 1.

[3], p. 46A (quartics), nos. 1, 2, 3, 4, 5, 6; p. 46B, no. 55; p. 51, nos. 3, 4, 5.

[4], p. 243, no. 8; p. 257, nos. 6, 7.

[5], p. 246, no. 23; p. 251, nos. 32, 35; p. 258, no. 27; p. 263, no. 23; p. 275, no. 5; p. 276, nos. 23, 24, 25, 26; p. 281, nos. 4, 9, 10, 11, 12, 13, 14, 15, 18.
[6], p. 223, nos. 6, 7, 8; p. 226, nos. 30, 31; p. 230, no. 28; p. 234, nos. 5, 6, 9, 14, 31, 32, 35, 36; p. 239, no. 10; p. 255, nos. 25, 26; p. 262, no. 4.
[7], p. 200, nos. 12, 15, 26; p. 204, nos. 6, 15; p. 213, no. 20; p. 229, nos. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
[8], p. 63, nos. 1(c), (d).

3. Miscellaneous devices. The following devices are helpful though they are not as frequently applicable as the two main devices.

(a) Factoring by grouping terms other than even and odd powered terms.
[3], p. 46, no. 2. $x^3+6x^2+3x+18=x^2(x+6)+3(x+6)=(x^2+3)(x+6)$.
[5], p. 262, nos. 3, 7; p. 246, no. 18; p. 286, nos. 4, 7.
[6], p. 221, no. 32; p. 234, no. 8.
[7], p. 200, no. 3.
[8], p. 63, no. 1(a).

(b) Completing the square on ax^4+bx^3.
[1], p. 331, no. 9; p. 336, no. 7.
[6], p. 255, no. 25. (We explain this problem since it saves the reader the task of finding 4 real roots by Horner's method.)

$$x^4 - 6x^3 + 8x^2 + 2x - 1 = 0$$

$$x^4 - 6x^3 + 9x^2 = 9x^2 - 8x^2 - 2x + 1$$

$$(x^2 - 3x)^2 = (x - 1)^2, \qquad x^2 - 3x \pm (x - 1) = 0,$$

$$x^2 - 2x - 1 = 0, \qquad x^2 - 4x + 1 = 0.$$

(c) Factoring as the difference between two squares.
[1], p. 336, no. 8.
[3], p. 46B, no. 32; $x^4=\pm(4x+1)^2$.
[5], p. 258, no. 29; $x^2=\pm(2x-3)$. p. 263, no. 27; $(x^2-7)^2=57$. p. 275, no. 6; $(x^2+4x)^2=25$.
[8], p. 93, no. 1(d). $x^2=\pm\sqrt{2}(x-3)$.

(d) Factoring of the type $(ax^2+bx)^2+c(ax^2+bx)+d$.
[1], p. 324, no. 14.

$$\begin{aligned} x^6 - 8x^5 + 25x^4 - 36x^3 + 20x^2 &= 0 \\ x^2(x^4 - 8x^3 + 16x^2 + 9x^2 - 36x + 20) &= 0 \\ x^2[(x^2 - 4x)^2 + 9(x^2 - 4x) + 20] &= 0 \\ x^2(x^2 - 4x + 5)(x^2 - 4x + 4) &= 0. \end{aligned}$$

(e) Factoring involving the sum or difference of two cubes.
[3], p. 46, no. 3;

$$y^3 - 2y + 4 = y^3 + 8 - 2y - 4 = (y + 2)(y^2 - 2y + 4) - 2(y + 2)$$
$$= (y + 2)(y^2 - 2y + 2).$$

[6], p. 223, no. 3;

$$x^3 + 4x^2 + 8x + 8 = x^3 + 8 + 4x^2 + 8x = (x + 2)(x^2 - 2x + 4) + 4x(x + 2)$$
$$= (x + 2)(x^2 + 2x + 4).$$

4. Suggestion to authors of college algebra texts. The author of this article suggests that the authors of college algebra books which are to be used as text books devote a paragraph near the beginning of the chapter on theory of equations to devices such as are discussed here. Then if the following exercises contain problems to which special devices apply the student can save time by applying them. Needless to say there are many problems which demand some method of approximating roots, or the method for finding rational roots. Is it not sound teaching to strive for efficiency?

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Editorial Note. This paper illustrates the artificiality of most of our instruction in the Theory of Equations. The solution of a polynomial equation with arbitrary real coefficients is usually a messy business which can be handled only by some method of approximation. Our textbook examples, however, are selected to have relatively "nice" answers. This gives the students a false impression of the true difficulties of the problem.

Because of the complications of an approximate solution, it is certainly desirable to look for linear or quadratic factors before using a method of approximation. The theorem on rational roots is designed to identify certain linear factors, and the devices mentioned by Professor Dinkines are helpful in finding certain quadratic factors. The student should be warned, however, that these methods are much more likely to be of use in textbook problems than in problems derived from scientific situations. The probability that a polynomial with integral coefficients will factor into polynomials of lower degree with integral coefficients is zero!

I believe that the authors of the textbooks mentioned above intended to

present exercises which illustrate the need for approximate methods. In constructing their equations, however, they seem to have worked backwards and to have formed their polynomials by multiplying together linear and quadratic factors. This made it easy for them to know the exact answers, which the students were expected to approximate by other methods. As a further hint to authors of textbooks, let me suggest that exercises intended for approximate solution be so constructed that the more elementary methods of solution mentioned above are not applicable.

C.B.A.

CORRECTION

LEE LORCH, Fisk University

In the article *The principal term in the asymptotic expansion of the Lebesgue constants*, this MONTHLY, vol. 61, 1954, pp. 245–249, the following correction should be made: on page 246, line 29, sin t should read $|\sin t|$.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NOTE ON THE RECURRING PERIOD OF THE RECIPROCAL OF AN ODD NUMBER

K. SUBBA RAO, Maharajah's College, Vizianagram

In this paper, I show how to obtain the recurring period of $1/n$, where n is any odd number and $(n, 5)=1$, by multiplication instead of by division. I also derive the form of n in order that $1/n$ may have the maximum recurring period $\phi(n)$, where $\phi(n)$ is Euler's totient function I further append a few results to Daljit Singh's results [1] concerning the reciprocal of a prime.

1. It has been shown by D. R. Kaprekar [2] that if p be any odd prime, other than 5, and a be the least integer such that $N(=ap+1)$ is divisible by 10, then the last $(p-1)$ digits in

$$\frac{N^{r+1}-1}{p} = a(1 + N + N^2 + \cdots + N^r), \qquad r \geqq p-1,$$

will represent a recurring period of the fraction $1/p$. These $(p-1)$ digits will

either be the actual recurring period of $1/p$ or will contain the recurring portion an integral number of times. a has the values 9, 3, 7 or 1 according as the prime ends in 1, 3, 7, or 9. The number a is called the first figure and the number $N/10$ is called the constant multiplier, and Kaprekar has shown how the recurring period of $1/p$ can be obtained by the "carrying forward" process. The same process can be applied to obtain the recurring period of $1/n$, where n is any odd number, not necessarily a prime, and $(n, 5)=1$. For, 10 is prime to n and if 10 belongs to the exponent e modulo n, 10^e-1 is divisible by n, that is, a number with e nines will be divisible by n and the number of digits in the recurring period of $1/n$ is e. Thus if n is of the form $(10m+1)$, $a=9$ and $N=90m+10$; the first number is 9, the constant multiplier is $(9m+1)$ and the "carrying forward" process gives the recurring period of $1/n$. Similarly when n is of the forms $(10m+3)$, $(10m+7)$ and $(10m+9)$, a takes the values 3, 7 and 1 respectively, and the constant multiplier has the values $(3m+1)$, $(7m+5)$ and $(m+1)$ respectively.

2. A. A. Krishnaswami Ayyangar [**3**] has given a set of conditions which must hold in order that q may be a prime with a maximum recurring period $(q-1)$ for $1/q$. I now discuss the corresponding problem when q is an odd number. I prove the following

THEOREM. *If n is an odd number with a maximum recurring period $\phi(n)$ for $1/n$, then n must be of the form p^m, where $m \geqq 1$ and p is a prime other than* 5.

Proof. From the foregoing discussion, we see that if e is the exponent to which 10 belongs modulo n, then the recurring period consists of e digits. If $e=\phi(n)$, then the recurring period is the maximum and it is equal to $(n-1)$, if n is an odd prime $(\neq 5)$. It follows that the recurring period of $1/n$ is a maximum when 10 is a primitive root of n. But it is known [**4**] that there exist primitive roots of a number only when it is 2, 4, $2p^m$ or p^m, where p is an odd prime. Since n is odd, it follows that n must be of the form p^m and the theorem is proved.

COROLLARY. *Every number of the form p^m (p an odd prime, other than* 5, *and $m \geqq 1$) which has* 10 *for a primitive root has the maximum recurring period.*

3. THEOREM. *There are infinitely many odd numbers whose reciprocals have the maximum recurring period.*

Proof. By the theorem in paragraph 2 above, every number of the form p^m (p an odd prime, other than 5, and $m \geqq 1$) which has 10 for a primitive root, has the maximum recurring period. It is known that if g is a primitive root of the odd prime p, and if the number $g^{p-1}-1$ is not divisible by p^2, then g is a primitive root of p^m, for any positive exponent m. If we now consider the least odd prime 7 which has 10 for a primitive root, it can easily be seen that 10^6-1 is not divisible by 49 $(=7^2)$, from which it follows that 10 is a primitive root of 7^m for any positive exponent m. Hence the theorem is proved.

4. It has been shown [**5**] that the recurring period of $1/109$ can be derived

from the first 108 terms of the Fibonacci series: 1, 1, 2, 3, 5, 8, $\cdots$ by multiplying them in order by 1, 10, 10^2, $\cdots$ and taking their sum up to 108 digits reckoning from the unit's digit. Since the number 109 is of the form $10m+9$, we have, from paragraph 1 above, $a=1$ and the constant multiplier is 11. By putting 1 as the first figure, multiplying it by 11 and applying the "carrying forward" process, we get all the 108 digits in the recurring period of 1/109. Kaprekar states [5] that "the recurring period of 1/109 can thus be built up from the terms of the Fibonacci series and it is worthwhile investigating the result of applying a similar method to the series of Lucas: 1, 3, 4, 7, 11, 18, 29, $\cdots$." It may be noted that in the recurring period of 1/109, the digits from the 101st onwards (reckoned from right to left) are the numbers of Lucas. In fact, the series of Lucas is also of the Fibonacci type and all types of Fibonacci series can be found in the recurring period of 1/109, if it is written twice. Thus we do not get anything new by applying the "carrying forward" process to Lucas numbers. It is highly remarkable that by applying this process to all types of Fibonacci series we are led to the recurring period of 1/109 and it is to be noted that 109 is the only prime having this property.

5. An appendix to Daljit Singh's results *Concerning the reciprocal of a prime* [1]: by the Theorem in paragraph 2 of this paper, it follows that for all numbers of the form p^m (p an odd prime $\neq 5$, $m \geqq 1$) with a maximum recurring period, Singh's results hold true. I give below an example to illustrate the properties stated in Singh's paper for numbers of the form $10n+9$ $(=p^m)$.

$$p^m = 7^2 = 49.$$

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$q(a)$	2	0	4	0	8	1	6	3	2	6	5	3	0	6
$r(a)$	2	20	4	40	8	31	16	13	32	26	15	3	30	6
a	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$q(a)$	1	2	2	4	4	8	9	7	9	5	9	1	8	3
$r(a)$	11	12	22	24	44	48	39	47	29	45	9	41	18	33
a	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$q(a)$	6	7	3	4	6	9	3	8	7	7	5	5	1	0
$r(a)$	36	17	23	34	46	19	43	38	37	27	25	5	1	10

In this particular case, there exist exactly 8 pairs $q(a)=r(a)=b$ where b takes all values from 1 to 9 except the value 7. This should be the case since 7 is not prime to 7^2. For all other values of $p(\neq 5)$ and if p^m is of the form $10n+9$ and has 10 for a primitive root, there exist exactly 9 pairs $q(a)=r(a)=b$ where $1 \leqq b \leqq 9$. By the Theorem in paragraph 3 above, it follows that there are infinitely many numbers of the form $(10n+9)$ having the property stated above.

6. Very little is known about those primes whose reciprocals have the maximum recurring period. Hardy and Wright in their *Introduction to the Theory of Numbers*, page 114, give the first six such primes. D. R. Kaprekar and A. A. Krishnaswami Ayyangar point out two more such primes [5], namely 109 and 487. Since all primes having 10 for a primitive root possess (by the foregoing discussion) this property, it follows from the Table of page 300 of Trygve Nagell's *Introduction to Number Theory* that, among the first 150 odd primes, there are 53 primes whose reciprocals have the maximum recurring period. It seems very likely that there are infinitely many odd primes having this property, though I am unable to prove it.

References

1. Daljit Singh, Concerning the reciprocal of a prime, this MONTHLY, vol. 61, 1954, pp. 32–34.
2. D. R. Kaprekar, Some properties of Demlo numbers and their applications to recurring decimals, The Math. Student, vol. 7, June, 1939.
3. Current Science, Sept. 1948.
4. L. E. Dickson, Introduction to the Theory of Numbers, p. 20.
5. D. R. Kaprekar, Demlofication of Fibonacci numbers, Bombay Univ. Jour., November, 1945. Also vide Demlo Numbers, pp. 75–82, published by D. R. Kaprekar, Khare House, Devlali Camp, India, 1948.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

NEWTON-COTES QUADRATURE FORMULAS

D. S. GREENSTEIN, University of Pennsylvania

In a recent note in this MONTHLY,* Morduchow pointed out the absence in the literature of a simple proof that the Newton-Cotes quadrature formula for $2n+1$ points gives exact results for polynomials of degree not exceeding $2n+1$. He also gave a proof which depends on Newton's forward interpolation formula.

The present note presents a simple proof depending only on the fundamental theorem of algebra. This proof might be presented to an elementary calculus class at the time Simpson's rule is discussed.

Let $f(x)$ be a polynomial of degree not exceeding $2n+1$. Let $p(x)$ be that polynomial of degree not exceeding $2n$ which interpolates $f(x)$ at the points

* Morduchow, Morris, A note on Newton-Cotes quadrature formulas, this MONTHLY, vol. 62, 1955, p. 33.

$x_0, x_0+h, x_0+2h, \cdots, x_0+2nh$ (*i.e.*, $f(x_0+ih)=p(x_0+ih)(i=0, \cdots, 2n)$). We wish to show that

$$\int_{x_0}^{x_0+2nh} f(x)dx = \int_{x_0}^{x_0+2nh} p(x)dx.$$

We shall prove this by showing that the integral between the same limits of the error polynomial $Q(x)=f(x)-p(x)$ is zero.

Since $Q(x)$ is at most of degree $2n+1$ and $Q(x_0+ih)=0$ $(i=0, \cdots, 2n)$, it follows that

$$Q(x) = C(x - x_0)(x - x_0 - h) \cdots (x - x_0 - 2nh).$$

Making the substitution $u=x-x_0-nh$, we find that

$$\int_{x_0}^{x_0+2nh} Q(x)dx = C\int_{-nh}^{nh} u(u^2 - h^2)(u^2 - 4h^2) \cdots (u^2 - n^2h^2)du,$$

which clearly vanishes, since we are integrating an odd function between symmetric limits.

ON THE CAUCHY CONVERGENCE CRITERION

BURNETT MEYER, University of Arizona

The following proof of the Cauchy convergence criterion seems somewhat simpler than those commonly in use in that Dedekind's Theorem is used directly. No use is made of the upper and lower limits or of the Bolzano-Weierstrass Theorem. It is assumed that a Cauchy sequence and the limit of a sequence have been defined.

THEOREM: *If* $\{x_n\}$ *is a Cauchy sequence of real numbers, then there is a real number* x *such that* $\lim_{n\to\infty} x_n=x$.

Proof. For every $\delta>0$, there is an N such that if $m, n>N$, $|x_m-x_n|<\delta/2$. Hence, for all $m>N$, $x_{N+1}-\delta/2<x_m<x_{N+1}+\delta/2$.

Consider the following partition of the real numbers: In class L place all real numbers y such that y is less than an infinite number of the x_n; in R are all y such that y is less than at most a finite number of the x_n. Since $(x_{N+1}-\delta/2)\in L$ and $(x_{N+1}+\delta/2)\in R$, neither L nor R is empty. Hence, the above partition is a cut which determines a real number x, the largest element of L or the smallest of R.

Since $(x-\delta/2)\in L$ and $(x+\delta/)2\in R$, there are an infinite number of the x_n in the open interval $(x-\delta/2, x+\delta/2)$. Let x_m be an element of the sequence in this interval with $m>N$, and let x_n be any element of the sequence with $n>N$. Then

$$| x - x_n | \leqq | x - x_m | + | x_n - x_m | < \delta/2 + \delta/2 = \delta,$$

so that $\lim_{n\to\infty} x_n=x$.

A SEQUENCE DEFINED BY A DIFFERENCE EQUATION

LOUIS BRAND, University of Cincinnati

The sequences defined by a difference equation of Riccati type have the most diverse and interesting properties, especially as regards the distribution of their cluster points. For this reason and in view of their relation to continued fractions, it is hoped that the following elementary treatment will be of service in supplying examples of how sequences behave.

The difference equation

$$(1) \qquad x_{n+1} = \frac{ax_n + b}{cx_n + d}, \qquad c \neq 0, \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

may be reduced to the form

$$(2) \qquad y_{n+1} = \alpha - \frac{\beta}{y_n}, \qquad \alpha = \frac{a + d}{c}, \qquad \beta = \frac{D}{c^2},$$

by the substitution $x_n = y_n - d/c$. When y_1 is given, we shall examine the character of the sequence defined by (2). When $y_n = 0$, y_{n+1} is not defined; in this case we write $y_{n+1} = \infty$, $y_{n+2} = \alpha$, and shall call the sequence *convergent* if its terms from then on have but a single cluster point.

Evidently y_{n+1} can be written as a continued fraction with n β's:

$$(3) \qquad y_{n+1} = \alpha - \frac{\beta}{\alpha -}\,\frac{\beta}{\alpha -}\cdots\frac{\beta}{y_1}.$$

But instead of dealing with this continued fraction we prefer to reduce (2) to a linear difference equation. Putting $y_n = z_{n+1}/z_n$ in (2) we have

$$(4) \qquad z_{n+2} - \alpha z_{n+1} + \beta z_n = 0.$$

If k_1, k_2 are the roots of its characteristic quadratic

$$(5) \qquad k^2 - \alpha k + \beta = 0,$$

namely $\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$, the general solution of (4) is

$$(6) \qquad z_n = C_1 k_1^n + C_2 k_2^n \qquad \text{when } k_1 \neq k_2,$$

$$(7) \qquad z_n = (C_1 + C_2 n) k_1^n \qquad \text{when } k_1 = k_2.$$

Since $y_n = z_{n+1}/z_n$, y_n depends only on *one* essential constant, which is fixed by the value of y_1.

We shall confine our discussion of (2) to the case when α, β and y_1 are real; then $\{y_n\}$ is a real sequence. There are three cases to consider depending on the character of the roots k_1, k_2.

Case 1. $\alpha^2>4\beta$, k_1, k_2 *real and unequal.*

The general solution of (2) is, from (6),

$$y_n = \frac{C_1k_1^{n+1} + C_2k_2^{n+1}}{C_1k_1^n + C_2k_2^n}\,. \tag{8}$$

When $C_1=0$, $y_n=k_2$ for all n; and if $C_2=0$, $y_n=k_1$ for all n. When $y_1\neq k_1$ or k_2, neither constant is zero; and

$$y_n = k_2\frac{(k_1/k_2)^{n+1} + C}{(k_1/k_2)^n + C}, \qquad C\neq 0. \tag{9}$$

If $\alpha=0$ the roots of (5) are $\pm\sqrt{-\beta}$, $k_1/k_2=-1$ and (9) shows that $y_{n+2}=y_n$. The sequence thus consists of two terms y_1 and $y_2=-\beta/y_1$ repeated indefinitely.

If $\alpha\neq 0$, let $|k_2|>|k_1|$; then $|k_1/k_2|^n\to 0$ and $y_n\to k_2$. We thus have the

THEOREM. *If* $\alpha\neq 0$, $\alpha^2>4\beta$ *and* $y_1\neq k_1$, *then*

$$y_{n+1} = \alpha - \frac{\beta}{y_n} \to k_2$$

the root of $k^2-\alpha k+\beta=0$ *of greater absolute value. Moreover if* $\alpha\neq k_1$, *the infinite continued fraction*

$$\alpha - \frac{\beta}{\alpha-}\,\frac{\beta}{\alpha-}\cdots \to k_2. \tag{10}$$

For example, with $\alpha=1$, $\beta=-1$,

$$y_n \to \tfrac{1}{2}(1+\sqrt{5}) = 1 + \frac{1}{1+}\,\frac{1}{1+}\cdots$$

if $y_1\neq\frac{1}{2}(1-\sqrt{5})$.

Case 2. $\alpha^2=4\beta$, $k_1=k_2=\frac{1}{2}\alpha$.

The general solution of (2) is, from (7),

$$y_n = k_1\frac{C_1 + C_2(n+1)}{C_1 + C_2 n}\,. \tag{11}$$

Evidently $y_n\to k_1$ irrespective of the value of y_1 and the continued fraction (10) converges to $\alpha/2$.

For example, with $\alpha=\beta=4$,

$$y_{n+1} = 4 - \frac{4}{y_n} \to 2.$$

Case 3. $\alpha^2<4\beta$, k_1, k_2 *conjugate complex.*

The general solution of (2) is still given by (8); and if we write

$$k_1 = r(\cos\theta + i\sin\theta), \qquad k_2 = r(\cos\theta - i\sin\theta),$$

it takes the form

$$y_n = r\,\frac{C_1\cos(n+1)\theta + C_2\sin(n+1)\theta}{C_1\cos n\theta + C_2\sin n\theta}. \tag{12}$$

From (5) we have

$$k_1k_2 = r^2 = \beta, \qquad k_1 + k_2 = 2r\cos\theta = \alpha;$$

hence we may take

$$r = \sqrt{\beta}, \qquad \theta = \cos^{-1}\frac{\alpha}{2\sqrt{\beta}} \qquad (0 < \theta < \pi). \tag{13}$$

If we choose γ in the interval $-\frac{1}{2}\pi < \gamma \leqq \frac{1}{2}\pi$ so that

$$\frac{\cos\gamma}{C_1} = \frac{\sin\gamma}{C_2} \quad \text{or} \quad \gamma = \tan^{-1}\frac{C_2}{C_1},$$

(12) becomes

$$y_n = r\,\frac{\cos(n\theta + \theta - \gamma)}{\cos(n\theta - \gamma)} \tag{14}$$

in which γ plays the role of arbitrary constant. In particular when $\gamma = 0$ ($C_2 = 0$),

$$y_n = r\,\frac{\cos(n+1)\theta}{\cos n\theta}, \qquad y_1 = r\,\frac{\cos 2\theta}{\cos\theta} = \alpha - 2\,\frac{\beta}{\alpha};$$

and when $\gamma = \pi/2$ ($C_1 = 0$),

$$y_n = r\,\frac{\sin(n+1)\theta}{\sin n\theta}, \qquad y_1 = r\,\frac{\sin 2\theta}{\sin\theta} = \alpha.$$

For values of γ other than 0 or $\pi/2$ we write (14) in the form

$$y_n = r\cos\theta - r\sin\theta\tan(n\theta - \gamma) \tag{15}$$

and consider the subcases when θ/π is rational or irrational.

Subcase 3.1. *$\theta/\pi = p/q$, rational.*

We assume that p and q are coprime. Then since

$$\tan(q\theta + \theta - \gamma) = \tan(p\pi + \theta - \gamma) = \tan(\theta - \gamma),$$

we see that $y_{q+1} = y_1$. Thus the sequence contains only q distinct terms $y_1, y_2, \cdots, y_q$, which repeat indefinitely in this order. The sequence has q cluster points and tends to no limit.

For example, the sequences $\{y_n\}$ defined by

$$y_{n+1} = 1 - \frac{1}{y_n}, \qquad y_{n+1} = 2 - \frac{2}{y_n}, \qquad y_{n+1} = 3 - \frac{3}{y_n}$$

have the periods 3, 4, 6 respectively.

Subcase 3.2. *$\theta/\pi = \lambda$ irrational.*

All terms of the sequence are now distinct. For $y_m = y_n$ implies that

$$\tan(m\theta - \gamma) = \tan(n\theta - \gamma), \qquad (m - n)\theta = N\pi,$$

where N is an integer, and θ/π would be rational.

When $\theta/\pi = p/q$, $\{y_n\}$ has q cluster points. We now show that when θ/π is irrational, $\{y_n\}$ has every point of the real continuum as cluster point. To this end we appeal to

KRONECKER'S THEOREM. *If λ is irrational, the set of points $x_n = n\lambda - [n\lambda]$ is dense in the interval* (0, 1); *that is, all points of* (0, 1) *are cluster points of the set.**

Proof. With $\theta = \lambda\pi$, $\gamma = c\pi$,

$$\tan(n\theta - \gamma) = \tan(n\lambda - c)\pi = \tan(n\lambda - [n\lambda] - c)\pi = \tan(\xi_n - c)\pi$$

where the set $\xi_n = n\lambda - [n\lambda]$ is dense in the interval (0, 1) and c is a fixed number between $-\frac{1}{2}$ and $\frac{1}{2}$. Hence the point set $(\xi_n - c)\pi$ is dense in the interval $(-c\pi, (1-c)\pi)$; and since this interval is of length π, the set $x_n = \tan(\xi_n - c)\pi$ is dense in the interval $(-\infty, \infty)$.

Now $\theta \neq 0$ or $\pi/2$ and hence in (15) $\sin\theta$ and $\cos\theta$ are both nonzero. Thus the point set x_n generates another

$$y_n = r\cos\theta - r\sin\theta\, x_n$$

which is dense in the interval $(-\infty, \infty)$. In other words, every point of the real continuum is a cluster point of the sequence $\{y_n\}$.

We state these results in the

THEOREM. *When α, β are real, $\alpha^2 > 4\beta$, and $\theta = \cos^{-1}\alpha/2\sqrt{\beta}$, the sequence $\{y_n\}$ defined by*

$$y_{n+1} = \alpha - \frac{\beta}{y_n}, \qquad y_1 \text{ real},$$

has a finite or infinite number of cluster points according as θ/π is rational or irrational. More specifically,

(a) *when $\theta/\pi = p/q$ a rational fraction in its lowest terms, the sequence takes on just q distinct values $y_1, y_2, \cdots, y_q$, which are thereafter repeated in this order;*

(b) *when θ/π is irrational, the terms of the sequence form a dense set in the entire real continuum.*

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* Hardy and Wright, *The Theory of Numbers*, Oxford, 1938, Theorem 439, p. 364.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1176. *Proposed by J. L. Brenner, State College of Washington*

Given a, b, c, d real and $ad-bc=1$. Show that

$$Q = a^2 + b^2 + c^2 + d^2 + ac + bd \neq 0,\ 1,\ -1.$$

E 1177. *Proposed by W. R. Utz, University of Missouri*

Describe the three types of plane loci of points the product of whose distances from a point and a line is constant.

E 1178. *Proposed by A. J. Goldman, Princeton University*

Prove that there exists a positive constant c with the following property: If T is any triangle whose area exceeds c, then the product of the lengths of the sides of T is greater than the area of T. What is the best possible value of c?

E 1179. *Proposed by C. D. Olds, San Jose State College*

Let f be an operator such that $f(z)=(|z|+z)/2$, and define $f^2(z)=f\{f(z)\}, \cdots, f^n(z)=f\{f^{n-1}(z)\}$. Evaluate

$$\lim_{n\to\infty} f^n(i),$$

where $i=\sqrt{-1}$.

E 1180. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Determine the 2319th digit in the expansion of 1000!.

SOLUTIONS

Pythagorean Parallelopipeds

E 1146 [1955, 40]. *Proposed by P. B. Johnson, Occidental College*

Show that any rectangle whose edges and diagonal are measured in integers can be made the base of a rectangular parallelopiped whose three edges and main diagonal are measured in integers.

Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J. Let c be the diagonal of the rectangle. If c is odd, take $(c^2-1)/2$ as the integral height and the integral main diagonal will be $(c^2+1)/2$; if c is even, take $(c^2-4)/4$ as the integral height and the integral main diagonal will be $(c^2+4)/4$.

Also solved by W. B. Carver, R. L. Caskey, G. B. Charlesworth, R. J. Cormier, Hüseyin Demir, Monte Dernham, Fred Discepoli, E. I. Gale, Michael Goldberg, William Googe, Vern Hoggatt, R. T. Hood, J. M. Howell, A. R. Hyde, Blair Kinsman, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, L. V. Mead, T. F. Mulcrone, C. S. Ogilvy, Sidney Penner, Walter Penney, J. V. Pennington, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., J. R. Slagle, D. R. Sudborough, A. V. Sylwester, R. O. Virts, Chih-yi Wang, and the proposer. Late solution by F. W. Saunders.

The proposer pointed out that we have here an aid to the problem of finding easy teaching examples—in this case vectors whose direction cosines are rational. H. W. Becker called attention to generalizations of the problem appearing in Dickson's *History of the Theory of Numbers*, vol. II, p. 319 and p. 509, and *Mathematics Magazine*, vol. 28, 1955, p. 154. Ogilvy wondered if it is possible for either or both of the other face diagonals to be integers also. In this connection see E 525 [Feb. 1943].

A Set of Integral Triangles

E 1147 [1955, 40]. *Proposed by E. P. Starke, Rutgers University*

If $\cos \alpha$ is rational $(0<\alpha<\pi)$, prove there are infinitely many triangles with integer sides having α as one angle. In particular, given $\cos \alpha=r/s$, find a three-parameter solution for the sides a, b, c.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. By the law of cosines

$$s(c+a)(c-a) = b(2cr - bs).$$

This will be satisfied if

$$ms(c+a) = n(2cr - bs) \quad \text{and} \quad n(c-a) = mb,$$

or if

$$na + mb - nc = 0 \quad \text{and} \quad msa + nsb + (ms - 2nr)c = 0.$$

It follows that

$$a = ts(m^2+n^2) - 2tmnr, \quad b = 2tn(nr - ms), \quad c = ts(n^2 - m^2).$$

This problem has been solved previously by Züge, *Archiv Math. Phys.* (2) 17, 1900, 354. See Dickson, *History of the Theory of Numbers*, vol. II, p. 215.

Also solved by W. J. Blundon, W. B. Carver, Hüseyin Demir, Michael Goldberg, D. C. B. Marsh, W. V. Parker, Walter Penney, J. V. Pennington, R. J. Wisner, and the proposer.

Two Equiareal Triangles

E 1148 [1955, 40]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let a, b, c be arbitrary points on the sides BC, CA, AB of triangle ABC, and let A', B', C' be the reflections of A, B, C in the midpoints of the segments bc, ca, ab. Show that triangles abc and $A'B'C'$ have equal areas.

Solution by Hüseyin Demir, Zonguldak, Turkey. Let a', b', c' be the reflections of a, b, c in the midpoints of BC, CA, AB. Since, by a well known property, abc and $a'b'c'$ have equal areas, we shall prove that $a'b'c'$ and $A'B'C'$ have equal areas. From $\overrightarrow{aB'}=\overrightarrow{Bc}=\overrightarrow{c'A}$, $\overrightarrow{aC'}=\overrightarrow{Cb}=\overrightarrow{b'A}$ we get $b'c'=B'C'$. Similarly $c'a'=C'A'$, $a'b'=A'B'$, and triangles $a'b'c'$ and $A'B'C'$ are actually congruent.

Also solved by W. B. Carver, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, C. S. Ogilvy, C. F. Pinzka, Roscoe Woods, and the proposer.

Pinzka called attention to two similar results in R. A. Johnson, *Modern Geometry* (1929), p. 80. Carver, Hyde, Ogilvy, and Woods gave simple solutions using oblique coordinates.

Editorial Note. The above solution shows that triangles $a'b'c'$, $A'B'C'$ are not only congruent, but also homothetic. It follows that if a, b, c are collinear on a line L, then A', B', C' are also collinear on a line L' parallel to the reciprocal transversal of L. Consequently, if L is a Simson line of triangle ABC, then L and L' are perpendicular.

First Perfect Square After 2^n

E 1149 [1955, 40]. *Proposed by A. S. Gregory, University of Illinois*

For each $n=1, 2, \cdots$ find the least positive integer which when added to 2^n yields a perfect square.

Solution by Michael Goldberg, Washington, D. C. Let $2^n+c=a^2$. For the smallest positive c, a^2 must be taken as the next square larger than 2^n. Then

$$c = a^2 - 2^n = ([2^{n/2}] + 1)^2 - 2^n.$$

When $n=2k$, $c=(2^k+1)^2-2^{2k}=2^{k+1}+1$. When $n\neq 2k$, the expression does not simplify.

Also solved by Leon Bankoff, W. B. Carver, Fred Discepoli, I. A. Dodes, A. J. Goldman, A. R. Hyde, M. S. Klamkin, L. J. Lange, D. C. B. Marsh, P. A. Piza, Azriel Rosenfeld, C. M. Sandwick, Sr., D. R. Sudborough, A. V. Sylwester, K. B. Williams, and the proposer. Some of these solutions were not complete. Late solution by Hüseyin Demir.

The proposer considered the more general problem: Let A, B be fixed integers greater than 1. For each $n=1, 2, \cdots$ find the least positive integer which when added to A^n yields a perfect Bth power.

SOLUTIONS

An Improper Integral

4588 [1954, 350]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Let $n>1$. Evaluate

$$\int_0^\infty (\sqrt[n]{x^n+1} - x)dx.$$

Solution by R. A. Rosenbaum, Wesleyan University. The limit tests show that the given integral diverges for $n \leqq 2$ and converges for $n>2$. For $n>2$, call the value of the integral, I. Then integration by parts gives

$$2I = \int_0^\infty (x^n + 1)^{1/n-1}dx.$$

The substitution $x^n+1=1/y$ leads to the beta-function

$$I = \frac{1}{2n}\int_0^1 y^{-2/n}(1-y)^{1/n-1}dy = \frac{1}{2n}\,\mathrm{B}\left(1-\frac{2}{n}, \frac{1}{n}\right).$$

Also solved by Ranko Bojanić, Leonard Carlitz, R. V. Esperti, H. E. Fettis, Martin Kruskal, Viktors Linis, A. E. Livingston, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

Separable Metric Spaces

4589 [1954, 350]. *Proposed by Casper Goffman, Wayne University*

It is well known that the interval (0, 1) is the union of a set of the first category and a set of measure zero. Generalize this result to arbitrary separable metric spaces.

Solution by the Proposer. A generalization is furnished by the Theorem: *If S is a separable metric space, then $S=E\cup Z$, where E is of the first category in S and Z is zero dimensional in the Hausdorff sense.* This is the required generalization since every Hausdorff zero dimensional set in (0, 1) has Lebesgue measure zero.

To prove this theorem, let $a_1, a_2, \cdots, a_n, \cdots$ be dense in S. For every n and m let σ_{nm} be a sphere of center a_n and radius $(m\cdot 2^{nm})^{-1}$. Let

$$T_m = \bigcup_{n=1}^\infty \sigma_{nm}, \qquad T = \bigcap_{m=1}^\infty T_m.$$

(a) For every m, $S-T_m$ is nowhere dense in S, so that $S-T$ is of the first category.

(b) Let $\epsilon>0$. An easy calculation shows that for every $0<\eta<\epsilon$, T may be covered by a sequence of spheres $\sigma_1, \sigma_2, \cdots, \sigma_n, \cdots$ whose radii $r_1, r_2, \cdots,$

$r_n, \cdots$ are all less than η, such that $\sum_{n=1}^{\infty} r_n^{\epsilon} < \eta$, whence T is of Hausdorff ϵ-dimensional measure zero. In other words, T is a Hausdorff zero dimensional set.

Divisors of the Fermat Numbers

4590 (1954, 350). *Proposed by Paul Erdös, University of Notre Dame*

Fermat's conjecture that all numbers of the form

$$F_n = 2^{2^n} + 1$$

are prime was proved wrong by Euler. Show, however, that $\sum 1/d \to 0$ as $n \to \infty$, where d ranges over all the divisors of F_n except 1.

Solution by Ranko Bojanić, Mathematical Institute, Belgrade. Since

$$\sum_{d|n} \frac{1}{d} - 1 = \frac{\sigma(n)}{n} - 1,$$

where $\sigma(n)$ is the sum of all divisors of n, including 1 and n, the required result is equivalent to the statement that

$$\frac{\sigma(F_n)}{F_n} \to 1, \qquad n \to \infty. \tag{1}$$

We have first, if $n = p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$,

$$1 > \frac{n}{\sigma(n)} = \prod_{p|n} \frac{1 - p^{-1}}{1 - p^{-\alpha-1}} \geqq \prod_{p|n} (1 - p^{-1}). \tag{2}$$

Now let p denote a prime divisor of F_n $(2 < p \leqq F_n)$. Then p is of the form $2^{n+1}k + 1$ and hence

$$p > 2^n. \tag{3}$$

From (2) and (3) it follows that

$$1 > \frac{F_n}{\sigma(F_n)} \geqq (1 - 2^{-n})^{\omega},$$

where ω is the number of different prime factors of F_n. Since

$$F_n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{\omega}^{\alpha_{\omega}} \geqq 2^{(\alpha_1 + \alpha_2 + \cdots + \alpha_{\omega})n} \geqq 2^{\omega n},$$

we have

$$\omega n \leqq \frac{\log (2^{2^n} + 1)}{\log 2} = 2^n + \frac{\log (1 + 2^{-2^n})}{\log 2},$$

or $\omega \leqq 2^n/n + 1$. Hence

$$1 > \frac{F_n}{\sigma(F_n)} \geqq (1 - 2^{-n})^{1+2^n/n} \to 1, \qquad n \to \infty,$$

and so (1) is proved.

We observe finally that from (1) follows immediately

$$\frac{\phi(F_n)}{F_n} \to 1, \qquad n \to \infty,$$

where $\phi(n)$ is the number of integers not exceeding n and prime to n.

The same results hold also for the numbers of the form $M_p = 2^p - 1$, where p is a prime: we have

$$\frac{\sigma(M_p)}{M_p} \to 1, \qquad \frac{\phi(M_p)}{M_p} \to 1, \qquad p \to \infty.$$

Also solved by Armand Brumer, and Leo Moser and J. Lambek (jointly).

RECENT PUBLICATIONS

Edited by E. P. Vance, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Introduction to Modern Algebra and Matrix Theory. By R. A. Beaumont and R. W. Ball. New York, Rinehart and Company, Inc., 1954. 12+331 pages. $6.00.

This is a carefully written introduction to the concepts and techniques of modern algebra, with particular emphasis on the theory of matrices. The topics covered are the following: Chapter I: elementary aspects of matrices and determinants; Chapter II: groups of transformations; Chapter III: vector spaces; Chapter IV: elementary group theory; Chapter V: rings and fields; Chapter VI: polynomials and the algebraic theory of fields; Chapter VII: matrices with polynomial elements; Chapter VIII: canonical forms.

This book is designed mainly for the intelligent third and fourth year undergraduate, whose major interest is in mathematics, the physical sciences, or certain of the social sciences. Considerable emphasis is placed on rigor, and the average student will find more than enough material here to occupy his time and tax his ability. A number of exercises are included in an effort to make the abstractions more meaningful. The material has been arranged so that a one-semester course on the theory of matrices can be given independently. Such a

course would be based on Chapters I, II, III, VII, and VIII, and need not stress abstractions.

Matrix theory and abstract algebra have become an integral part of every well-rounded undergraduate mathematics curriculum. This has resulted in the publication in recent years of a number of worthwhile accounts. It is not our purpose here to compare each of the available texts with the one by Beaumont and Ball. The better ones will doubtless compete with each other, and anyone planning a course along these lines is urged to look at all of the available material with considerable care before selecting the book best suited to his particular needs.

H. J. RYSER
The Ohio State University

Universal Mathematics. By the 1954 Summer Writing Group.

Part I of this "book of experimental text materials" was published in September 1954 by the University of Kansas Book Store. Part II will be available in September 1955 as a publication of the Tulane University Book Store. The writing of these lithographed books has been an activity of the Association's Committee on the Undergraduate Mathematical Program. Persons wishing examination copies of either part should communicate with Professor G. B. Price, Mathematics Department, University of Kansas, Lawrence, Kansas.

C.B.A.

NEW BOOKS RECEIVED

Transform Calculus with an Introduction to Complex Variables. By E. J. Scott. New York, Harper and Brothers, Publishers, 1955. 8+330 pages. $7.50.

Numerical Methods. By A. D. Booth. New York, Academic Press, Inc., 1955. 7+195 pages. $6.00.

An Introduction to Stochastic Processes. By M. S. Bartlett. New York, Cambridge University Press, 1955. 14+312 pages. $6.50.

Binomial Coefficients. Edited by J. C. P. Miller. (Vol. 3 of the Royal Society Mathematical Tables). New York, American Branch of Cambridge University Press, 1954. 8+162 pages. $5.50.

Proceedings of the First Conference on Training Personnel for the Computing Machine Field. Edited by Arvid W. Jacobson. Detroit, Michigan, Wayne University Press, 1955. 104 pages. $5.00.

Advanced Mathematics for Engineers, Third Edition. By H. W. Reddick and F. H. Miller. New York, John Wiley and Sons, Inc., 1955. 14+548 pages. $6.50.

First Course in Algebra for Colleges. By L. J. Adams, New York, Henry Holt and Company, 1955. 6+217 pages. $3.00.

Integers and Theory of Numbers. By A. A. Fraenkel. New York, Scripta Mathematica, 1955. 102 pages. $2.75.

The Theory of Numbers. By B. W. Jones. New York, Rinehart and Company, 1955. 11+143 pages. $3.75.

College Algebra. By P. R. Rider. New York, The Macmillan Company, 1955. 14+397 pages. $4.00.

Slide Rule Operations. By D. R. Sudborough and H. W. Zeoli. Ann Arbor, Michigan, J. W. Edwards, Publisher, 1954. 76 pages. $1.50.

Business Mathematics. By J. A. Mira and George Hartmann. New York, D. Van Nostrand Company, 1955. 6+341 pages. $4.50.

Higher Transcendental Functions, Vol. 3. Sponsored by California Institute of Technology, The Bateman Project Staff. Edited by A. Erdelyi. New York, McGraw-Hill Book Company, 1955. 17+292 pages.

General Topology. By J. L. Kelley. New York, D. Van Nostrand Company, 1955. $8.75.

Plane Trigonometry. By C. R. Wylie, Jr., New York, McGraw-Hill Book Company, 1955. 381 pages. $4.00.

Parmi Les Belles Figures de la Geometrie dans l'espace. By Victor Thébault. Paris, Librairie Vuibert, 63, Boulevard Saint-Germain, 1953. 286 pages.

College Algebra, 4th Edition. By W. L. Hart. Boston, D. C. Heath and Company, 1953. 10+480 pages.

Curso de Analisis Matematico, Tomo III. By Cristobal de Losada y Puga. Lima, Peru, 1954, Imprenta Santa Maria, Calle de Santa Catalina, 661. 814 pages.

Tables of Sines and Cosines for Radian Arguments. By U. S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, 43. Washington, D. C., 1955. 278 pages. $3.00.

Analytic Geometry, 2nd Edition. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1955. 9+310 pages. $3.75.

Analytic Geometry. By N. H. McCoy and R. E. Johnson. New York, Rinehart and Company, Inc., 1955. 14+301 pages. $4.00.

Calculus. By W. L. Hart. Boston, D. C. Heath and Company, 1955. 13+626 pages. $5.50.

Transactions of Symposia in Applied Mathematics, Vol. 2. Symposium on Computing Mechanics, Statistics and Partial Differential Equations, held at the University of Chicago, Apr. 29–30, 1954. New York, Interscience Publishers, Inc., 1954. 216 pages. $5.00.

Plane Algebraic Curves. By E. J. F. Primrose. New York, St. Martin's Press, Inc., 103 Park Ave., New York 17, 1955. 7+111 pages. $3.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEMORIAL TO PROFESSOR E. L. POST

A memorial to the late Professor Emil L. Post is being established at The City College of New York. The form of the memorial will depend to some extent on the response to the project. Any who so desire may send contributions to Professor H. P. Wirth, Chairman of the Memorial Committee, or to Professor B. P. Gill, Treasurer, The City College, New York 31, New York.

PERSONAL ITEMS

Professor V. O. McBrien of the College of the Holy Cross was official delegate of the Association at the inauguration of President A. B. Bronwell of Worcester Polytechnic Institute on April 30, 1955.

Professor J. I. Tracy of Texas Christian University was the official representative of the Association at the inauguration of President W. M. Tate of Southern Methodist University on May 5, 1955.

Dr. R. C. Blanchfield, National Research Council Fellow at Massachusetts Institute of Technology, has been awarded a National Science Foundation Postdoctoral Fellowship for 1955–1956.

Dr. Claude Chevalley of Columbia University has been awarded a Guggenheim Fellowship.

Assistant Professor Edwin Halfar of the University of Nebraska was awarded a research grant-in-aid by the University Research Council for summer work.

Dr. W. C. Hamilton, formerly a graduate fellow at the Crellin Laboratory of Chemistry, California Institute of Technology, is now a National Science Foundation Post-doctoral Fellow at the Mathematical Institute, Oxford, England.

Professor M. Gweneth Humphreys, chairman of the Department of Mathematics of Randolph-Macon Woman's College, is a recipient of a Faculty Fellowship awarded by the Fund for the Advancement of Education. Dr. Humphreys, who is on sabbatical leave from the College during 1955–56, is studying at the University of British Columbia. Also, Dr. Humphreys plans to study through short visits the undergraduate programs in several colleges and universities.

Professor Marston Morse of the Institute for Advanced Study was awarded the honorary degree of Doctor of Science by the University of Maryland on March 25, 1955.

Professor Ivan Niven of the University of Oregon has been awarded a Faculty Fellowship by the Fund for the Advancement of Education and is on leave of absence at the University of California, Berkeley.

Professor H. B. Ribeiro of the University of Nebraska was awarded a research grant-in-aid by the University Research Council for summer work.

Dr. W. R. Wasow of the University of California at Los Angeles has been awarded a Fulbright fellowship and is on leave of absence in Rome.

Associate Professor G. W. Whitehead of Massachusetts Institute of Technology is on leave of absence for the year 1955–1956 on a Guggenheim Fellowship at Oxford University and at the University of Paris, France.

Brown University announces the following: Mr. Paul Slepian, formerly an assistant at the University, and Dr. Heini Halberstam of University College of the Southwest of England have been appointed to instructorships; Associate Professor W. S. Massey has been promoted to a professorship; Assistant Professor F. M. Stewart has been promoted to an associate professorship; Professor Bjarni Jonsson is on leave of absence for the year 1955–1956 and is spending the year at the University of California as Visiting Associate Professor.

Dartmouth College announces that it is inaugurating a new program for gifted mathematics students.

Fresno State College reports: Professor F. R. Morris, formerly chairman of the Department of Mathematics and head of the Physical Science Division, has retired; Associate Professor Roy Dubisch has been promoted to a professorship and appointed Chairman of the Department of Mathematics. Assistant Professors John Christopher of Pacific University and R. D. Stalley of Iowa State College have been appointed to instructorships.

Illinois Institute of Technology announces: Assistant Professor Alfonso Shimbel of the University of Chicago and Dr. C. A. Nicol of the University of Texas have been appointed to instructorships.

Knox College reports the following: Professor Rothwell Stephens, head of the Department of Mathematics, has been appointed Hitchcock Professor of Mathematics; Dr. W. C. Ross, Jr., of the State University of Iowa and Mr. Dwain Small, formerly a teacher at Richmond, Indiana, have been appointed to instructorships.

At Massachusetts Institute of Technology: Associate Professor A. P. Calderon of Ohio State University and the Institute for Advanced Study has been appointed to an associate professorship; Assistant Professor N. C. Ankeny of Johns Hopkins University and Dr. L. N. Howard, recently Higgins Lecturer at Princeton University, have been appointed to assistant professorships; Dr. A. P. Mattuck, National Science Foundation Fellow at Harvard University, and Dr. W. F. Reynolds, previously an instructor at the College of the Holy Cross and research fellow at Harvard University, have been appointed C. L. E. Moore Instructors; Dr. Hartley Rogers, Jr., formerly Benjamin Peirce Instructor at Harvard University, and Dr. J. J. Levin of Purdue University have been appointed Visiting Lecturers; Assistant Professor Kenkichi Iwasawa has been promoted to an associate professorship; Professor Raphael Salem has been appointed to a professorship at the Sorbonne, France; Professor Norbert Wiener

is on leave of absence for the year 1955–1956 and will spend most of the time in India.

At the University of Wisconsin, a conference entitled "The Computing Laboratory in the University" was held August 17–19, 1955.

Washington University announces: Associate Professor I. I. Hirschman, Jr., has been promoted to a professorship; Assistant Professor H. Margaret Elliott has been promoted to an associate professorship.

Wayne University reports the following: Associate Professor W. F. Eberlein of the University of Wisconsin has been appointed Visiting Professor for 1955–1956; Assistant Professor S. I. Goldberg of Lehigh University has been appointed to an assistant professorship; Mr. Robert Kuller, formerly an instructor at Dartmouth College, and Mrs. Patricia J. Wells, recently an assistant at Michigan State University, have been appointed to instructorships; Associate Professor Benjamin Epstein is on sabbatical leave for 1955–1956 and is at Stanford University; Associate Professor Y. W. Chen, while on leave of absence for the first semester, is at the Institute for Advanced Study; Miss Winifred Burroughs has been appointed Visiting Lecturer at Wheaton College, Massachusetts.

Mr. W. R. Allen, formerly a mathematician at the University of Chicago, Institute for Air Weapons Research, has a position as an associate at the Forrestal Research Center, Princeton University.

Assistant Professor F. J. Arena of North Dakota State College has been appointed to an assistant professorship at Canisius College.

Mr. P. H. Arnold, previously a staff member at the Sandia Corporation, Albuquerque, New Mexico, is now a computing analyst at North American Aviation, Columbus, Ohio.

Associate Professor R. W. Barnard of the University of Chicago has retired.

Associate Professor P. M. Batchelder of the University of Texas has retired.

Mr. R. E. Bayles, formerly an actuarial assistant for John Hancock Mutual Life Insurance Company, Boston, Massachusetts, is at the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. R. V. Benson has been appointed to an instructorship at Los Angeles City College.

Dr. W. J. Berger, recently an assistant at Carnegie Institute of Technology, has a position as a mathematician with R.C.A. Service Company, Patrick Air Force Base, Cocoa, Florida.

Assistant Professor R. R. Bernard of Yale University has been appointed to an associate professorship at Davidson College.

Professor R. W. Brink of the University of Minnesota is now Acting Assistant Dean of the College of Science, Literature and the Arts.

Dr. Eleazer Bromberg, previously assistant chief of the Computing Center, Institute of Mathematical Sciences, New York University, has been appointed to an associate professorship in the Graduate School of Arts and Sciences, New York University.

Assistant Professor J. L. Brown, Jr., of Pennsylvania State University has been promoted to an associate professorship.

Associate Professor R. H. Bruck of the University of Wisconsin has been promoted to a professorship.

Dr. R. J. Buehler, formerly a staff member of the Sandia Corporation, Albuquerque, New Mexico, is a project associate at the Naval Research Laboratory, University of Wisconsin.

Associate Professor Hobart Bushey of Hunter College has been promoted to a professorship.

Reverend A. E. Cahill, instructor at Belmont Abbey College, has been promoted to an assistant professorship.

Professor W. L. Carter of the Territorial College of Guam has been appointed to an assistant professorship at the University of Cincinnati.

Mr. J. T. Clausen, Jr., chief of Methods and Data Analysis, White Sands Proving Ground, Las Cruces, New Mexico, has accepted a position as an aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. J. W. Coy of Michigan State University has a position as an analytic statistician at White Sands Proving Ground, Las Cruces, New Mexico.

Assistant Professor Avron Douglis of New York University has been promoted to an associate professorship.

Professor W. L. Duren, Jr., of Tulane University has been appointed Professor of Mathematics and Dean of the College of Arts and Sciences, University of Virginia.

Dr. L. K. Durst of Rice Institute has been promoted to an assistant professorship.

Mr. J. S. Elston, associate actuary at Travelers Insurance Company, Hartford, Connecticut, has retired.

Mr. D. O. Etter, formerly a graduate student at Tulane University, has accepted a position as a chemist in the Control Laboratory, International Lubricant Corporation, New Orleans, Louisiana.

Dr. A. G. Fadell of the University of Buffalo has been promoted to an assistant professorship.

Mr. A. M. Fleishman, recently a mathematician at the United States Naval Proving Ground, Dahlgren, Virginia, has a position as an engineer for the Radio Corporation of America, Camden, New Jersey.

Mr. G. C. Francis, previously a lecturer at Columbia University, is now a mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Dr. José Gallego-Diaz of Escuela Especial de Ingenieros Agronomos, Madrid, Spain, has been named Professor of General Physics.

Dr. T. M. Gallie, Jr., formerly a research instructor at Duke University, is now a research engineer for Humble Oil and Refining Company, Houston Research Center, Texas.

Mr. S. I. Gass, formerly a mathematician with the Computation Division, Headquarters, United States Air Force, Washington, D. C., has accepted a position as an applied science representative with the I.B.M. Corporation, Washington, D. C.

Mr. A. L. Gilmore, Jr., previously a radar instructor at Keesler Air Force Base, Biloxi, Mississippi, has accepted a position as a mathematician at Air Proving Ground, Eglin Air Force Base, Florida.

Assistant Professor Seymour Ginsburg of the University of Miami has a position as a research analyst at Northrop Aircraft Corporation, Hawthorne, California.

Mr. M. L. Goldwater, recently an engineer for Librascope, Glendale, California, is now a research engineer for J. B. Rea, Santa Monica, California.

Mr. R. M. Gordon, formerly a mathematical analyst for Lockheed Aircraft Corporation, Burbank, California, is now an electronic applications specialist and instructor in programming with National Cash Register Company, Dayton, Ohio.

Mr. W. T. Gregorzak, previously a teaching assistant at Rutgers University, has been appointed Research Staff Assistant at Johns Hopkins University Radiation Laboratory, Baltimore, Maryland.

Mr. J. R. Hadley, formerly a manufacturing control manager for Richardson Company, Indianapolis, Indiana, is employed as a chief industrial engineer by Refrigeration Appliances, Chicago, Illinois.

Professor E. E. Haskins, head of the Department of Physics of Norwich University, has been appointed Chairman of the Department of Mathematics of Clarkson College of Technology.

Assistant Professor G. C. Helme of Pratt Institute has been promoted to an associate professorship.

Dr. P. S. Herwitz, previously a research associate for the Institute for Cooperative Research, Johns Hopkins University, has a position as a mathematician for the I.B.M. Corporation, Washington, D. C.

Associate Professor S. T. Hu of Tulane University has been appointed to a professorship at the University of Georgia.

Mr. J. H. Kaplan, previously a student at Temple University, is employed as a mathematician at Frankford Arsenal, Philadelphia, Pennsylvania.

Miss Carolyn E. Kappel, formerly a student at Carleton College, is a technical assistant at the Bell Telephone Laboratories, New York, New York.

Mr. Edgar Karst of Independence, Missouri, is now a computing analyst, Electronic Computer Section, Great Lakes Pipe Line Company, Kansas City, Missouri.

Dr. Maurice Kennedy, formerly a graduate student at California Institute of Technology, has been appointed Assistant Lecturer at University College, Dublin, Ireland.

Mr. R. B. Kiltie, previously a graduate student at New York University, has been appointed to an assistant instructorship at Newark College of Engineering.

Mr. G. J. Kleinhesselink, recently a research analyst for Northrop Aircraft Corporation, Hawthorne, California, is now a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Professor R. E. Langer of the University of Wisconsin is on leave of absence and has a contract with the Department of the Air Force, Air Research and Development Command.

Mr. H. D. Lechner, recently a student at the University of Kansas, has a position as an applied science representative for I.B.M. Corporation, Oklahoma City, Oklahoma.

Associate Professor Marguerite Lehr of Bryn Mawr College has been promoted to a professorship.

Dr. C. E. Lemke, previously a research associate at General Electric Company, Knolls Atomic Power Laboratory, Schenectady, New York, has a position as an engineer with the R.C.A. Victor Division of the Radio Corporation of America, Camden, New Jersey.

Mrs. Isabel S. Macquarrie, visiting lecturer at the Summer School of Mount Allison University in 1955, has been appointed to an instructorship at Wellesley College.

Professor W. G. Madow of the University of Illinois is on leave for the year 1955–1956 and is Visiting Professor at Stanford University.

Mr. Saul Mandel, formerly a student at the University of Oklahoma, has a position as a design engineer at Aerojet-General, Azusa, California.

Dr. N. M. Martin, previously a research associate at Willow Run Research Center, University of Michigan, has been appointed a member of the technical staff of Ramo-Wooldridge Corporation, Los Angeles, California.

Mr. A. L. Mayerson, principal actuary in the New York State Insurance Department, New York City, is now in Paris on a Fulbright scholarship.

Acting Assistant Professor John McCarthy of Stanford University has been appointed to an assistant professorship at Dartmouth College.

Dr. Garner McCrossen of the University of Colorado has a position as a mathematician at Holloman Air Development Center, Holloman Air Force Base, New Mexico.

Mr. Emanuel Mehr of Brooklyn College has been appointed to the position of Associate Engineering Mathematician in the College of Engineering, New York University.

Professor T. E. Mergendahl of Tufts College is retiring with the title of Professor Emeritus.

Mr. R. B. Merrill, formerly a sales correspondent for the Eagle-Picher Company, Chicago, Illinois, is employed as a statistician by the Industry Export Service, Cincinnati, Ohio.

Associate Professor L. I. Mishoe of Morgan State College has been promoted to a professorship.

Assistant Professor D. R. Morrison of Tulane University has accepted a position as staff member at the Sandia Corporation, Albuquerque, New Mexico.

Mr. J. D. Munn of Mississippi Southern College has been promoted to an assistant professorship.

Mr. M. P. O'Donnell, lecturer at the University of Queensland, Brisbane, Australia, is on leave of absence for two years at Clare College, Cambridge, England.

Mr. L. A. Ondis, II, formerly a junior scientist at Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has been promoted to an associate scientist.

Mr. W. L. Phillips, Jr., previously a graduate student at Purdue University, is now a graduate assistant at the University.

Dr. J. H. Powell of the University of Detroit has been appointed to an assistant professorship at Western Michigan College of Education.

Professor J. F. Randolph, chairman of the Department of Mathematics of the University of Rochester, is on leave of absence for the academic year 1955–1956 as Visiting Professor at the American University, Beirut, Lebanon.

Associate Professor P. K. Rees of Louisiana State University has been promoted to a professorship.

Professor P. R. Rider of Washington University has retired with the title of Professor Emeritus.

Miss Marilyn A. Rogers, formerly a student at Carleton College, is now an engineering assistant at General Electric Company, Schenectady, New York.

Mr. W. G. Rouleau, previously a mathematician for Army Map Service, Washington, D. C., has a position as a mathematician and programmer with Engineering & Research Corporation, Riverdale, Maryland.

Associate Professor L. J. Savage of the Committee on Statistics, University of Chicago, has been promoted to a professorship.

Dr. Seymour Schuster of the Polytechnic Institute of Brooklyn has been promoted to an assistant professorship.

Dr. D. H. Shaffer of Carnegie Institute of Technology has a position as a research mathematician at Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Associate Professor C. Eucebia Shuler of the University of South Carolina has been promoted to a professorship.

Mr. R. F. Smith of Syracuse University has been appointed to an assistant professorship at the University of Vermont.

Mr. S. P. Spaulding, formerly an ordnance engineer at the Naval Torpedo Station, Newport, Rhode Island, has accepted a position as an operations research analyst at Electric Boat, Division of General Dynamics Corporation, Groton, Connecticut.

Miss Dorothy M. Swan of Monticello College has been appointed to an assistant professorship at Teachers College at Cortland, New York.

Professor J. A. Ward of the University of Kentucky has been granted a leave of absence for the year 1955–1956 to take a position as a mathematician with the Division of Technical Analysis, Holloman Air Force Base, New Mexico.

Dr. Warren Weaver, director of the Division of Natural Sciences, Rockefeller Foundation, has been promoted to Vice-President for Natural and Medical Sciences.

Dr. J. W. Weihe of the University of California has accepted a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Mr. Howard Young, recently in the United States Army, has a position as an actuarial trainee for Metropolitan Life Insurance Company, New York, New York.

Dr. J. W. Young, previously a mathematician with the I.B.M. Corporation, Atlanta, Georgia, is now a mathematician for I.B.M. Corporation, in Poughkeepsie, New York.

Assistant Professor G. C. Zader of The Citadel has been appointed Director of Admissions and Placement, Rose Polytechnic Institute.

Professor Emeritus R. C. Archibald of Brown University died on July 26, 1955. He was a charter member of the Association, and was President in 1922.

Professor Emeritus H. T. R. Aude of Colgate University died on June 2, 1955. He was a member of the Association for thirty years.

Professor J. W. Campbell of the University of Alberta died on January 23, 1955. He was a member of the Association for thirty-eight years.

Dr. A. P. Cowgill of Syracuse, New York, died on March 22, 1955.

Mr. C. H. Dennison, chemist for Archer Rubber Company, Milford, Massachusetts, died on March 15, 1955. He was a member of the Association for thirty years.

Mr. William Douglas of Courtenay, British Columbia, Canada, died on January 5, 1955.

Professor Henri Fehr of Geneva, Switzerland, died on November 2, 1954. He was Secretary of the International Commission on Mathematics Instruction and was honorary President of this Commission.

Professor H. T. Guard, head of the Department of Mathematics of Colorado Agricultural and Mechanical College, died on June 8, 1955.

Professor J. R. Kline of the University of Pennsylvania died on May 2, 1955. He was a member of the Association for thirty-five years.

Professor H. R. Phalen, head of the Department of Mathematics of the College of William and Mary, died on May 30, 1955. He was a charter member of the Association.

Professor C. H. Richardson of Bucknell University died on March 13, 1955. He was a member of the Association for thirty-seven years.

Professor Emeritus L. L. Smail of Lehigh University died on January 26, 1955. He was a member of the Association for thirty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1955 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	C. B. Read, University of Wichita
Missouri	F. F. Helton, Central College
Ohio	L. L. Lowenstein, Kent State University
Pacific Northwest	Ivan Niven, University of Oregon
Southeastern	F. W. Kokomoor, University of Florida
Southwestern	M. S. Hendrickson, University of New Mexico
Upper New York State	J. F. Randolph, University of Rochester
New England Region	G. B. Thomas, Jr., Massachusetts Institute of Technology

Ballots were received from one-third of those eligible to vote in the above elections for sectional governors. The proportion of votes cast is always much higher in the sectional elections than in the annual elections for national officers. The highest percentage of votes cast was in the Kansas Section where votes were received from 46% of the members of the section.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1955 COMBINED MEMBERSHIP LIST

The Mathematical Association of America issues each year a Combined Membership List jointly with the American Mathematical Society. It is expected that the members of the Society for Industrial and Applied Mathematics will also be included in the 1955 List.

All members of the Association will receive a copy of the 1955 Combined Membership List sometime in December as one of the privileges of membership.

The office of the Association should be notified promptly of all changes in rank, position, and address which have not previously been reported. The final date for receipt of changes is October 15. Any errors in the 1954 Membership List should also be reported before October 15.

H. M. GEHMAN, *Secretary-Treasurer*

CONTINUATION OF THE PROGRAM OF VISITING LECTURERS

The National Science Foundation has granted the Association the sum of $20,500 for the continuation of the Program of Visiting Lecturers for a period of approximately eighteen months, beginning on or about February 1, 1956. The Program will be administered by the Association's Committee on Visiting

Lecturers. The present committee consists of Professors G. B. Huff, B. W. Jones, and D. E. Richmond, Chairman. The Committee is authorized to select the lecturers and to arrange their itineraries.

REPORT OF THE COMMITTEE ON THE UNDERGRADUATE MATHEMATICAL PROGRAM

1. General statement. The Committee on the Undergraduate Mathematical Program was appointed by President E. J. McShane in January 1953. Since it was directed to consider the problems of making available in our society the values of modern mathematics, the word *program* was used in its name rather than *curriculum*. The Committee was instructed to attack the problem with broader scientific and cultural objectives than could be expressed through another mere study of curriculum revision.

At the Kingston meeting in September 1953, the Committee reported to the Board of Governors that there exists a widespread dissatisfaction with the existing undergraduate program in mathematics, complemented by a remarkable unanimity of feeling about the nature of the deficiencies in the present program and the general nature of the program which should replace it. Believing that the reformation of college mathematics cannot be accomplished by the adoption of a new curriculum emanating from any committee, we recommended a widespread program of "doing" to overcome the inertia of the enormously ponderous structure which carries onward the present program with all of its deficiencies. The findings of the Committee and its recommendation to get on with the "doing" was adopted by the Board of Governors. The Committee, reorganized to include C. V. Newsom, who is also chairman of the important Joint Committee on Teacher Education in Mathematics, was charged with organizing the "doing" phase of the work.

This charge to go ahead with the much-talked-about revolution in the mathematics program is one which we would not dare accept as our own private responsibility. In all fields we must learn the ways in which mathematics can contribute to scientific and social thinking and support the existing technology. To this end we must enlist the aid of the men in all these fields who have the greatest wisdom and scientific insight; for the history of previous attempts at curricular revision in mathematics strongly suggests that they failed because these attempts were made upon a narrowly pedagogical and organizational basis, without the participation of the real intellectual leaders in mathematics and in the fields which use mathematics. In particular, we must find the men with ideas, who, by experimental writing and teaching, can beat into shape new courses and their textbooks, so that the rest of us can teach new and (we hope) better courses in mathematics. The Committee must guard against the adoption of arbitrary opinions of its own or of others in order that any new program or movement which we foster shall be true to the development of mathematics itself.

In the existing state of affairs there are a large number of conditions which bear upon any movement to modify the traditional program, whether by natural

forces or by arbitrary effort. From the scientific standpoint, one must take into account the extended mathematical needs of modern engineering and physical science. At the same time such sciences as econometrics, physiology, sociology, and genetics seem to demand, in part at least, entirely new mathematics. Moreover, the emergence of new mathematical technologies based upon high-speed digital automata presents serious problems of policy. For the intriguing gadgetry of these machines tends to obscure the potentially great magnitude of their impact upon men. This current development based upon the old subjects of symbolic logic, numerical analysis, and combinatorial theory is another example of the fact that the work of the creative mathematician tends to be applied much less directly than that of the empirical scientist and, even so, largely through the mathematical knowledge which technical workers in other fields acquire in school and college.

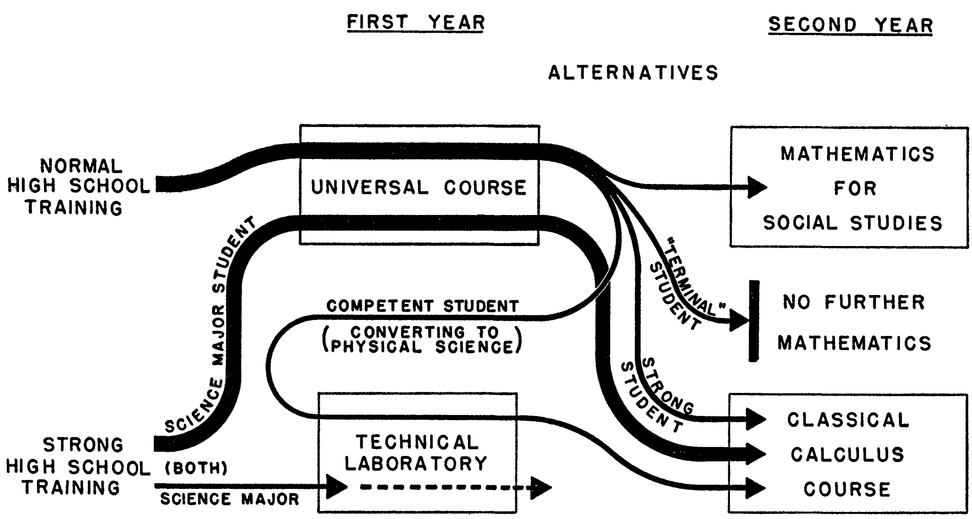
Thus, turning to the educational conditions affecting our problem, one finds that probably the central one is the desertion of elementary teaching by the best mathematicians, old and young. Besides this, there is the rigid sequential organization of the traditional program with each course depending heavily on technical prerequisites, the compartmentation in college programs with attendant difficulties for the student who transfers or changes program, the growing tendency to repeat high school courses in college, the terminal course idea, the emphasis upon cultural and liberal aspects in education today, the current popularity of logic, the recent studies of articulation of school and college, the incipient movement to train a special breed of college mathematics teachers by means of special education programs, and the complicated relationships of the policies of foundations, which might support the writing of new text material, to the publishing companies, and the royalty rights of authors. There are, of course, many mathematicians who still believe strongly in "the old time religion" and cannot see what this fuss is all about.

In view of this complex of inertial elements, it is the opinion of this Committee that, in mathematics, a broad, coordinated attack will be needed in contradistinction to the textbook revolution which was successful in the teaching of English literature. In elementary mathematics, new ideas introduced by local writing and teaching efforts have always turned out to be largely overpowered by the self-propagation of the traditional but retrogressive stock. Moreover, we believe that for very cogent reasons we must continue to seek *one, universal freshman course* for all reasonably qualified students, presupposing intermediate algebra and high school geometry, and ignoring the question of remedial courses for students who enter college without adequate mathematical training in school. Only by means of such a universal course can the best principles of liberal education be served. Only in this way can we avoid the error of forcing the immature student, upon entering college, to make choices which will seriously restrict his freedom of development in later years, and in this way we will attempt to make sure that the subjects taught are really the most valuable ones. We are confident that the mathematics which survives such a selection process from the common

intersection of all of demands of the various disciplines, including higher mathematics, will turn out to have the values of generality and abstractness which have always characterized good mathematics. Also we are confident that this selection process will naturally sift out essentially the same set of ideas which would be chosen as fundamental if one considered only the development of mathematics itself.

2. A universal freshman course and sequels. The proposed universal course would contain all of the theory prerequisite to any sophomore course in mathematics, covering this, with appropriate exercises and problem solving, in three hours per week for a year. In most schools of engineering, it should be supplemented by technical problem-solving work in additional "laboratory" class meetings.

To assure ourselves of the existence of a mathematics course for freshmen which meets these specifications, the Committee has attempted to construct one in outline. We do not propose our course as the solution but perhaps a schematic diagram might illustrate the general direction of our efforts. The "technical



laboratory supplement," which might be two hours per week, would presuppose a stronger high school training, perhaps three units, and this strong high school training, jointly with the universal freshman course, would form the prerequisite for the sophomore calculus. The laboratory itself would not be a prerequisite to sophomore calculus, though it could serve a remedial function which might fit many weakly prepared students. These supporting functions of the technical laboratory work are indicated by the dotted arrow pointing towards calculus. This scheme would continue to make essential use of strong mathematical training in high school without shutting the door to a good student who enters college without such training.

This Committee thought of the universal course as being made up of two subjects. (Arrange them as you will.) One subject would consist of functions, graphs and elementary calculus, covering differentiation and integration of polynomials, logarithms, exponential functions and general powers, with a minimum of formulas and with emphasis on ideas. The other subject would take off from set concepts, set up the fundamental language of theoretical mathematics and proceed, via some of the simpler abstract algebraic systems (but not the full real number system), to probability with emphasis upon the binomial distribution. If possible, this should lead to the sampling problem and here make a junction with statistics without actually getting into statistics. We believe that the universal course should contain little or no statistics but it should lay the foundation for this subject in a much more thorough way than the traditional course does. Engineers and physical scientists now need these probability ideas as well as the social scientists.

Now in this scheme, what becomes of the traditional college algebra, trigonometry, and analytic geometry? The essential algebra in college algebra would be covered in the universal course. Numerical trigonometry might be handled in the technical supplement. Analytic trigonometry might form the first subject of the sophomore calculus, which would be time enough for its uses in mechanics. Euclidean analytic geometry is well known to be not a proper prerequisite to calculus, which requires the more general geometry, the geometry of the cartesian product of the real line and the real line, *i.e.*, "graphs." This includes straight lines, slopes and the (weak) area of a rectangle. Thus the first real treatment of *Euclidean* analytic geometry would be in the sophomore course. (This beautiful old subject needs a proper revival in American college mathematics in a way which neither the hurried freshman nor sophomore courses permit!) Moreover, the graphs, or affine geometry, of the Universal Course, is not a negligible geometry.

The schematic diagram indicated a branching in the second year with a course called mathematics for social sciences and statistics appearing parallel to the traditional calculus. This is a long-needed development. For the calculus has not been a suitable universal second course. Actually, however, this alternate second course will probably turn out to meet many needs other than those of the social science student and potential statistician. For example, the prospective high school teacher might find it better for his needs than calculus. What the syllabus of such a new course should contain is an open question. Statistics? Multivariable algebra? Postulational models? Scales of measurement? Maximum problems in several variables?

Returning to the technical laboratory of the first year, we think of the first semester being devoted to numerical trigonometry and to numerical methods, with considerable practice in graphical methods and the use of tables of logarithms and exponentials. Worded problems would form an important part of it, with calculus drill beginning after mid-semester. Since relatively little calculus

would appear in the second semester of the universal course, the technical laboratory for the second semester would become largely a calculus drill course. Thus engineering students would enter the sophomore course in analytic geometry and calculus with almost a year's study of elementary calculus behind them, a good part of which would have been based upon a strong high school training.

By these means and by providing at minimum cost an appropriate outlet for those who want and need less technical mathematics, we hope to strengthen the foundations of the classical sophomore calculus course and enable it to be started at a relatively advanced point and to proceed into the methods and applications of calculus. This should clear the way for more vector methods, at least in the second semester, and perhaps permit the calculus to include more differential equations, mechanics, and numerical integration than has been possible in the past. Finally, we point out that the principle of the Universal Course provides for an efficient conversion of those students who discover, after entering college, that they want to study engineering or physical science. The so-called "terminal course" often makes this change of plan prohibitive in cost.

The Committee has nothing of its own to report on courses beyond the sophomore year. It has been seeking out interesting new courses which are being offered in various institutions, some of which are mentioned in Section 3 of this report. A number of the discussions of the Committee have centered on the revival of geometry.

3. Summary of known activities. For the answers to the questions about the proposed sophomore course in mathematics for social scientists, and other questions on mathematics basic to social science, our Committee is relying upon another group, *viz.*, the Committee on the Mathematical Training of Social Scientists which developed into the 1953 Summer Institute on Mathematics for Social Scientists at Dartmouth, Professor William G. Madow, Chairman. This very significant project was described at the Association Christmas Meeting in Baltimore by several members of the Institute: W. G. Madow, R. M. Thrall, R. R. Bush, and Howard Raiffa.* In addition to this group, K. O. May of Carleton College is working on an interesting book on mathematics for social scientists which has much new material. Also W. G. Madow is individually writing an elementary book on mathematics for social scientists. This is significant in view of the author's experience in the Dartmouth Summer Institute. Another body of material which might go into the Mathematics for Social Science is the semester course in statistics as constructed by S. S. Wilks for Princeton freshmen to follow a semester of calculus.

Since there existed no such Committee on the important questions relating to the mathematical training for the engineering of today and of the future, a joint committee was appointed by the Association and the American Society for

* See this MONTHLY, vol. 61, pp. 550–561.

Engineering Education. Professor G. B. Thomas of M.I.T. is Chairman. This Committee will search out the engineering leaders who have the knowledge and foresight to tell us what forms of mathematical training are likely to be most valuable in the engineering of the next twenty years. To assist this committee in its work it is hoped that some conferences on modern mathematics and modern engineering will be arranged, possibly with the sponsorship of the National Science Foundation. R. S. Burington, who is a member of the joint committee, and also president of the Mathematics Section of A.S.E.E., arranged an extremely interesting program on this subject at the 1954 summer meeting of A.S.E.E. in Urbana, Illinois. It is hoped that the proceedings will soon be available in print.*

At a somewhat higher level, training in applied mathematics has been the subject of a study directed by F. J. Weyl for the Division of Mathematics of the National Research Council. Two symposia were held in connection with meetings of the Society. Proceedings of these symposia will be available in print soon.** This work was supported by the National Science Foundation.

The idea of early introduction of calculus, inherent in the proposed Universal Course, is by no means new or radical. This is the form of the first college course implied by both of the studies of articulation of school and college supported by the Fund for the Advancement of Education. The first report appeared in *General Education in School and College*, Harvard University Press (1952). The mathematical part was reprinted in AMERICAN MATHEMATICAL MONTHLY.§ The mathematical part of the second study is known as the Brinkmann Committee, whose results were reported to the Association at the Baltimore meeting, December 1953.§§ The older textbooks which form prototypes for at least the first half of the Universal Course include those of F. L. Griffin, *Mathematical Analysis;* and Milne-Davis, *Introductory College Mathematics*, the latter being interesting also for its sections on numerical methods. The use of such books with an early introduction to calculus dispensing with the traditional "preparation" is increasing and meeting with good success, notably in the California schools, and in Minnesota, following the University of Minnesota. Other recent books on freshman calculus of special interest are those of D. E. Richmond of Williams College, E. G. Begle of Yale, and Karl Menger of Illinois Institute of Technology. Begle's book has for this level perhaps the most modern treatment of the formal theory available in print. Menger's book contains a radical attack on the conceptual ideas of the variable and proposes a solution using the identity function as a variable. This book may have great influence, for notable simplifications are achieved.

A unique text in many ways is the preliminary edition of Allendoerfer-Oakley

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* See this MONTHLY, vol. 62, 1955, pp. 385–392.

** See this MONTHLY, vol. 61, No. 7, Part II, Slaught Memorial Paper, No. 3.

§ See this MONTHLY, vol. 60, 1953, pp. 380–383.

§§ See this MONTHLY, vol. 61, 1954, pp. 319–323.

*Principles of Mathematics.** This text contains the best material on sets, logic, and Boolean algebra of any informal general mathematics text.

A number of books have been written or are in writing which seek to introduce mathematics in a more formal and logical manner than is conventional. These include *Fundamental Mathematics*, written by the College Mathematics Staff of the University of Chicago. It aims at an organization and explication of basic mathematical concepts, built up from elements of logic and set theory. Also at Chicago, the Department of Mathematics is preparing texts for undergraduate courses. In some respects Herman Meyer of the University of Miami, in his *Mathematical Analysis*, pushes the postulational development of the traditional freshman material farther and more consistently than any other text. Also in this field, the Johnson, McCoy and O'Neill *Fundamentals of College Mathematics* has been used experimentally in a number of places.

At the University of Kansas this summer, the Kansas Summer Writing Group under the direction of G. B. Price is writing a preliminary edition of text material for the Universal Course as designed by the Committee. This text material will be tried out on a full scale at Tulane and on a smaller scale at several other institutions next year. This non-commercial writing was supported by the University of Kansas and the Social Science Research Council.

Courses which show interesting developments, and of which we are aware, include Artin's honors course for specially selected freshmen at Princeton. This goes further in calculus than conventional courses do in two years. The Stanford University department is constructing an experimental honors course in calculus for social scientists, having a course in logic taught in the philosophy department as a prerequisite. Most mathematicians are familiar with Pólya's work, also at Stanford, on heuristic methods in mathematics. His forthcoming book on *Mathematics and Plausible Reasoning*** is awaited with much interest. New courses in appreciation and understanding of mathematics are being offered, but the returns are not yet in on the support for them. Besides courses based upon the books of R. L. Wilder, E. R. Stabler, Courant-Robbins, and Kershner-Wilcox, we know that W. Feller at Princeton and A. M. Gleason at Harvard are developing courses of this character. At Brown University, H. Federer, Jonsson, and K. G. Lister have been developing a four-semester course in elementary mathematical analysis from a modern point of view, based on text material written by Federer and Jonsson. A significant feature is the exercises for students in mathematical writing, with complete quantification and logical detail. Among the more advanced undergraduate courses, A. W. Tucker has a well developed undergraduate course in combinatorial topology and R. L. Bing at Wisconsin has one on elementary set topology.

New courses and programs for training teachers of mathematics, either for high school or college teaching, are too numerous to mention in detail here,

* Printed edition, McGraw-Hill Book Company, 1955.

** Princeton University Press, 1954.

especially since they are in the province of the Newsom Joint Committee of M.A.A. and N.C.T.M. Briefly we may indicate the internship program for prospective college teachers, sponsored by the Fund for the Advancement of Education at nine universities, and new graduate programs, such as that at Yale. Also the special emphases of the University of Wisconsin and Notre Dame departments in this direction are noteworthy. The first Summer Conference for College Mathematics Teachers last year at Boulder was organized by B. W. Jones. New ones are being directed this summer by Ivan Niven at Oregon, E. A. Cameron at North Carolina, and for high school teachers by C. B. Allendoerfer in Seattle, all sponsored by the National Science Foundation. A college lectureship program for the coming year is being set up with a grant from N.S.F. The Internationale Matematische Unterrichts Kommission, founded by Felix Klein, is being revived at the Amsterdam International Congress. Saunders MacLane, S. S. Cairns, and A. M. Gleason represent the United States in that activity.

4. An appeal to all mathematicians. We here appeal to mathematicians, teachers, scientists, and engineers to give us the benefit of your counsel. Tell us what you are doing and what you think is needed. Tell us also specifically what this "dead wood" is which everyone says we can cut out of the present program. Give us copies of your notes or syllabi for new courses and tell us about your experiments in teaching or writing. We are particularly concerned to have the results of self-study analyses which departments of mathematics have conducted. Ultimately any complete job of writing must be in the form of writing for the classroom text with exercises. Anything short of that is likely to be of very transient value. If possible, send us seven copies of any such material you have which you consider to be significant for a reformulation of undergraduate mathematics, especially now the freshman program, but ultimately reaching on to the advanced work and back into the schools. This information will be directly helpful in carrying out the work itself and indirectly it will provide us with a basis upon which we can assess the amount of financial assistance which will be needed in furthering the program on a national basis.

Direct approaches to the physiologists, physicists, and earth scientists have been initiated with a view to getting counsel from them, analogous to that which we sought from the engineers and social scientists. Scientists in other fields will be approached later, and the results will be presented in another report.

Meanwhile, in mathematics itself there is the most urgent need for the active participation of the very best living mathematicians in the work of reformulation of our school and college programs. The last successful reformulation of the program in mathematics was accomplished in the early part of this century and was founded upon the ideas of such outstanding mathematicians and teachers as Felix Klein, Poincaré, Boltzmann, E. H. Moore, J. W. A. Young, John Perry, and many others. That a number of subsequent reform efforts have been failures or at most minor successes may be attributed to their lack of mathematical

leadership; for the teaching organizations tried to go it alone. This Committee is seeking to enlist the active participation of our able mathematicians in this work but, independently of our efforts, many established and young mathematicians who formerly thought primarily in terms of research are turning their attention to the teaching of mathematics. For it is realized that the economic basis of the mathematical profession and its relation to the general welfare is still largely founded on teaching. If graduate departments of mathematics do not furnish the teachers of college mathematics, and these teachers are qualified in some other way, we may expect that the values of modern mathematics will be lost to all but a few specialists. Moreover, the economic basis of the profession will be destroyed, as it was in the profession of classics. Unfortunately, many of the leaders of the mathematical profession are in universities where they are insulated from the signs of deterioration which are apparent in the average college and hence do not have much direct experience with the problem.

5. Suggested Special Undertakings. Finally, we list a few possible subjects for special projects which will contribute to the general objectives. We hope that individual mathematicians and departments will undertake them. If financial assistance is needed, this Committee will undertake to be of such assistance as it can to those who plan to submit a proposal asking for it. The necessary coordination of these financial appeals is the business of a committee under the chairmanship of Dean Mina Rees of Hunter College. A partial list of special problems on which work is needed is: (1) The writing of several manuscripts for a universal freshman course and classroom trials of these text materials. (2) The simplification of calculus so that it can be taught to freshmen generally. (3) Mathematics for genetics and physiology. (4) Mathematics for the earth sciences. (5) Theory for first year students. For example, how far should one go into logic? How well do rigorous postulational treatments get across? (6) Correct and teachable set theory and probability ideas for the first year. (7) The role, if any, of the normal probability distribution. (8) Dimensional analysis and measurement. (9) Vectors and linear operators as early as possible and in calculus. (10) Heuristic methods and problem solving to teach students to *make* the abstraction rather than to teach them previously abstracted theory. (11) The resetting of geometry in the curriculum. (12) Calculus technique making simplest use of numerical methods, galleries of graphs and tables of integrals. (13) Assembling and preparing readings, other supplementary materials for use in elementary courses, and guides which will make the mathematics library more accessible to students. (Here H. L. Meyer and I. Wirszup of the Chicago College Staff are doing good work.) (14) Undergraduate "research." (15) New courses in pure mathematics for undergraduates (such as R. L. Bing's topology course). (16) Combinatorial analysis and digital theory. (17) Boolean algebra. (18) Courses in computation. (19) Number theory for undergraduates and perhaps upper high school students. (20) Mathematics in the arts. (21) Complex numbers and functions in the undergraduate curriculum. (22) Trigonometry. What

is its place and value? (23) Grassman algebras and exterior products. Shall the 3-dimensional vector product be retained in courses? (24) School-college articulation effecting the advancement of well-trained students. With college credits? (25) Training for applied mathematics. (26) Morale building in high school mathematics. (27) Training for actuarial science. (28) Tukey's "mathematical engineering." (29) Reformulation of standard undergraduate programs.

6. Publication Policy. In conclusion, we state our policy on copyrights and relations to commercial publishers. This Committee will not get into the textbook business and will not approve or disapprove any textbook. Writers of experimental text materials, when subsidized by grants under the jurisdiction of this Committee will assign all royalty interests to the Association or the foundation making the grant until the expiration of the period of the grant is reached or the repayment of the grant has been effected. Ultimately we expect that any new textbooks, which may result wholly or in part from our efforts, will be published by normal methods. We must treat whatever ideas come to us as belonging to the public domain, though we hope we will always acknowledge credit where it is due. With these understandings we invite every mathematician to contribute his own ideas to the general effort.

Along with the undersigned members of the Committee, President E. J. McShane has worked as hard as any member of it, and R. C. Yates contributed greatly as an active member in 1953.

August 16, 1954

W. L. DUREN, JR., *Chairman*
C. V. NEWSOM
G. B. PRICE
A. L. PUTNAM
A. W. TUCKER

ADDENDUM

The Committee on the Undergraduate Program is being reorganized with a new Chairman, E. J. McShane, who originally appointed it. The new Committee will also include John G. Kemeny of Dartmouth. W. L. Duren will remain an ex officio member as President of the Association.

The grant of $25,000 made in March 1955 by the Ford Foundation, and announced elsewhere in this MONTHLY, gives the Committee its first dependable financial support. This amount of money is generous for a start but the magnitude of the general effort is so great that these funds are small in comparison.

The Committee met in Lawrence in January 1955 with further support from the University of Kansas. Here a week's work was done on Universal Mathematics, Part II, particularly on applications of the Boolean set operations. A new Summer Writing Group is now being planned for 1956. Its location has not been determined. A critical account of the Tulane experience with Universal Mathematics, Part I, has been prepared for publication in this MONTHLY.

April, 1955

W. L. DUREN, JR.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 124 persons have been elected to membership by the Board of Governors on applications duly certified.

GRACE L. ABHAU, Student, Kent State University.

EUGENE ALBERT, M.A. (Brooklyn C.) Instr., Brooklyn College.

HARMON ALEXANDER, M.A. (N.Y.U.) Van Nuys, Calif.

W. O. ALEXANDER, JR., B.S. (Houston) Grad. Student, University of Houston.

ELIZABETH A. AULT, Student, Ursinus College.

A. F. BAKENHUS, Student, University of Houston.

LEONARD BANNER, B.A. (Brooklyn C.) Math. Teacher, Lubavitcher High School, Brooklyn, N. Y.

G. E. BILLS, A.B. (W. Va. U.) Lt. (j.g.) United States Navy.

L. M. BOSTICK, B.S. (Eastern New Mexico) Grad. Asst., Eastern New Mexico University.

R. D. BOYD, Student, Queen's University.

J. S. BRADLEY, Student, Mississippi Southern College.

W. S. BUSH, B.A. (Mississippi) Grad. Fellow, University of Mississippi.

RAYMOND CARPENTER, M.A. (Columbia) Asso. Professor, Northeastern State College, Tahlequah, Okla.

J. R. CASTILLO-NUNEZ, Student, Agricultural and Mechanical College of Texas.

W. E. CHRISTILLES, Student, St. Mary's University.

LEROY COOPER, B.S. (Philander Smith) Tulsa, Okla.

J. B. DOUGLASS, Student, Texas Christian University.

REV. E. D. EICHMAN, Math. Teacher, Epiphany Apostolic College.

A. H. ETTELSON, Student, University of Buffalo.

D. L. FAASS, Student, University of Houston.

HERBERT FARKAS, Student, City College of the City of New York.

F. D. FARLEY, B.S. (Concord) Grad. Student, West Virginia University.

F. G. FENDER, Ph.D. (Pennsylvania) Professor, Rutgers University.

A. M. FENSTERMACHER, Student, Knox College.

LEAH FINE, Student, Agnes Scott College.

P. M. FITZPATRICK, M.S. (Catholic) Pvt., United States Army.

J. L. C. FORD, JR., Student, Montana State College.

T. S. FRANK, Student, Lawrence College.

R. G. FRYER, B.S. (St. Lawrence) Teaching Fellow, University of Buffalo.

A. V. GAFARIAN, B.S.E. (Michigan) Grad. Student, University of California at Los Angeles.

D. T. GIANTURCO, Student, University of Buffalo.

M. C. GILLILAND, B.S. (U.C.L.A.) Grad. Student, University of California at Los Angeles.

HARRISON GIVENS, JR., B.S. (Yale) Equitable Life Assurance Society, New York, N. Y.

JUDITH GORENSTEIN, Student, Massachusetts Institute of Technology.

L. A. GRAHAM, A.M., M.E. (Columbia) Engr., Graham Transmissions, Menomonee Falls, Wisc.

E. A. GREEN, Student, Queen's University.

MRS. ELZIE A. GREENE, M.A. (Arkansas) Staff Member, Sandia Corp., Albuquerque, N. M.

KATHLEEN HAMLIN, Student, Wayne University.

J. V. HANCOCK, Student, Memphis State College.

W. J. HARTMAN, Student, University of Colorado.

LT. COL. J. C. HEMPSTEAD, C. E. (Iowa S.C.) Asst. Professor, United States Air Force Academy.

W. F. HERRIN, Student, Central College.

T. W. HILDEBRANDT, S.M. (M.I.T.) Grad. Fellow, Oak Ridge Institute of Nuclear Studies.

W. A. HOCKINGS, B.S. in Chem. (Michigan C. of Mining and Tech.) Res. Chemist, Calumet & Hecla, Calumet, Mich.

R. F. HOUSE, Student, Texas Christian University.

H. C. HOWARD, JR., B.A. (Wooster) Teaching

Asst., Carnegie Institute of Technology.

LORETTA A. HUMPHREYS, Student, University of Oregon.

G. R. HUNT, M.A. in Ed. (New Mexico) Asst. Professor, San Angelo College.

M. J. HURLEY, JR., Student, University of Arizona.

RONALD JACOBOWITZ, Student, City College of the City of New York.

J. D. JACOBS, Sycamore, Va.

E. S. KEEPING, D.I.C. (London) Professor and Head of the Department of Mathematics, University of Alberta.

C. H. KEIM, Midshipman, United States Naval Academy.

D. F. KIMBALL, Student, Brown University.

D. R. KING, Student, Rutgers University.

T. C. KIPPS, M.A. (California) Instr., University of Santa Clara.

H. P. KUANG, M.S. (Minnesota) Teaching Asst., University of Minnesota.

L. J. LANGE, B.S. (Regis) Grad. Student and Instr., University of Colorado.

R. A. LEACH, B.A. (Roosevelt) Engr., Natural Gas Pipeline Company of America, Chicago, Ill.

JOANN LETOHA, Student, Kent State University.

W. C. LORDAN, B.A. (Wesleyan U.) Grad. Student, University of Wisconsin.

J. H. P. MAECHER, M.S. (Miami) United States Army.

R. T. J. MAHONEY, Student, University of Buffalo.

D. B. MAIER, Student, Rutgers University.

R. L. MANCHESTER, B.A. (Buffalo) Math. Instr., Franklinville Central School, N. Y.

M. J. MANSFIELD, M.S. (Purdue) Res. Asst., Statistical Lab., Purdue University.

SAMUEL MAREIN-EFRON, Student, Harvard University.

A. V. MARTIN, Ph.D. (Duke) Asso. Professor, University of New Mexico.

T. D. MCADAM, A.B. (Washburn) Instr., Washburn University of Topeka.

C. A. MCCARTHY, Student, University of Rochester.

BROTHER ANDREW MCCAULEY, B.A. (Manhattan) Grad. Student, Catholic University; Instr., De La Salle College.

L. L. MCCELVEY, B.A. (A.&M.C. of Texas) 2nd Lt., Pilot, United States Air Force.

J. B. MCGRORY, Student, University of the South.

G. W. MCLAUGHLIN, B.A. (Washington S.C.) Private Tutor, Harrington, Me.

J. P. MENARD, B.A. (St. Michael's) Grad. Teaching Asst., Catholic University.

D. M. MERRIELL, Ph.D. (Chicago) Head, Department of Mathematics, Robert College, Istanbul, Turkey.

AARON MILLER, M.S. (Washington U.) Retired Math. Teacher, Indianapolis, Ind.

W. G. MILLER, Ph.D. (Florida) Professor, Clemson College.

MISS MERLE MITCHELL, M.A. (New Mexico) Instr., University of New Mexico.

J. L. NEWTON, Student, University of Colorado

R. V. NOLAN, B.A. (Buffalo) Teaching Fellow, University of Buffalo.

R. E. OBERDORFER, Student, Kent State University.

A. J. O'CONNOR, Schenectady Sign Service, N. Y.

T. J. O'NEIL, B.A. (Buffalo) Grad. Student, University of Buffalo.

R. C. O'NEILL, M.A. (Columbia) Lt. (j.g.) U.S.N.R.

MARY F. OVERFIELD, Student, Kansas State Teachers College.

J. H. PADGETT, M.A. (North Carolina) Instr., Armstrong College.

HIRAM PALEY, Student, University of Rochester.

G. L. PATE, M.A. (George Peabody) Instr., University of Georgia, Atlanta Division.

G. P. PATERNOSTER, Structural Detailer, Harry L. Dovell & Co., Chicago, Ill.; Student, Roosevelt University.

SIDNEY PENNER, Student, City College of the City of New York.

R. L. PRATT, Student, Washington University.

SUSAN PYEATT, Student, Catholic University.

J. T. ROBERTSON, Student, Memphis State College.

J. B. ROGERS, M.A. (Michigan) Math., Rand Corporation, Santa Monica, Calif.

HUGO ROSSI, Student, City College of the City of New York.

G. L. ROWLAND, A.B. (U.C.L.A.) Instr., University of New Mexico.

RUTH L. ROYER, M.S. (Iowa S.C.) Instr., Chico State College.

T. W. ROZELLE, M.A. (Michigan) Asso. Professor, Wisconsin State College.

A. A. SAGLE, Student, University of Washington.
R. W. SCOTT, B.A. (Carleton) Northfield, Minn.
C. R. SELIGER, A.B. (Johns Hopkins) Teaching Asst., Rutgers University.
R. E. SHAFER, Student, University of California, Berkeley.
O. T. SHANNON, M.Litt. (Pittsburgh) Asst. Professor, Agricultural, Mechanical & Normal College, Pine Bluff, Ark.
L. A. SHEPP, Student, Polytechnic Institute of Brooklyn.
A. J. SILBERGER, Student, University of Rochester.
M. G. SMITH, B.S. (Oklahoma) Teaching Asst., University of Oklahoma.
DANIEL SOKOLOWSKY, M.S. (Wisconsin) Asst. Professor, Antioch College.
S. E. SPIELBERG, Student, University of Pennsylvania.
J. C. STUELPNAGEL, Student, Yankton College.
REV. T. J. TAYLOR, M.S. (Notre Dame) Instr., St. Ambrose College.
GEORGIA M. THOMPSON, Student, University of Arizona.
GABRIEL TSIANG, M.S. (Notre Dame) Lecturer, Southern Illinois University.
J. B. VIEAUX, Student, Agricultural and Mechanical College of Texas.
T. C. WALKER, Student, Montana State University.
H. M. WEITKAMP, M.A. (Cincinnati) Teacher, Cincinnati Public Schools.
L. W. WESLEY, B.A. (Minnesota) Supervisor, Engg. Dept., North American Aviation, Columbus, Ohio.
ROBERT E. WHEELER, Airman, United States Air Force.
FREDERICK R. WHITE, Student, University of Buffalo.
K. F. WILSON, B.S. (Florida) Grad. Asst., University of Florida.
R. J. WILSON, Student, St. Mary's University.
T. J. WOLINSKI, Student, St. John's College.
G. F. WOODLIEF, JR., Student, Duke University.
J. L. WULFF, A.B. (Sacramento S.C.) Grad. Student, Sacramento State College.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 26, 1955, at Michigan State College in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor R. V. Churchill of the University of Michigan presided at both morning and afternoon meetings and at the luncheon and business meetings.

A total of sixty-one persons attended the meetings including forty-four members of the Association:

Bess E. Allen, W. D. Baten, F. A. Beeler, J. E. Bellardo, C. H. Butler, A. T. Butson, R. V. Churchill, S. D. Conte, Helen E. Core, J. W. Coy, D. E. Deal, E. R. Deal, S. F. Dice, H. G. Falahee, J. S. Frame, J. W. Gaddum, E. W. Goings, V. G. Grove, L. W. Gunter, Frank Harary, G. E. Hay, Fritz Herzog, J. D. Hill, Walter Hoffman, R. D. James, Leo Katz, M. A. Laframboise, G. P. Loweke, M. T. MacNeil, L. E. Mehlenbacher, D. C. Morrow, H. W. Nace, A. L. Nelson, E. A. Nordhaus, E. S. Northam, George Piranian, J. E. Powell, J. H. Powell, E. D. Rainville, R. W. Schenkel, F. C. Sherburne, Jr., B. M. Stewart, C. K. Tsao, J. L. Ullman.

The nominating committee consisting of Professors B. M. Stewart. Chairman, and D. C. Morrow proposed Professor C. C. Richtmeyer of Central Michigan College of Education as Chairman and Professor S. D. Conte of Wayne University as Secretary-Treasurer for the year 1955–1956. The slate was elected unanimously.

Professor C. H. Butler was appointed chairman of a committee of the

Michigan Section of the MAA which is to cooperate with a similar committee from the Michigan Council of Teachers of Mathematics with the purpose of improving the quality of mathematical teaching in the secondary schools and of encouraging students to continue their mathematical training.

The following papers were presented at the morning and afternoon sessions:

1. *Some properties of n-convex functions*, by Professor R. D. James, Michigan State College.

A convex function $f(x)$ is one for which the curve $y=f(x)$ between x_1 and x_2 lies below the chord joining the points (x_1, y_1), (x_2, y_2). It is well known [Hardy, Littlewood, and Pólya, *Inequalities*, Theorem 111] that a continuous convex function has left-hand and right-hand derivatives. A convex function may also be defined as one whose divided differences of order 2 are all non-negative, and this concept may be generalized. A function whose divided differences of order n are all non-negative will be called an n-convex function. With this definition a 2-convex function is convex in the ordinary sense. It can be shown that an n-convex function has differential coefficients of order r, $1 \leqq r \leqq n-2$ and left-hand and right-hand differential coefficients of order $n-1$.

2. *An application of the Cauchy-Hadamard formula*, by Professor J. L. Ullman, University of Michigan.

In a first course in function theory the Cauchy-Hadamard formula for finding the radius of convergence of a power series is introduced, and it is pointed out that a singularity of the function defined by the series must lie on the circle of convergence. Our purpose is to show how the Cauchy-Hadamard formula can be used for arriving at deeper properties of functions. The results discussed are a) a method for finding all the poles of a meromorphic function, b) the location of the limit points of the totality of the zeros of a meromorphic function and its derivatives, and c) a proof of a theorem due to Jentzsch, together with some generalizations.

The exposition is based on the following references: a) A. Hurwitz, *Sur une Theorème de M. Hadamard*, Comptes rendus des séances de l'Academie des Sciences, vol. 128, 1899, pp. 350–353; b) George Pólya, *Über die nullstellen sukzessiver derivierten*, Mathematische Zeitschrift, vol. 12, 1922, pp. 36–60; c) J. L. Ullman, *Hankel determinants whose elements are sections of a Taylor series*, Part I, Duke Mathematical Journal, vol. 18, 1951, pp. 151–156, Part II, same journal, vol. 19, 1952, pp. 155–164.

3. *On complementary graphs*, by Professor E. A. Nordhaus and Dr. J. W. Gaddum, Michigan State College, presented by Professor Nordhaus.

Best possible bounds are established for the sum and product of the chromatic numbers of an unoriented finite linear graph and its complementary graph. Estimates are also given for upper and lower bounds for the chromatic number for graphs having a prescribed number of nodes and edges, employing a result of Turan. Some results are found concerning the density of graphs by correlating recent work done by Zykov, Greenwood and Gleason.

4. *On strengthening and weakening points of directed graphs*, by Professor Frank Harary, University of Michigan.

Four degrees of connectedness: strong, unilateral, weak, and disconnected, are defined for directed graphs (digraphs). These satisfy: every strong digraph is unilateral; every unilateral digraph is weak. Let $C_3 = \{\text{strong digraphs}\}$, $C_2 = \{\text{unilateral digraphs which are not strong}\}$, $C_1 = \{\text{weak digraphs which are not unilateral}\}$, $C_0 = \{\text{disconnected digraphs}\}$ be the disjoint connectedness categories. Let Π_{ij} be the set of all points P such that digraph $\mathcal{D} \in C_i$ and $\mathcal{D} - P \in C_j$. A point $P \in \Pi_{ij}$ is called a strengthening point of $\mathcal{D}$ if $i > j$, and a weakening point if $i < j$. It is shown that Π_{31} is the only empty class among the Π_{ij}. Characterizations of other classes Π_{ij} are given and strengthening lines are also discussed.

5. *Stability of numerical methods in the solution of differential equations* by Professor Saul Rosen, Wayne University. Presented by title.

6. *An engineering degree with specialization in mathematics*, by Professor G. E. Hay, University of Michigan.

An expository paper dealing with the mathematical content of an engineering degree which emphasizes mathematics. This degree is currently offered at the University of Michigan.

7. *A note on power series which diverge everywhere on the unit circle*, by Professor Fritz Herzog, Michigan State College.

Lusin has shown that there exist power series $\Sigma a_n z^n$, with $\lim a_n = 0$, which diverge at all points of the circle $|z| = 1$. In this note, a very simple example of a power series of this kind is constructed, which has the additional property that its coefficients are real and non-negative. Moreover, the example can be modified in such a way that the real part of the power series yields an example of a cosine series $\Sigma a_n \cos n\theta$, with $a_n \leqq 0$ and $\lim a_n = 0$, which diverges for all real θ.

8. *On a certain family of subsets of a set*, by Professor Ben Dushnik, University of Michigan, introduced by the Secretary.

A condition is stated under which a set ϵ can be represented as the union of a specified number of members of a certain family of subsets of ϵ. The condition is related to the generalized continuum hypothesis.

9. *Some properties of the Fejêr polynomials*, by Professor Fritz Herzog, Michigan State College, and Professor George Piranian, University of Michigan, presented by Professor Piranian.

The polynomials

$$P_n(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^n}{1} - \frac{z^{n+1}}{2} - \cdots - \frac{z^{2n-1}}{n}, \quad n = 1, 2, \cdots,$$

were introduced by Fejêr, who showed that they are uniformly bounded on the unit circle. The authors show that, as $n \to \infty$,

$$\lim M_n = 2 \int_0^\pi \frac{\sin t}{t}\, dt,$$

where M_n is the maximum of $|P_n(z)|$ on the unit circle. They also obtain the following results on the location of the roots of $P_n(z) = 0$. The point $z = 1$ is the only positive root and the only root on the unit circle. Each of the sectors $(2k-1)\pi/n < \arg z < (2k+1)\pi/n$, $k = 1, 2, \cdots, n-1$, contains exactly two roots. There are, therefore, no negative roots when n is odd. But there are two negative roots when n is even, and their distance from the point $z = -1$ is asymptotically equal to $(\log n)/n$, as $n \to \infty$. Finally, all roots, other than $z = 1$, lie at a distance from the unit circle which is less than $(4+\epsilon)(\log n)/n$ and greater than $[(2e)^{1/2} - \epsilon]/n$, if n is sufficiently large.

10. *The abstract definition of conditional probability and conditional expectation*, by Dr. Shu-Teh C. Moy, Wayne University, introduced by the Secretary.

The definition of the conditional expectation of a random variable, or the conditional probability of an event with respect to another random variable, is based on the Radon-Nikodyn theorem of derivatives. For random variables which take values in a separable, reflexive Banach space, a similar theorem of derivatives is proved. Based on this theorem, the conditional expectation of a Bochner integrable function can be similarly defined.

11. *A continued fraction for periodic rent, logarithms, and roots*, by Professor J. S. Frame, Michigan State College.

The continued fraction defined recursively by setting $f_k(n, r) = f_k = (n^2-k^2)r^2/((2k+1)(2+r)+f_{k+1})$, is rapidly convergent if $x = nr < 1$ and $r < 1$, and it converges for all $r > -1$ when $n = 0$. It provides a rapidly convergent expression for the total repayment $T_{n,r} = n/a_{\overline{n}|r} = 1+(n+1)r/2+f_1(n, r)/2$, on a unit loan repaid in n equal installments with compound interest at rate r per period. In fact, if $x = nr < 1$ and $r < 1/2$ the fourth convergent has an error of about $2(x/10)^{10}$ and gives $T_{n,r}$ to ten significant figures. Letting n approach 0 yields $\ln(1+r) = r/T_{0,r}$, convergent for all $r > -1$. Setting $n = x/r$, and letting r approach 0, gives the familiar continued fraction $f_1 = x \coth(x/2) - 2$ where $f_k = x^2/(4k+2+f_{k+1})$. Finally a rapidly convergent root extraction method is defined by $(c^m+r)^{1/m} = c+2cr/(mb-r-F_1)$ where $b = 2c^m+r$ and $F_k = (k^2m^2-1)r^2/((2k+1)mb-F_{k+1})$.

12. *The Jacobi integral in the restricted problem of three bodies applied to elliptic orbits*, by Professor G. P. Loweke, Wayne University.

The Jacobi integral in the restricted problem of three bodies can be applied to elliptic orbits of the finite masses at their maximum and minimum distances by observing that at these extreme positions the orbits of the two masses reduce to circles. At minimum distance corresponding to perihelion the angular acceleration is zero, changing from positive to negative at this point, and the motion is perpendicular to the radius vector. At a point corresponding to aphelion this condition again exists, the angular acceleration changing from negative to positive at this point. The surfaces of zero relative velocity for elliptic orbits can now be compared at these two points. It can be shown that the error involved by assuming a circular solution is a function of the eccentricity of the orbits of the finite masses, e^3. When applied to the major planets the maximum per cent error by disregarding the eccentricity is as follows: For the orbit of Mercury, 75%; Venus, 2%; Earth, 5%; Mars, 30%; Jupiter, 15%; Saturn, 18%; Uranus, 15%; Neptune, 3%; Pluto, 95%.

S. D. CONTE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 11–12, 1955, at Tennessee Polytechnic Institute, Cookeville, Tennessee. Professor W. L. Williams, Chairman of the Section, and Professor R. H. Moorman, Vice-Chairman, presided over the general sessions; Professors R. W. Cowan, L. A. Dye, Tomlinson Fort, C. G. Phipps, E. B. Shanks and W. S. Snyder presided over subsections.

There were about 180 in attendance including the following 99 members of the Association:

R. H. Ackerson, S. A. Anderson, R. W. Ball, R. C. Boles, R. D. Boswell, Jr., Floyd Bowling, M. G. Boyce, C. L. Bradshaw, J. P. Brewster, G. M. Brown, J. W. Brown, W. Evelyn Brown, C. W. Bruce, B. F. Bryant, L. Virginia Carlton, H. P. Carter, R. S. Christian, D. H. Clanton, B. G. Clark, J. A. Cooley, C. L. Cope, R. J. Cormier, R. W. Cowan, H. J. Dark, J. M. Doran, L. A. Dye, E. D. Eaves, F. A. Ficken, M. K. Fort, Jr., Tomlinson Fort, Emma W. Garnett, R. L. Gay, I. C. Gentry, S. T. Gormsen, W. W. Graham, E. H. Hadlock, O. G. Harrold, Jr., T. W. Hildebrandt, A. T. Hind, Jr., J. H. Hoelzer, A. S. Householder, G. B. Huff, J. A. Hyden, John Jones, Jr., F. W. Kokomoor, Stephen Kulik, H. T. LaBorde, J. W. Lagrone, C. G. Latimer, H. L. Lee, D. B. Lowdenslager, G. H. Lundberg, Elna B. McBride, J. B. McGrory, S. W. McInnis, E. J. McShane, R. C. Meacham, G. W. Medlin, E. P. Miles, Jr., D. D. Miller, R. H. Moorman,

J. A. Nohel, J. D. Novak, C. R. Partington, G. L. Pate, P. B. Patterson, Lillian G. Perkins, C. G. Phipps, R. B. Plymale, Ellen F. Rasor, L. T. Ratner, H. A. Robinson, J. U. Russell, E. B. Shanks, D. C. Sheldon, A. R. Sloan, H. L. Slonecker, Jr., C. B. Smith, W. S. Snyder, D. E. South, F. W. Stallard, E. L. Stanley, L. W. Stark, W. G. Stokes, W. L. Strother, J. M. Thomas, C. W. Thomson, J. Clifton Thurman, A. W. Tucker, R. Z. Vause, Jr., Ruric E. Wheeler, M. C. Wicht, W. L. Williams, R. A. Willoughby, R. L. Wilson, F. J. Witt, M. D. Woodard, F. L. Wren, B. K. Youse.

The following officers were elected for the coming year: Chairman, Professor R. H. Moorman, Tennessee Polytechnic Institute; Vice-Chairman, Professor D. F. Barrow, University of Georgia; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *Three dimensional relaxation*, by Professor Ray Kinslow, Tennessee Polytechnic Institute, introduced by the Secretary.

The method of systematic relaxation of constraints has been found very useful in solving two-dimensional and axially symmetrical field problems. A technique is described by which the relaxation method may be applied to general three-dimensional fields. The relaxation net, which is cubical, is drawn oblique in such a manner that the computer need not visualize the three dimensions, but may perform the operations as if a hexagonal net had been employed in a plane-potential problem. A problem involving Laplace's equation is illustrated.

2. *Transposition theorems for linear equations and inequalities*, by Professor A. W. Tucker, Princeton University (By invitation).

In euclidean vector n-space the closed convex set $\{U \mid U = A_1x_1 + \cdots + A_px_p + B_1y_1 + \cdots + B_qy_q;\ x_i \geqq 0, \sum x_i = 1;\ y_j \geqq 0\}$ contains $U=0$ *or* a shortest vector $\neq 0$ that makes an acute angle with each A_i and a nonobtuse angle with each B_j. So, $AX+BY=0$ has a solution $X \geqq 0, \sum x_i = 1$, $Y \geqq 0$ *or* $U'A > 0$, $U'B \geqq 0$ has a solution U [Motzkin, Thesis, 1936]. Cases: (1) B vacuous [Gordan, *Math. Ann.*, vol. 6, 1873, Stiemke, *Math. Ann.*, vol. 76, 1915; Dines, *Bull. A.M.S.* vol. 42, 1936]; (2) $A = -B_0$ (a vector) [Farkas, *Crelle*, vol. 124, 1902; Weyle, *Comm. Math. Helv.*, vol. 7, 1935]; (3) $B = I$ (identity matrix) [von Neumann-Morgenstern, *Theory of Games*, 1944]; (4) $A = S+I$, $B=(S, I)$, S a skew-symmetric matrix [new sharpened existence theorem for games and linear programming].

3. *Mathematics for computers*, by Dr. A. S. Householder, Oak Ridge National Laboratory.

Suggestions are made for modifying the content of conventional courses in mathematics which, without detriment to the mathematical quality of the courses, would provide better preparation for students in the sciences and for those who will find employment as mathematicians in industrial and governmental laboratories. These suggestions include the development of analytical geometry in terms of matrices, the parallel development in calculus of differentials and of finite differences, of analytical and of numerical quadrature, of differential and of difference equations, and the introduction of the qualitative theory of equations with de-emphasis of closed solutions.

4. *On the equivalence of the usual definitions of conic sections*, by Professor G. B. Huff, University of Georgia.

In the standard textbooks on analytic geometry it is customary to begin by defining a conic section as a section of a right circular cone by a plane. The parabola, ellipse, and hyperbola are then defined again and studied as plane loci. Professor Huff pointed out the existence of simple figures which permit heuristic proofs of the equivalence of the two kinds of definitions.

5. *Mathematical education behind the iron curtain*, by Professor Stephen Kulik, University of South Carolina.

The paper presents the system of elementary and university education in the U.S.S.R. and in the Ukraine in particular. Mathematical programs and methods of teaching and organization of research work in mathematics are discussed.

6. *A new course in fundamental concepts of mathematics*, by Professor R. H. Moorman, Tennessee Polytechnic Institute.

Tennessee Board of Education requires that all who are certified to teach in Tennessee shall have had three quarter hours in *Fundamental Concepts of Mathematics*. Professor Moorman has developed such a course around the meaning and history of *number, measurement, the function concept*, and *the nature of proof*. An attempt is made to explain mathematics to students majoring in elementary education, physical education, music, *etc.*, who have previously taken no mathematics in college and almost none in high school. The course is proving to be a good orientation into mathematics for general education students.

7. *Stability of the motion of an infinitesimal body*, by Professor W. L. Williams, University of South Carolina.

In this paper, the motion of an infinitesimal body, subject to the attraction of three bodies with finite masses, is studied with particular emphasis on the areas where the motion is stable.

8. *An inductive approach to the real number system*, by Professor C. G. Phipps, University of Florida.

It is fashionable in modern mathematics to emphasize the axiomatic method. Consequently authors seldom show the inductive approach to the axioms used. This report outlines such an approach to the axioms of the real number system as an aid in a better understanding of mathematical processes.

9. *Trigonometry for calculus students*, by Mr. R. D. Boswell, Jr., University of Georgia.

The sine and cosine functions are defined by the usual power series expansion. Using elementary theorems of calculus, the basic properties of these functions are proved.

10. *Algebra courses for juniors and seniors*, by Professor R. W. Ball, Alabama Polytechnic Institute.

In the curriculum for junior and senior mathematics majors, it is suggested that the specialized courses in the theory of equations, matrices and determinants, *etc.* be replaced by a reasonably abstract course in modern algebra. Among the advantages in this replacement are economy of course offering for schools with a limited graduate program, a workable description of the nature of numbers for those who plan to teach arithmetic and beginning algebra, an early introduction to the notion of a mathematical proof, and a course which is often pleasantly different from the previous courses taken by the student.

11. *Excruciating problems*, by Professor R. A. Willoughby, Georgia Institute of Technology.

A number of examples of freshman and sophomore problems are presented. These problems tend to trap students who are sloppy in their analysis of problems and who depend blindly on memorized procedures.

12. *Invariants of certain linear differential equations*, by Professor R. W. Cowan, University of Florida.

Coefficients of the normal form of several linear differential equations are shown to be invariant under a linear transformation of the dependent variable. A proof by mathematical induction is supplied to enable one to determine simply the adjoint equation. Relations are established involving invariants and the coefficients of the adjoint equation.

13. *Note on Liapounov's second method*, by Professor J. A. Nohel, Georgia Institute of Technology.

Consider the system of nonlinear differential-functional equations (1) $x'=f(t, X(t))$, $'=d/dt$, where x and f are real vectors with n components, t is a real scalar, and $X(t)$ denotes the set of functions $\{x_h[u_k(t)]\}$, $(h=1, \cdots, n, k=1, \cdots, m)$. The functions f_j, $j=1, \cdots, n$, and u_k, $k=1, \cdots, m$, are given. In addition to assumptions which guarantee existence and uniqueness of solutions (see J. Franklin, "*On the existence of solutions of systems of functional differential equations*," Proc. of the A.M.S., vol. 5, 1954) suppose that $f(t, 0)=0$; then $x=0$ is a solution of (1). Sufficient conditions for the stability and asymptotic stability of the zero solution are given using a slight generalization of Liapounov's "Second Method" for ordinary differential equations.

14. *Differential equations with interfaces between regions*, by Mr. C. L. Bradshaw, Oak Ridge National Laboratory.

Reactor calculations in two or three dimensions give rise to the mathematical problem of determining the fundamental eigenvalue of a system of equations. A simple problem in two dimensions and two regions, where complete analytic solutions are possible, has been solved. Iterative methods for solving this problem have been evaluated as to rapidity of convergence and ease of preparation for a high speed computer.

15. *What is a solution of a differential equation?*, by Professor Tomlinson Fort, University of South Carolina.

In this paper the author considers the "solutions" usually given in text-books. He shows that not only are definitions frequently carelessly worded but that many functions not mentioned in the more commonly used text-books must be considered solutions under the definitions that the authors do give.

16. *A note on uniform spaces, by* Mr. B. F. Bryant, Vanderbilt University.

Let X be compact uniform space which is not metrizable, and let the collection $\{V_\alpha\}$ be the subsets of X^2 which determine the topology of X. It was shown that if f is a self-homeomorphism of X, then for each V_α there exists (x_α, y_α) such that the orbit of (x_α, y_α) under the homeomorphism $g(x, y)=(f(x), f(y))$ is contained in V_α. This implies that if X is a compact uniform space on which it is possible to define an unstable self-homeomorphism, then X is metrizable.

17. *A note on vector spaces*, by Mr. R. H. Ackerson, Alabama Polytechnic Institute.

Let V be a vector space of dimension n over a field F, and A a linear transformation of V into itself. Let $R(A)$ denote the range of A, with dimension $r(A)$; and $N(A)$ denote the null space of A. The following theorem is proved: Let $I(A)=R(A)\cap N(A)$. The dimension of $I(A)=r(A)-r(A^2)$. The result is obtained without use of canonical forms of matrices. Some corollaries are given, including the equivalence of the three conditions $V=R(A)\oplus N(A)$, $I(A)=0$, and $r(A)=r(A^2)$.

18. *Essential mappings*, by Professor M. K. Fort, Jr., University of Georgia.

Let A be a finite subset of the plane P. Then there exists a continuous mapping f of a circle

S into $P-A$ such that f is not homotopic to a constant in $P-A$, but such that for every proper subset B of A, f is homotopic to a constant in $P-B$.

19. *Mappings with property "T,"* by Professor E. B. Shanks, Vanderbilt University.

A triple of real numbers has property "T" provided each one of the triple is the number of units in the length of one side of some fixed triangle (such a triple will be called a triangle of real numbers). A mapping of the set of positive real numbers into itself has property "T" provided each triangle of real numbers is mapped into a triangle of real numbers. Results were presented relative to the class of mappings so defined.

20. *What is linear programming?*, by Professor F. A. Ficken, University of Tennessee.

A "transportation problem" gives warehouses $\alpha(\alpha=1, \cdots, m)$ with capacities w_α, factories $j(j=1, \cdots, n)$ with products p_j, and unit shipment costs $c_{\alpha j}$ to α from j, and requires shipments $s_{\alpha j} \geqq 0$ such that $\sum_j s_{\alpha j}=w_\alpha$, $\sum_\alpha s_{\alpha j}=p_j$, and the total shipping cost $\sum_{\alpha,j} c_{\alpha j}s_{\alpha j}$ shall be a minimum. The general problem of linear programming requires an extreme value of a linear functional subject to a given system of linear constraints (equalities or inequalities). The object here is to illustrate the problem and in extremely simple terms to sketch and illustrate the "simplex method" of arranging calculations leading to a solution.

21. *Committee on the undergraduate mathematics program*, by Professor E. J. McShane, The University of Virginia. (By invitation.)

Professor McShane reported on the earlier meetings of the Committee on the Undergraduate Mathematical Program and on the material prepared by the group working last summer in Lawrence, Kansas. He also described the nature of Part II of the experimental text materials, now in preparation at Tulane University, and gave some examples of the type of applications to genetics, political science, *etc.* which it will contain.

22. *An orthotropic plate with a parabolic notch*, by Professor C. B. Smith, University of Florida.

A large orthotropic plate contains a notch in the form of a parabola with its principal axis parallel to an axis of elastic symmetry of the plate. The plate is taken to be in a state of plane stress under the action of forces applied along the boundary of the notch. The resulting stress distribution is discussed theoretically by using functions involving two complex variables.

23. *A note on functional completeness II*, by Professor R. A. Willoughby, Georgia Institute of Technology.

Given the set $S=\{0, 1, 2, 3\}$ and the operation $\times$ such that $a\times 0=0\times a=0$, $a\times 1=1\times a=a$, $a\times a=a$, $2\times 3=3\times 2=0$, there does not exist a $0\to 1$ cyclic permutation $\frown$ on S for which $(S, \times, \frown)$ is functionally complete.

24. *A basic set of polynomial solutions for the Euler Poisson Darboux equation*, by Professors E. P. Miles, Jr. and Ernest Williams, Alabama Polytechnic Institute, presented by Professor Miles.

The authors consider the Cauchy Problem for the E.P.D. equation: $u(x_1, x_2, \cdots, x_m, 0)=f(x_1, x_2, \cdots, x_m)$, $u_t(x_1, x_2, \cdots, x_m, 0)=0$, $L_k[u]=\sum_{i=1}^m u_{x_ix_i}-u_{tt}+kt^{-1}u_t=0$. If $k>0$, f is analytic, and a unique analytic solution exists, the homogeneous polynomials of degree n in the expansion of $u(x_1, x_2, \cdots, x_m, t)$ are linear combinations of the following set of solutions for $L_k(u)=0$. For any set of non negative integers b_j such that $b_0=0$, $\sum_{j=0}^m b_j=n$ let

$$H^{*n}(k, t, x_1, \cdots, x_m) = \sum \left[n!(a_0/2)! \Big/ \prod_{j=0}^{m} a_j![(b_j - a_j)/2]!t^{\{a_0\}} \prod_{j=1}^{m} x_j^{a_j} \right]$$

where $t^{\{a_0\}} = [1 \cdot 3 \cdot 5 \cdots (a-1)/(1+k)(3+k) \cdots (a-1+k)]t^{a_0}$ and the summation extends over all a_j such that (1) $a_j \equiv b_j$, mod 2, $j=0, 1, 2, \cdots m$, (2) $\sum_{j=0}^{m} a_j = n$, and (3) $a_j \leqq b_j$, $j=1, 2, \cdots m$. For $k=0$ these polynomials form a portion of the authors' basic sets for the wave equation.

25. *A theorem on disks*, by Professor O. G. Harrold, Jr., University of Tennessee.

Let J be a topological line (unbounded in both directions) in three-space. Does there exist a disk (closed, topological 2-cell) meeting J in a single point whose boundary links J? It is proved that if there is some homeomorphism h of space on itself such that the image of some pair of points on J defines a rectifiable sub-arc of $h(J)$, such a disk exists, even a tame one.

26. *A sound mathematics curriculum for a well-staffed four year liberal arts college*, by Professor S. T. Gormsen, Rollins College.

Professor Gormsen led a general discussion relative to courses which all mathematics and science majors should take and the mathematics courses recommended for majors in other fields. It was agreed that more mathematics on all levels, and not less, should be taught. We should give more 20th and less 17th century mathematics. Linear algebra is as important perhaps as calculus.

27. *Generalized limit procedures*, by Professor E. J. McShane, The University of Virginia. (By invitation.)

This talk was devoted to the presentation of a theory of limits which is of considerable generality and at the same time is intended to be easily understood by beginners. A "system of stages" is a collection of one or more sets, each having one or more points in it, such that if S_1 and S_2 are members of the system, some set S_3 of the system is contained in both S_1 and S_2. A "limit process" consists of a function f and a system $\mathfrak{S}$ of stages in the domain D on which f is defined; if f is real-valued, it has L as limit if for every $\epsilon > 0$, there is a stage S in the system $\mathfrak{S}$ such that $|f(x) - L| < \epsilon$ for every x in S. The usual theorems on limits of sums, *etc.*, can be proved easily. The theory covers the applications to sequences, series, integrals, *etc.*, without mental contortions.

28. *A simple proof of the Euler transformation*, by Professor Tomlinson Fort, University of South Carolina.

The Euler transformation of a convergent infinite series is formally established by means of the summation by parts formula. The remainder after n terms appears in a form which can be shown to approach zero in two or three lines.

29. *On a series for* $(1+k)^k$, by Professor D. E. South, University of Florida.

Using the moment generating function, one can show that the kth moment about the origin of a Bernoulli distribution has the form $\sum_{i=1}^{k} A_{ki} r^{(i)} p^i$. In order to determine the coefficients A_{ki}, the fundamental relation between the operators Δ and E of the calculus on finite differences and the theorem on the nth difference of a polynomial were used to show that $(1+k)^k$ can be expressed as

$$\sum_{i=0}^{k} (-1)^{i+2} \binom{k+1}{i+1} (k-i)^k, \quad k \text{ being a positive integer.}$$

30. *Machine computation of definite integrals*, by Mr. C. L. Gerberich, Oak Ridge National Laboratory, introduced by the Secretary.

Very often in the field of scientific computations there will arise definite integrals that will not

lend themselves to an analytic treatment. It is therefore necessary to pick from among the varied numerical methods those which will best fit the needs of a high speed digital computer. Of the methods studied it was found that Gauss's quadrature formula is by far the superior as a general purpose method. The only type problems that show the Gauss method at a disadvantage are those with integrands given only at equally spaced points. Such problems may arise when the integrand is given by experimental data.

31. *Computing higher mathematical functions*, by Mr. S. G. Campbell, Oak Ridge National Laboratory, introduced by Dr. A. S. Householder.

The most generally useful analytic methods for calculating higher mathematical functions are series methods (power series, asymptotic series, hypergeometric series, *etc.*), approximating polynomials (particularly Legendre and Tchebycheff polynomials), and continued fractions. Numerical methods include interpolation, successive approximation, numerical quadrature, and numerical solution of differential equations. None of these methods possesses over-all superiority. The use in modern science of increasingly complicated higher mathematical functions, often with complex argument, and of high speed digital computers to evaluate such functions, indicates that an increasing percentage of students who become professional mathematicians will need to know and be able to evaluate these computational techniques.

32. *An involutorial transformation of order 5 associated with a pencil of twisted cubics*, by Professor L. A. Dye, The Citadel.

A 1-1 correspondence between a pencil of twisted cubics and a pencil of quadric surfaces is used to determine an involutorial cremona transformation of order 5. The base of the web of homoloids consists of two twisted cubics, one of which is double, and 5 simple lines, three of which are parasitic.

33. *A note on starlike functions*, by Professor M. C. Wicht, North Georgia College.

The author applies Royster's condition for starlikeness (*Duke Journal*, September, 1952) of mappings with respect to several families of directed curves including the solutions of $\dot{z}-Pz=Q$, where $\dot{z}=dz/dt$, $P=P(t)$, $Q=Q(t)$ and $z=x(t)+iy(t)$ with $x(t)$ and $y(t)$ real. The condition reduces to

$$(Px+Q)S_h+yS_v=0$$

where $S_h=$Real Part $[-iF'(z)/F(z)]$ and $S_v=$Real Part $[F'(z)/F(z)]$. The relations between this condition for starlikeness on these families and those for starlikeness on families of vertical and horizontal lines was noted.

34. *On matrices all of whose characteristic roots lie within the unit circle*, by Professor H. T. LaBorde, The University of the South.

A problem arising in economics is one of determining sufficient criteria that a given matrix have all of its characteristic roots in or on the unit circle. If $A=(a_{ij})$ is a real square matrix of order n, let $R_i=\sum_{j=1}^{n}|a_{ij}|$, $S_i=\sum_{j=1}^{n}|a_{ji}|$ $(i=1, 2, \cdots, n)$. It is proved here that a matrix can have as many as one-half the R_i and one-half the S_i greater than unity and still have all of its characteristic roots lie within the unit circle.

35. *On the construction of a ternary quadratic form*, by Professor E. H. Hadlock, University of Florida.

Given the invariants Ω and Δ associated with a ternary quadratic form, let certain restrictions be imposed on the first coefficient of the form and the third coefficient of its reciprocal form. Then

formulas are obtained for each of the five remaining coefficients of the form and also for each of the five remaining coefficients of the reciprocal form.

36. *Lipschitz conditions for a piece-wise smooth function*, by Professor W. S. Snyder, University of Tennessee.

Professor Snyder demonstrated the theorem: Let R be a convex body in euclidean N space, E^N, and f a real-valued continuous function defined on R. Let S be a Lipschitzian surface in E^N, *i.e.*, $S=G(Q)$ where G is a Lipschitzian vector function defined from the unit cube Q of E^{N-1} to E^N. Assume there is a constant M such that at each $x \in R-S$ f has first partial derivatives whose absolute values do not exceed M. Then f satisfies a Lipschitz condition in R in the sense that if L is a segment and $L \subset R$ then on L, $|\Delta f| \leq M|L|\sqrt{N}$.

37. *A geometric approach to the derivation of formulae for derivatives of an analytic function using polar coordinates*, by Professor S. T. Gormsen, Rollins College.

This paper illustrates a rigorous geometric approach to the derivation of the one-directional formulae for the derivatives of analytic functions using polar coordinates. As a biproduct a direct derivation of the necessary Cauchy-Riemann conditions are established.

38. *On some questions related to convexity*, by Professor L. T. Ratner, Vanderbilt University.

Several theorems pertaining to convexity of functions and of sets are generalized and extended to include approximate convexity. Sets which are approximately convex according to either of two definitions are characterized in new terms, and the exact relationship between the two definitions of approximate convexity is determined. Finally, a study is made of local approximate convexity.

39. *Subsets of Peano spaces*, by Professor W. L. Strother, University of Miami.

By $S^1(X)$ Professor Strother denotes the space of non-null closed subsets of the space X and for $n>1$ he defines $S^n(X)$ to be the space of non-null closed subsets of $S^{n-1}(X)$. He establishes the theorem: If X is a Peano space, then $S^n(X)$ is a Peano space.

H. A. ROBINSON, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-second annual meeting of the Iowa Section was held jointly with the sixty-seventh annual meeting of the Iowa Academy of Science and the twenty-second convention of the Junior Academy of Science at St. Ambrose College, Davenport, Iowa, on April 15 and 16, 1955. The Chairman, Professor H. T. Muhly, the Vice-Chairman, Professor F. A. Brandner, and Reverend T. J. Taylor presided in turn. The officers elected are: Chairman, Professor F. A. Brandner, Iowa State College; Vice-Chairman, Professor R. S. Jacobsen, Luther College; and Secretary, Professor Fred Robertson, Iowa State College.

The registered attendance was 39 including the following 28 members:

H. G. Apostle, Barbara J. Beechler, R. H. Bing, F. A. Brandner, E. L. Canfield, J. O. Chellevold, E. W. Chittenden, Byron Cosby, Jr., A. T. Craig, W. M. Davis, Kathryn P. Ellis, J. J. L. Hinrichsen, R. V. Hogg, Rev. J. A. Hratz, L. A. Knowler, O. C. Kreider, H. T. Muhly, E. N. Oberg, Fred Robertson, W. C. Ross, Jr., Hazel M. Rothlisberger, D. E. Sanderson, J. A. Schumaker, R. C. Seber, F. M. Stein, H. P. Thielman, H. J. Weiss, Roscoe Woods.

The following papers were presented:

1. *The approximate solution of integro-differential equations*, by Mr. F. M. Stein, Iowa Wesleyan College.

It is known that, under certain conditions, the integro-differential equation

$$U(u) \equiv L(u) - \int_a^b h(x, t)u(t)dt = f(x)$$

with boundary conditions $U_i(u)=0$, $(i=1, 2, \cdots, m)$ has a unique solution of the form $u(x) = \int_a^b H(x, t)f(t)dt$, where $H(x, t)$ is the Green's function of the problem. This paper examines the existence, uniqueness and conditions for convergence of a polynomial which satisfies the boundary conditions and the criterion that

$$\int_a^b |f(x) - U[P_n(x)]|^r dx$$

shall be the least for $r>0$. An analogous problem is also examined when the approximating function is a trigonometric sum. When $r=2$ the coefficients are actually determined.

2. *Bending of a rectangular plate with even and odd order boundary conditions*, by Mr. J. P. Li, Iowa State College, introduced by the Secretary.

Bending of a rectangular plate with two adjacent edges clamped can be solved systematically by Fourier method. The deflection of the plate is taken in the form $w=w_1+w_2$ where w_1 is the deflection of a simply supported plate and w_2 represents a correction state which is chosen as two infinite series of particular solutions of the biharmonic equation. By adjusting the values of coefficients of w_2 by ordinary Fourier analysis, all of the specified boundary conditions can be completely satisfied. Examples are worked out for a rectangular plate subjected to uniformly distributed load with two adjacent edges clamped and the others either simply supported or free.

3. *Beams of uniform strength subjected to uniformly distributed loads*, by Mr. J. P. Li and Mr. W. A. Gross, Iowa State College, presented by Mr. Gross, who was introduced by the Secretary.

Beams of uniform strength in bending have the same maximum flexural stress at any cross section. Shapes, deflections, and weights for cantilever, simple, and fixed beams of uniform strength are derived taking into account the weights of the beams. Explicit solutions are given for certain beams of rectangular cross sections and constant height or constant width.

4. *A network for representing elastic bodies in spherical coordinates*, by Mr. W. A. Gross, Iowa State College.

A network representing elastic bodies having linear stress relations and described in spherical coordinates is developed. The case of axial symmetry is treated. Equilibrium is satisfied by a finite difference grid which assures compatibility of strain. Usual boundary conditions may be fitted.

5. *A remark on integrally closed local domains*, by Professor H. T. Muhly, State University of Iowa.

Let L be a complete two dimensional local domain which is integrally closed in its quotient field, and let m be the ideal of non-units in L. Assume that L contains a field k over which the residue field L/M is finite and algebraic. Under these assumptions it follows that if x, y is any system of parameters for L, then L is a regular module over the power series ring $k\{x, y\}$ in the sense that L admits an independent basis over $k\{x, y\}$.

6. *A note on function spaces*, by Professor H. A. Dye, State University of Iowa, introduced by the Secretary.

This is to call attention to a theorem, characterizing compact spaces in terms of allied structures, which does not seem to be stated in the extensive literature on this subject. For X a compact Hausdorff space, denote by $P(X)$ the (multiplicative) group of all positive continuous functions on X. Call a group isomorphism ϕ of $P(X)$ on $P(Y)$ bounded if, under both ϕ and its inverse, the unit sphere is transformed into a bounded set. The theorem asserts that each bounded isomorphism between $P(X)$ and $P(Y)$ is implemented (in the natural sense) by a homeomorphism of X on Y.

7. *On testing hypotheses about a certain type of truncated distribution*, by Professor R. V. Hogg, State University of Iowa.

Let the results of an experiment be described by the known positive density $g(x)$. Due to some limitations, possibly physical, we are able only to observe results of the experiment between two fixed, but unknown, values, say c and d. We consider two hypotheses: 1) that m_0 is the median of this truncated distribution; and 2) that $c=c_0$, $d=d_0$. We test these hypotheses against all possible alternatives by use of the likelihood ratio λ. If n is our sample size, we find that for the first hypothesis the principal nth root of λ, say t, has the density $(n-1)^{n-2}$, $0<t<1$, if the hypothesis is true. In the second case, t has the density $n(n-1)t^{n-2}(1-t)$, $0<t<1$. In both instances, the interesting fact to observe is that while the computation of t depends upon the known density $g(x)$, the distribution of t is free of this density.

8. *A property of the median*, by Professor A. T. Craig, State University of Iowa.

In a regular case of estimation, let a random variable have non-degenerate probability density which depends on a parameter θ for which there exists a sufficient statistic. It is proved that the probability density of the median of a random sample cannot be free of θ. Thus the median and the sufficient statistic are always stochastically dependent.

9. *Some remarks on the December Meeting of the Board of Governors of the Mathematical Association of America*, by Professor E. N. Oberg, State University of Iowa (Section Governor).

10. *Designing a mathematics curriculum for students of the non-physical sciences*, by Mr. Robert Seber, State University of Iowa. (By invitation.)

We consider a mathematics curriculum with respect to (I) topical content, (II) style of presentation, and (III) illustrative material. Attention is called to two types of course sequences built about a fixed core of topics and denoted as "traditional" and "unified." Dissatisfaction with current curricula is centered upon (II) and upon the non-inclusion and non-emphasis of topics not represented in the core.

We view a curriculum as a collection of statements presented verbally and symbolically. Certain terms of mathematics are the referents of terms used in the non-physical sciences. It is in this sense that mathematics is applied in these sciences. This application is considered in two categories: (A) quantitative description of phenomena of non-physical science and (B) mathematical models to schematize non-physical science phenomena.

A course sequence with a prerequisite of college algebra and trigonometry is proposed and considered relative to (I), (II), and (III). This sequence includes topics of algebra, geometry, analysis, and statistics. The course structure is classified as "unified." The standard core of topics is retained but is not treated as extensively. Emphasis is placed on the style of presentation which is for each new topic on "explication" in the sense of Carnap. Topics of modern algebra, statistics and probability are an integral part of the course. Each topic is considered with reference to its descriptive

background, formalization as a mathematical system, and application in describing phenomena of non-physical science. Types of illustrative material relative to applications are presented in detail for some topics.

11. *Developing new mathematics*, by Professor R. H. Bing, University of Wisconsin. (By invitation.)

This discussion emphasizes the fact that mathematics is an alive and growing subject. The following things are considered: examples of new mathematics recently discovered, unsolved problems that are being attacked, the amount of new mathematics being developed, experiences of researchers in developing new mathematics, the thrill that students experience in inventing mathematics, the role of teachers in leading students to see the beauty of mathematics.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Georgetown College, Georgetown, Kentucky, on April 30, 1955. Professor Charles Hatfield, Chairman of the Section, presided at both the morning and afternoon session.

There were 49 persons in attendance, including the following 32 members of the Association:

R. C. Brown, Jr., H. W. Burnette, Esther A. Compton, J. B. Cornelison, V. F. Cowling, H. H. Downing, J. C. Eaves, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Tadeusz Leser, E. J. Mickle, W. L. Moore, R. S. Park, Sallie E. Pence, Mary Pettus, J. D. Riley, G. G. Roberts, W. J. Robinson, D. C. Rose, M. I. Rose, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, K. E. Stoll, Louise C. Stolle, W. C. Swift, J. T. Vallandingham, J. A. Ward, T. M. Wright, W. M. Zaring.

At the business meeting the following officers were elected for the coming year: Chairman, Professor V. F. Cowling, University of Kentucky; Secretary-Treasurer, Professor A. W. Goodman, University of Kentucky; Traveling Lecturer, Professor J. C. Eaves, University of Kentucky.

The following papers were presented:

1. *Isometric mappings*, by Professor A. W. Goodman, University of Kentucky.

Definitions were given of metric space, Hilbert space, and isometric mapping, and an example was given of a space of four points which cannot be mapped isometrically on any sub-space of Hilbert space.

2. *Connected ideals in compact connected mobs*, by Professor W. M. Faucett, University of Kentucky, introduced by the Secretary.

A mob is a Hausdorff space that admits a continuous associative multiplication. It is known that if a mob S is connected and possesses a minimal ideal K, then K is connected. The purpose of this note is to state the result that if S is compact, connected, and $S^2=S$, then every maximal ideal of S is connected. An example is given to show that with the same hypothesis S may contain disconnected ideals.

3. *Heredity in mathematics*, by Professor R. W. Bagley, University of Kentucky.

Some elementary notions concerning hereditarily separable spaces are discussed.

4. *On the product of linear forms*, by Mr. W. M. Zaring, University of Kentucky.

Let a_{ij} and b_i, $i, j=1, \cdots, n$ be real numbers. It has been conjectured that

$$\left| \prod_{j=1}^{n} (L_i(X) - b_i) \right| \leq \frac{|D|}{2^n}$$

has integral solutions X_i, $i=1, \cdots, n$, where $D=|a_{ij}| \neq 0$ and $L_i(X)=\sum_{j=1}^{n} a_{ij}X_j$. Minkowski first proved the result for $n=2$. The author presents R. Bellman's proof (*A note on the product of linear forms*, this MONTHLY, vol. 51, 1944, p. 161) that the conjecture is true if the a_{ij}'s form an orthogonal transformation.

5. *The zeros of polynomials*, by Professor V. F. Cowling, University of Kentucky.

In this paper a discussion is given of the various types of problems concerned with the distribution of the zeros of polynomials. Particular emphasis is placed on those results in which limitations on the distribution of the zeros are obtained respectively in terms of the moduli and arguments of the coefficients.

6. *The Gaussian integral factors of* $10i$, by Mr. H. W. Burnette, University of Kentucky.

The speaker discusses, briefly, the integral factors of composite rational integers, such as 24, pointing out which factors are non-trivial. Then he defines the necessary quantities to show how Gaussian integers of the form $a+bi$, where $a=0$, in particular, $10i$, may be factored into 17 essentially different integral factorizations.

7. *The Silverman-Toeplitz theorem*, by Mr. W. C. Swift, University of Kentucky.

The Silverman-Toeplitz theorem is a well known result. Except for one complication, the proof is straightforward. Aided by an example, the method of resolving this difficulty is suggested. An application illustrates the theorem.

8. *Modern teaching of survey courses*, by Professor Arno Jaeger, University of Cincinnati, introduced by Professor G. M. Merriam, University of Cincinnati.

This is a report on an elementary survey course for students who need mathematics as a college requirement only and not as a prerequisite for other subjects. The instructor felt very strongly that an introduction to modern mathematics is more suitable to demonstrate the nature of mathematics than a further elaboration of the naturally limited concepts of high-school mathematics. Even the freshmen understood the modern ideas and conceptions presented rigidly and often in Bourbaki's language, but with examples from ordinary life if possible. The students enjoyed making contributions by composing lecture notes, giving oral reports, or constructing visual aids.

9. *On Kasner's circle*, by Mr. M. I. Rose, University of Kentucky.

Let $w=u+iv$ be a polygenic function of the complex variable $z=x+iy$. Let the complex number $z+\Delta z$ approach z along a line of slope m. Then Kasner showed that the points $\alpha(m)+i\beta(m)$

describe a circle in the derivative plane. In this paper we study an analogous problem for the case in which w is a function of a hypercomplex variable $z=1x+iy$, having a multiplication table

	1	i
1	1	i
i	i	$a+bi$

and find that the set of points $\alpha(m)+i\beta(m)$ describe a conic section determined by a and b.

10. *A summer school program for in-service teachers*, by Professor J. C. Eaves, University of Kentucky.

In most cases in-service teachers must obtain their mathematics training in summer classes. Some of the courses available to them are courses that they have already had; other courses are strictly at the research level and the participant finds that he has insufficient background to profit from them. This is an attempt to combine topics from all of the various courses, selecting from both classical mathematics and modern mathematics. This will give the teacher studying in the summer a broader coverage of interesting material and will give him early in his training some idea of the modern concept of mathematics.

11. *Bounded variation and absolute continuity*, by Professor E. J. Mickle, Ohio State University. (By invitation.)

The definitions of bounded variation and absolute continuity for a real-valued function $y=f(x)$ were given. The generalization of these concepts by Tonelli to a function of two variables $z=z(x, y)$ were discussed. Examples were given to illustrate the difficulties encountered in generalizing these concepts to plane mappings. The results of Radó and Cesari in giving a complete generalization of bounded variation and absolute continuity were stated.

A. W. GOODMAN, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Morgan State College, Baltimore, Maryland, on April 16, 1955. The Section joined with the American Chemical Society (Maryland Section), National Institute of Science, Beta Kappa Chi Scientific Society, Maryland Conference of Biology Teachers, and the American Association of Physics Teachers, in the Dedication Program for Calloway Hall and the new Science Quadrangle at Morgan State College. Professor C. H. Frick, Chairman of the Section, presided at the morning session of the Section.

There were 48 persons in attendance, including the following 40 members of the Association:

R. P. Bailey, T. J. Benac, R. O. Blummer, Jr., B. H. Buikstra, L. G. Campbell, P. L. Chessin, P. J. Federico, J. F. Foley, Gloria C. Ford, C. H. Frick, Michael Goldberg, R. A. Good, J. R.

Gorman, J. R. Hammond, Louise S. Hunter, C. F. Koehler, W. W. Leutert, R. M. Mason, Carol V. McCamman, L. I. Mishoe, T. W. Moore, G. A. Paxson, J. W. Popow, W. W. Proctor, R. S. Raven, R. W. Rector, Sister Rita, H. R. Smith, W. S. Soar, C. F. Stephens, M. F. Stilwell, W. J. Strange, E. G. Swafford, J. A. Tierney, M. M. Torrey, Marcelle M. Walker, P. M. Whitman, Beryl W. Williams, R. H. Wilson, Jr., A. W. Yonda.

The following officers were elected to serve for a period of one year: Chairman, Professor F. E. Johnston, George Washington University; Vice-Chairmen, Professor Ella C. Marth, Wilson Teachers College and Mr. W. H. Norris, Jr., Maury High School, Norfolk, Virginia; Secretary, Professor R. P. Bailey, U. S. Naval Academy.

Miss Carol V. McCamman reported that 81 high schools participated in the Second Annual High School Contest given April 6, 1955. Attention was called to the excellent results obtained by Mr. R. O. Blummer and Mr. Paul Chessin in the campaign to interest industrial organizations in the contest, which now boasts an extensive list of prizes. The Section approved an amendment to the By-Laws to include the Sectional Governor among the members of the Executive Committee.

The following papers were presented:

1. *Digitalization of war games*, by Dr. W. W. Leutert, Chief, Computing Laboratory, Aberdeen Proving Ground.

An outline was presented of how a battle may be simulated on a high speed digital computer.

2. *Rotors, plane and fancy*, by Mr. Michael Goldberg, Department of the Navy. (By invitation.)

The history and development of rotors, which are generalizations of curves and surfaces of constant width, was discussed. Meissner derived a Fourier series expression describing the totality of rotors touching all sides of the regular plane polygons. Mr. Goldberg described his kinematic methods for obtaining these rotors and showed how they are used for obtaining rotors in spherical polygons also. Meissner showed that non-spherical rotors exist for the tetrahedron, the cube and the octahedron, but not the dodecahedron and the icosahedron. The works of Euler, Minkowski, Fujiwara, Blaschke, Meissner, J. W. Green and others on these and related problems were discussed. A large collection of plane, spherical and solid models was used to illustrate the lecture.

At luncheon, the Section and the other scientific societies participating heard Dr. F. G. Watson of Harvard University speak on *A Laboratory for Learning*. The joint afternoon session was addressed by Dr. G. P. Harnwell, President of the University of Pennsylvania, who spoke on *Implications of Current Developments in Physics*.

R. P. BAILEY, *Secretary*

Correction: In the report of the Fall meeting of this section, there is an error in abstract 1A, this MONTHLY, vol. 62, p. 211; in the first line the symbol (m/k) should read $\binom{m}{k}$.

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The fourteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Queens College, Flushing, New York, on April 30, 1955. Professor A. B. Brown, Collegiate Vice-Chairman of the Section, presided at the morning session and Professor H. F. Fehr, Chairman of the Section, presided at the afternoon session.

There were 116 persons in attendance, including the following 92 members of the Association:

Joseph Andrushkiw, R. G. Archibald, W. W. Bessell, Jr., S. I. Birnbaum, D. H. Blanksteen, Samuel Borofsky, C. B. Boyer, A. B. Brown, Azelle Brown, J. H. Bushey, Jewell H. Bushey, A. J. Carlan, Audrey M. Carlan, G. B. Charlesworth, Charles Clos, P. J. Cocuzza, H. J. Cohen, T. F. Cope, Demetrios Counes, W. H. H. Cowles, D. R. Davis, I. A. Dodes, J. N. Eastham, J. E. Eaton, W. H. Fagerstrom, H. F. Fehr, J. M. Feld, William Forman, R. M. Foster, Brother Linus R. Foy, D. H. Frank, E. T. Frankel, Leona Freeman, M. P. Friedman, Bernard Greenspan, Harriet M. Griffin, Laura Guggenbuhl, C. M. Hebbert, M. J. Hellman, G. C. Helme, E. Marie Hove, Joseph Jablonower, Aida Kalish, O. J. Karst, L. S. Kennison, G. A. Keyes, H. S. Kieval, A. E. Kinney, Charles Koren, A. T. Kovitz, A. W. Landers, C. H. Lehmann, C. B. Maile, Jr., J. H. Manheim, May H. Maria, F. H. Miller, A. J. Mortola, P. M. Moskowitz, C. J. Oberist, Eugene Odin, L. F. Ollmann, C. F. Pinzka, Walter Prenowitz, James J. Quinn, H. W. Raudenbush, C. F. Rehberg, Susan L. Reid, Selby Robinson, N. J. Rose, M. F. Rosskopf, H. D. Ruderman, J. P. Russell, John Salerno, Charles Salkind, A. H. Sarno, Abraham Schwartz, Aaron Shapiro, E. I. Shapiro, James Singer, Sister M. Anita, Sister Maria Loyola, Morris Smith, Mildred M. Sullivan, R. L. Swain, F. B. Taylor, M. Virginia Terhune, L. F. Tolle, Annita Tuller, R. M. Warten, M. E. White, J. M. Wolfe, Leo Zippin.

The following officers were elected for the year 1955–56: Chairman, Professor A. B. Brown, Queens College; Collegiate Vice-Chairman, Professor Mina S. Rees, Hunter College; High School Vice-Chairman, Dr. Barnet Rich, Brooklyn Technical High School; Secretary, Dr. Azelle Brown, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting reports were given by the secretary, the treasurer, the Committee on Contests and Awards, and the Committee on Coordinating Mathematical Training.

The following report on the activities of the Committee on Contests and Awards was given by its chairman, Professor W. H. Fagerstrom.

There were 881 schools registered for the sixth annual mathematics contest, which is sponsored by the Metropolitan New York Section. Of this number 479 schools registered with the Metropolitan New York Section and 402 schools registered with the eight other state units that operate independently, using the same questions and rules of the contest as used by the Metropolitan New York Section. These registrations were distributed as follows: British Columbia—50 schools, Colorado—65 schools, Illinois—85 schools, Manitoba—5 schools, Oregon—61 schools, Upper New York—46 schools, Washington—60 schools, and Wyoming—30 schools.

The 881 schools were from nearly every state in the union, three provinces of Canada, Hawaii, and Scotland. There were over 23,000 students enrolled for

the contest. The winning school was not announced, since all of the scores were not known. Anyone desiring information about the contest should write to Professor W. H. Fagerstrom, City College, New York 31, N. Y.

Professor F. H. Miller reported the following recommendations for the Committee on Coordinating Mathematical Training: 1) three years of high school mathematics as a minimum for all college entrants; 2) separation of college-entrance and other students in the high schools; 3) greater stress on concepts and their application to computational procedures.

The group voted that the Committee should notify the Upper New York State Section that it approved such recommendations.

Dr. J. J. Theobald, President of Queens College, welcomed the people at the meeting, and then the following papers were presented:

1. *What the high school can do to recruit teachers of mathematics*, by Mr. L. W. Schlumpf, Andrew Jackson High School, St. Albans, introduced by the Secretary.

The shortage of qualified teachers of secondary mathematics poses an acute problem for all future training in this field. The problem promises to become even more serious. More efficient use of available personnel should be obtained by effecting different license requirements for teachers performing the two different basic jobs of high school mathematics: teaching those who continue their studies in mathematics and teaching those who are not academically minded. Every effort should be made to arouse the interest of able high school pupils in mathematics through segregation of them in honor classes, instruction of them by special methods, assignment to them of superior and enthusiastic teachers, and establishment of a comprehensive extra-curricular program in this field.

2. *Recruitment of teachers of mathematics*, by Mr. Joseph Jablonower, Board of Examiners, City of New York.

The shortage of teachers of mathematics and of students who are preparing to teach mathematics is already acute. With the increase in school population, and the depletion in the teaching ranks, we face a situation that calls for radical measures. Administration and boards of education have been yielding to the temptation of lowering requirements in professional preparation for teaching and reducing requirements as to knowledge of the subject.

Industries and educational foundations should be alerted to the needs of the situation and should get states and municipalities to recognize education as a major and indispensable industry. Industry and educational foundations need to work out wisely subventions that will assure trainees for the teaching of mathematics. Boards of education need to formulate salary schedules which will assure the competent teacher a scale of living respected by the community.

3. *What the colleges can do to recruit teachers of mathematics*, by Professor D. R. Davis, State Teachers College, Montclair, New Jersey.

Experience suggests the following measures to promote an effective program for the recruitment of teachers of mathematics: 1) provide a good mathematics curriculum, 2) recommend and treat teaching as a profession which requires a minimum of four years of college, 3) maintain a recruiting program among the high schools with the cooperation of local teachers and administrators, guidance personnel, college faculty representatives and college student representatives, 4) secure positive assistance from state and local organizations for scholarships, 5) picture carefully the advantages of the teaching profession, 6) give effective supervision to those doing practice teaching, 7) work for better salaries, buildings, equipment, and working conditions, 8) help direct

the efforts of all concerned toward elevating the teaching profession, that people may choose it because they feel it has dignity, purpose, meaning, and the respect and support of the public.

4. *Hilbert's fifth problem*, by Professor Leo Zippin, Queens College.

At the International Congress of Mathematicians, Paris 1900 (see *Göttinger Nachrichten* of that year) Hilbert proposed a number of distinct research programs, called problems. What we now know as the Fifth Problem (certain other questions having been forgotten) is the following: *is a locally-euclidean group necessarily isomorphic to a Lie group?* The last fifty years have witnessed the affirmative solution of this problem as part of the analysis of the structure of locally compact groups. The roll call of the names of those who made significant contribution to this larger problem is too long to be given here.

5. *How to do by arithmetic what you cannot do with the calculus*, by Professor M. G. Salvadori, School of Engineering, Columbia University, introduced by the Secretary.

The difficulties encountered in the analytical solution of problems of great practical importance leads the applied mathematician to the use of approximate solutions obtainable often by elementary arithmetical tools. The talk illustrated the application of numerical methods to a variety of linear problems.

E. MARIE HOVE, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The annual spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at the University of Kansas City, Kansas City, Missouri, on April 22, 1955. Professor Maria Castellani, Chairman of the Section, presided at the morning session, and Reverend W. C. Doyle, Rockhurst College, presided at the afternoon session.

There were 45 persons in attendance, including the following 30 members of the Association:

John J. Andrews, S. Louise Beasley, C. A. Bridger, Maria Castellani, John F. Daly, W. C. Doyle, D. H. Erkiletian, Jr., C. V. Fronabarger, J. D. Haggard, Nola L. A. Haynes, F. F. Helton, N. Q. Hubbard, G. H. Jamison, C. A. Johnson, L. O. Jones, P. S. Jones, C. E. Kelley, P. G. Kirmser, S. L. Levy, F. H. Lloyd, Marie A. Moore, O. J. Peterson, L. E. Pummill, Lois J. Roper, J. S. Rosen, Robert Schatten, W. R. Scott, R. G. Smith, W. A. Vezeau, Margaret F. Willerding.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Francis Regan, St. Louis University; Vice-Chairman, Professor H. D. Brunk, University of Missouri; Local Secretary-Treasurer, Professor Marie A. Moore, Harris Teachers College. Professor Margaret F. Willerding, Harris Teachers College, retained her position of Association Secretary for a fourth year.

The following program was presented at the morning session:

1. *The significance and derivation of the formula*, by Mr. N. Q. Hubbard, Lincoln High School, Kansas City.

The study of the formula covers an extensive field in the branches of mathematics. The re-

search on this problem extends from the dates of 1857 to 1955. The significance and derivation of the formula, as it appears in print, covers an abundance of research work on this topic. The validity has been shown by various citations in this study by noted scholars in the field of mathematics whose authenticity has met the criteria in educational research. This study shows very plainly the aids, new methods of procedures, principles and modern avenues of approach to algebraic solution in secondary mathematics. The significance and derivation of the formula stands out as one of the basic concepts of junior and senior high school mathematics of today.

2. *A differential equation applicable to population problems*, by Mr. C. A. Bridger, Bureau of Vital Statistics, Jefferson City.

The differential equation $dP = f(P)F(t)dt$, where f and F are polynomials, is assumed. Examples for simple cases are developed. The Pearl-Reed logistic occurs when f is a quadratic and F is a constant. On the basis of information from previous decades, estimates by counties for Missouri are made for 1960. In over half of the counties, the exponential form of the solution of the equation is used.

3. *Birth, death, and waiting in line*, by Dr. Ernest Koenigsberg, Midwest Research Institute, Kansas City, introduced by the Secretary.

The problems of waiting in lines or queues can be introduced in terms of birth and death processes. A queue system is described by three characteristics: 1) a birth process or input mechanism; 2) a queue discipline (*e.g.*, "first come, first served"); 3) a death process or exit mechanism.

Several birth and death processes are described and formulated, and the "exponential holding time" queue system is formulated in terms of the rate of change of the number of units at the service station (*i.e.*, those waiting in line and those being served). The "constant holding time" queue system is also formulated in terms of the number of ways that a given number of units can be at the service station. Various other cases are discussed: queuing with priorities, queuing with special service, random queue disciplines, and now Poisson birth processes.

4. *Reorientation in economic theory: linear and non-linear programming*, by Professor E. Altschul, University of Kansas City, introduced by the Secretary.

Theory of Games by von Neumann and Morgenstern has initiated a new trend in developing analytical tools in economic theory and applied economics. Human actions geared toward optimization of various goals represent a maximum problem entirely different from that of physics. In a society each participant has to maximize a function without having control over all variables, being forced to meet actions of opponents.

Von Neumann and Morgenstern proved that economic problems are not those resembling maximum problems in physics, but problems of maximizing under constraint conditions, leading to the analysis of relative maxima as expressed by linear inequalities. The traditional approach of calculus must be replaced by matrix analysis. The new minimax approach opens a wide avenue for economic generalizations and development of flexible tools in solving practical problems of optimization in managerial decision.

5. *Some sign and rank tests in statistics*, by Professor W. A. Vezeau, St. Louis University.

A short discussion of basic concepts in statistics was presented as background. Then a historical development of sign and rank tests was given. Reasons for the use of such tests by mathematics teachers were presented with particular stress upon the introduction of such tests as special topics or projects in certain classes of mathematics. A number of practical problems were discussed using some of the sign and rank tests.

6. *Rational function approximations for the exponential function*, by Mr. Y. L. Luke, Midwest Research Institute, Kansas City, introduced by the Secretary.

Employing Chebyshev polynomials, simple rational function approximations for the exponential function in the complex domain are obtained. The technique follows that of C. Lanczos and is known as the τ-method for the solution of linear differential equations with rational coefficients. Some numerical examples are present. Results are useful for stability and response studies of time delay control systems.

7. *Infinite symmetric groups*, by Professor W. R. Scott, University of Kansas.

Let X and Y be infinite cardinal numbers. The infinite symmetric group $S(X, Y)$ is the group of $1-1$ transformations of a set of order X onto itself in which fewer than Y elements are moved. The alternating group $A(X)$ is the subgroup of finite even permutations in $S(X, Y)$. Any homomorphism of any $S(X, Y)$ or $A(X)$ on to an $S(X, Y)$ or $A(U)$ is an isomorphism and $X=U$, $Y=V$. Any factor group of $S(X, Y)$ contains a subgroup isomorphic to $S(X, Y)$ if $Y\neq\aleph_0$. If $2^Z<X$ (Z may be finite) and $Y\neq\aleph_0$, then neither $S(X, Y)$ nor $A(X)$ contains a subgroup of index $\leq Z$.

8. *Phase plane solution of non-linear differential equations*, by Dr. S. L. Levy, Midwest Research Institute, Kansas City.

The ordinary electric bell's motion is analyzed in terms of an on-off servo-system. The analysis is completely graphic and is, furthermore, fully idealized so that it does not detract from the physical principles involved and the use of the phase-plane method. The response of a more realistic system is indicated. The result of the analysis shows a bell would not operate if it did not have "poor on-off control."

9. *The use of television in mathematics education*, by Professor P. S. Jones, University of Michigan. (By invitation).

There have been more than twenty-six television programs or series with mathematical content presented over stations in the United States. Most of these have been of a semi-popular "cultural" nature, designed to interest viewers in mathematics and to inform them about its role and importance. However, there have been several programs designed to teach particular mathematical topics (the slide rule, high school algebra, nomography, the teaching of arithmetic, measurement), and more programs sponsored by public schools showing the content and methods of teaching for public relations purposes.

Television offers opportunities for making individual skilled speakers or teachers and special equipment available to a large audience and hence can do much to increase interest in and appreciation of mathematics and to actually enrich the teaching of mathematics for school children as well as for adult viewers.

MARGARET F. WILLERDING, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 23, 1955. Professor W. R. Transue, Chairman of the Section, presided at the morning and afternoon sessions.

There were 111 persons registered in attendance including the following 89 members of the Association:

Grace L. Abhau, J. E. Adney, Jr., C. E. Amos, P. R. Annear, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Jack Belzer, J. H. Blau, G. M. Bloom, Foster Brooks, O. E. Brown, Emalou Brumfield, L. E. Bush, F. C. Calabrese, C. D. Calhoon, V. B. Caris, Dorothy I. Carpenter, A. B. Carson, R. A. Clark, E. H. Clarke, G. M. Clough, C. C. Crell, Wayne Dancer, H. F. Davis II, R. W. Emmert, W. S. Ericksen, P. L. Evans, Beverly R. Ferner, H. E. Fettis, W. T. Fishback, M. P. Fobes, G. R. Glabe, H. W. Godderz, R. P. Gosselin, L. J. Green, Arnold Grudin, Marshall Hall, Jr., H. G. Harp, H. L. Harter, R. G. Helsel, Raymond Huck, R. Y. Iwanchuk, S. J. Jasper, M. L. Johnson, Margaret E. Jones, J. W. Kaiser, J. S. Klein, L. C. Knight, Jr., D. M. Krabill, F. A. Kros, P. E. Lauderbach, Nathan Lazar, E. B. Leach, H. D. Lipsich, L. L. Lowenstein, H. R. Mathias, S. W. McCuskey, E. J. Mickle, L. H. Miller, Knox Millsaps, C. C. Morris, O. M. Nikodym, Emma J. Olson, H. S. Pollard, Gustave Rabson, R. F. Reeves, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, S. A. Rowland, H. J. Ryser, B. L. Schwartz, Nancy M. Scribano, R. L. Shively, E. T. Stapleford, R. R. Stoll, Ralph E. Thomas, H. E. Tinnappel, H. S. Toney, W. R. Transue, E. P. Vance, E. H. Wang, Marion D. Wetzel, F. B. Wiley, G. P. Williams, C. O. Williamson, Alberta Wolfe, B. J. Yozwiak.

The following officers were elected for the coming year: Chairman, Professor R. R. Stoll, Oberlin College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Dr. W. E. Deskins, Ohio State University; Program Committee, Chairman, Professor Wade Ellis, Oberlin College; Professor H. D. Lipsich, University of Cincinnati; Dr. E. B. Leach, Case Institute of Technology.

The following papers were presented:

1. *Approaches to measure and integration*, by Professor W. R. Transue, Kenyon College. (Chairman's address).

A comparative discussion is given of several diverse methods of development of the theory of measure and integration. Starting with a consideration of some properties of measurable and integrable sets and functions, and those of measure and integral, there follows a discussion of the use of these properties in the introduction of definitions of these entities. Using Lebesgue measure and integral as an example, an outline is given of 1) the classical development, 2) that based on the Lusin theorem and in use by the Tonelli school, and 3) the procedure of Bourbaki.

2. *The superiority of the trapezoidal rule for a certain class of definite integrals*, by Mr. H. E. Fettis, Wright Air Development Center.

By means of Poisson's summation formula, it can be shown that for certain integrands the error in applying the trapezoidal rule vanishes exponentially as the interval of integration tends to zero. For such functions, the trapezoidal rule can be shown to be superior to Simpson's Rule and other more complicated quadrature formulas. Examples are given of functions for which the magnitude of error can be predicted in advance in terms of the interval of integration.

3. *Note on isosceles orthogonality*. By Professor H. F. Davis, II, Miami University.

Following R. C. James, we say that vectors v and w are isosceles orthogonal if the length of $v+w$ equals the length of $v-w$. Then *a normed linear space is euclidean if and only if orthogonal complements are always subspaces*. A simple direct proof of this theorem is given. The author's interest in this topic arose in connection with directing a master's thesis at Miami University. He feels that original work exploring such concepts is more valuable to the student than a purely expository thesis would be.

4. *Moments of non-integral order*, by Dr. H. L. Harter, Aeronautical Research Laboratory, Wright Air Development Center.

Statisticians attach little meaning to moments of non-integral order and have, until recently, given them little thought. However, certain considerations in the theory of turbulence have led to expressions involving such moments, especially those of order one-half. Moments of order one-half have real values only for distribution functions which are identically zero for all negative values of the variable, and hence when taken about the mean they are always imaginary. Moments about the origin have been computed for several well-known distributions which are restricted to non-negative values, and the results are given in this paper. Absolute moments about the mean are also considered.

5. *Functions of several variables with preassigned derivatives*, by Dr. E. B. Leach, Case Institute of Technology.

The existence of a function of one real variable whose derivatives of all orders have given values at a point has been shown in a simple way by A. Rosenthal. This note makes the following extension: Let M be a C^∞ manifold of dimension m, regularly imbedded in R^n (where $m<n$). Then there is a C^∞ function f defined in R^n, whose derivatives of all order have given traces on M, provided the traces satisfy the "strip conditions" imposed by the chain rule for differentiating a composite function.

6. *Characterizations of determinant functions*, by Professor R. R. Stoll, Oberlin College. (By invitation).

Professor Stoll discussed sets of axioms for determinant functions devised by Schreier and Sperner, Menger and Kozin, Hensel, and Stephanos. Those of Schreier and Sperner were motivated by the following problem: To devise a function on nth order matrices which vanishes at A if and only if the homogeneous system of linear equations $AX=0$ has a nontrivial solution. The suggestion was made that one of these approaches might be more stimulating to undergraduates, and more in keeping with current emphasis than the traditional presentation.

7. *A new frontier: photogrammetry*, by Professor H. S. Toney, Wright Air Development Center and University of Dayton.

The field of photogrammetry offers a challenge to mathematicians to use applied mathematical methods to solve problems of loci and distance determinations needed for geodetic survey and airborne reconnaissance. Photogrammetry (only about 100 years old) is defined as the science of obtaining reliable measurements by means of photography (terrestrial or aerial). This introductory paper contains a brief history and a survey of problems in this field which should interest mathematicians.

8. *Social choice and impossibility theorems*, by Professor J. H. Blau, Antioch College.

Arrow's *Social Choice and Individual Values* has been widely quoted and has stimulated further research. In this note it is pointed out that the principal theorems are false. An *ordering* of a set of alternatives is a reflexive, connected and transitive relation on this set. A *social welfare function* (SWF) is a method of combining individual orderings to form a social ordering, subject to several natural conditions. Arrow's Theorems 2 and 3 assert that if certain degrees of variety in individual orderings are permitted, there can be no SWF. An explicit SWF which serves as a counter-example is given here. The error is discussed. On the other hand, if *all* individual orderings are permitted, there can be no SWF. Arrow's methods yield a proof.

9. *A note on Fourier coefficients*, by Professor R. P. Gosselin, Youngstown College.

Let $c(g)$ be the nth Fourier (exponential) coefficient of $g(x)$. Let $f(x)$ belong to L^q, q an integer $\geqq 2$. Let $c_n(f)$ be positive and decrease with $1/|n|$. By use of Parseval's formula, it is shown that $(c_n(f))^q \leqq A_q c_n(f^q)/(|n|+1)^{q-1}$, where A_q is a constant depending only on q. As an application of this inequality, a proof of the following result, due to Hardy and Littlewood (*Journal of the London Math. Soc.*, vol. 6, 1931, pp. 3–9), is obtained: If $f(x)$ belongs to L^r, $r\geqq 2$, and $c_n(f)$ decreases, then

$$\sum_{n=-\infty}^{+\infty} |n|^{r-2}(c_n(f))^r \leqq B_r \int^{2\pi} |f(x)|^r dx.$$

10. *The tensor form of the equations of hydrodynamics*, by Mr. W. H. Lane, Wright Air Development Center, introduced by the Secretary.

The tensor forms of the general energy equation and of the Navier-Stokes equations of motion for a viscous incompressible fluid are considered. The special case for spherical coordinates in which the velocity and temperature fields are assumed to be inversely proportional to the power of a radial vector is developed, and the resulting class of exact solutions is discussed.

11. *On ideals in the ring of linear multidifferential polynomials*, by Mr. Frank Levin, University of Cincinnati, introduced by Professor H. D. Lipsich.

The ring of linear multidifferential polynomials is a noncommutative ring of polynomials in several indeterminates. This ring is not a principal ideal ring, and, therefore, the results are stated ideal-theoretically. A basis of an ideal of multidifferential polynomials which corresponds to the basis of an ideal in the principal ideal ring of ordinary differential polynomials is the *canonical basis* of the ideal. With this basis one is able to provide an upper bound for the length of a basis of the ideal and to give a necessary and sufficient condition for solvability of multidifferential equations.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-eighth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on Friday afternoon and evening and Saturday forenoon, April 22 and 23, 1955. Professor Nathan Schwid, Chairman of the Section, presided at all three sessions.

There were 62 persons registered for the meeting, including the following 46 members of the Association:

J. W. Ault, G. E. Bardwell, C. F. Barr, B. C. Bellamy, W. E. Briggs, J. R. Britton, R. K. Butz, F. M. Carpenter, Sarvadaman Chowla, E. L. Crow, W. E. Dorgan, F. N. Fisch, H. T. Guard, Leota C. Hayward, Anna S. Henriques, Archie Higdon, J. E. Householder, Sr., P. F. Hultquist, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, R. B. Kriegh, L. J. Lange, E. B. McLeod, Jr., M. L. Madison, W. D. Marsland, Jr., W. E. Mientka, W. K. Nelson, Greta Neubauer, D. O. Patterson, J. W. Querry, O. M. Rasmussen, O. H. Rechard, A. W. Recht, Calvin A. Rogers, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, P. O. Steen, W. J. Thron, E. P. Tovani, V. J. Varineau, W. W. Varner, C. R. Wylie, Jr.

Officers elected at the meeting for 1955–1956 were: Chairman, Professor

C. R. Wylie, Jr., University of Utah; Vice-Chairman, Professor R. R. Gutzman, Colorado School of Mines; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *Remarks on the functional equation* $f[f(z)]=e^z-1$, by Professor W. J. Thron, University of Colorado.

If the functional equation has a solution $f(z)$ which is holomorphic in a sufficiently large neighborhood of the point $z=0$ then it can be shown that $f(0)=0$. Using this one can determine a unique formal power series solution $f(z)=\sum c_n z^n$. Let r be the radius of convergence of this series. By means of Picard's theorem, Hadamard's factorization theorem, and a result of Pólya, it is established that $r<\infty$. The statement that $r=0$ but that there exists a solution of the functional equation which is holomorphic for all z, not on the negative real axis, concludes the paper.

2. *Note on hemispheric numerical integration of the barotropic model*, by Major J. F. Blackburn, USAF Academy, and W. L. Gates, presented by Major Blackburn, who was introduced by the Secretary.

Under the assumption of frictionless, adiabatic flow in hydrostatic and quasi-geostrophic equilibrium, the barotropic equation was solved numerically for hemispheric flow. The method of solution was similar to the scheme outlined by Charney, Fjørtoft and von Neumann (*Tellus*, November, 1950) applied to a smaller area. The procedure consists of the cyclical calculation of the absolute vorticity advection at time t for each point of a finite difference grid, the solution of the resulting finite difference equations for $\partial z/\partial t$ at each point by a method of relaxation, and the calculation of the heights z at time $t+\Delta t$ using centered differences over a short time interval.

3. *Some simple geometrical properties of the space* L^2 by Mr. A. E. Labarre, Jr., University of Wyoming, introduced by the Secretary.

Geometrical interpretations of the Parseval and Riesz-Fischer theorems are given. The space L^2 is the direct product of the even functions of L^2 and the odd functions of L^2. How the operation of differentiation in L^2 can be interpreted as an orthogonal transformation is explained.

4. *The number of lattice points in an n-dimensional tetrahedron*, by Professors Sarvadaman Chowla and W. E. Mientka, University of Colorado, presented by Professor Mientka.

Let the $a_i (1\leqq i\leqq n)$ be positive integers relatively prime in pairs, and $A=\Pi_{i=1}^n a_i$. In this paper we find exact expressions (which are polynomials in η/A and the a_i) for (i) the number of solutions in non-negative integers x_i of $\sum_{i=1}^n a_i x_i=\eta$ whenever $\eta\equiv 0 \pmod{A}$, (ii) the number of lattice points in the tetrahedron bounded by the planes $\sum_{i=1}^n a_i x_i=\eta$ $(x_i\geqq 0)$ again provided $\eta\equiv 0 \pmod{A}$.

5. *A possible measure of asymmetry in a line*, by Professor Calvin A. Rogers, Colorado Agricultural and Mechanical College.

Eight requirements were set up, formally expressing intuitive convictions about the asymmetry in the x-axis of two points P_1 and P_2 with same abscissa and ordinates, y_1 and y_2. From these, it was deduced that one of the simplest functions satisfying all requirements was the fraction $(y_1+y_2)^2/(y_1^2+y_2^2)$.

6. *Remarks on the distribution of primes*, by Professors A. J. Kempner and Sarvadaman Chowla, University of Colorado, presented by Professor Kempner.

From the extended Prime Number Theorem two formulae are derived:

$$\frac{\pi(\delta x)-\pi(\gamma x)}{\pi(\beta x)-\pi(\alpha x)}=\frac{\delta-\gamma}{\beta-\alpha}\cdot\left\{1+\mathcal{C}(\alpha,\beta,\gamma,\delta)\cdot\frac{1}{\log x}\right\}+o\left(\frac{1}{\log x}\right)$$

and

$$[\pi(\delta x)-\pi(\gamma x)]-[\pi(\beta x)-\pi(\alpha x)]=[(\delta-\gamma)-(\beta-\alpha)]\cdot\frac{x}{\log x}+\mathcal{C}'(\alpha,\beta,\gamma,\delta)\cdot\frac{x}{\log^2 x}+o\left(\frac{x}{\log^2 x}\right).$$

Specialization of α, β, γ, δ leads to results concerning the distribution of primes.

7. *Knots and quadratic forms*, by Professor K. A. Hirsch, University of London and University of Colorado. (By invitation).

The speaker discussed certain topological properties of knots by considering the invariants of related quadratic forms.

8. *The power series coefficients of L-series*, by Dr. W. E. Briggs, University of Colorado.

Consider $L_k(s)=\sum_1^\infty \chi(n)n^{-s}$ where χ is a real non-principal character mod k. This series can be presented by the power series $\sum_1^\infty L^{(r)}(1)(s-1)^r/r!$. These coefficients can be determined by evaluating the r-th derivative of the defining series to obtain

$$L^{(r)}(1)=(-1)^r\sum_1^k \chi(t)\gamma_{r,k,t}, \quad\text{where}\quad \gamma_{r,k,t}=\lim_{x\to\infty}\left[\sum_{n\leqq x,n\equiv t(k)}\frac{\log^r n}{n}-\frac{\log^{r+1}x}{k(r+1)}\right].$$

Similarly the power series coefficients of the zeta function can be determined by considering

$$h(s)=\zeta(s)-\frac{s}{s-1}=s\int_1^\infty\frac{[x]-x}{x^{s+1}}dx=\sum_0^\infty\frac{h^{(r)}(1)}{r!}(s-1)^r$$

and evaluating the expression obtained by differentiating the integral r times. This gives $h^{(r)}(1)=(-1)^r\gamma_r$ for $r>0$ and $h(1)=\gamma_0-1$ where $\gamma_r=\gamma_{r,1,0}$.

9. *A problem in interpolation*, by M. L. J. Lange, University of Colorado.

Given a polynomial of degree n with coefficients in a field K, $f(x)=(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_k)^{m_k}$, and with the α_i in the root field K' of $f(x)$, and given a polynomial $g(z)$ of arbitrary degree with coefficients in K', the problem is to find a polynomial $h(x)$ of degree $\leqq n-1$ with coefficients in an extension field K'' such that $F(x)=g(h(x))-x$ is divisible by $f(x)$. The author showed that an $h(x)$ with the required properties exists if and only if for all α_i with $m_i>1$ the equation $g(z)-\alpha_i=0$ has at least one simple root. He also gave a method for actually constructing such a polynomial $h(x)$.

10. *An outline of the mathematics curriculum and schedule at the USAF Academy*, by Colonel Archie Higdon, USAF Academy.

The United States Air Force Academy mathematics curriculum consists of courses in college algebra, plane and spherical trigonometry, analytical geometry, differential and integral calculus, applied calculus, and elementary differential equations for a total of 21 quarter hours.

Students will be sectioned according to demonstrated ability in mathematics with an average of 12.5 students per section. They will be graded almost every day and daily preparation is mandatory. The top sections will cover some advanced topics in each course not required for those with less aptitude for mathematics. These top sections will contain many students who would be admitted with advanced standing in civilian schools. All students are required to complete all four years at the U. S. Air Force Academy regardless of previous college training.

11. *A new Poisson equation analog computer*, by Mr. W. W. Varner, University of Colorado.

A new computer has been completed at the University for the very rapid solution of the general second order partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + ku + F(x, y, u) = 0$$

with appropriate boundaries. It also solves the Poisson equation in three dimensions with a grid of 960 nodes or mesh points which can be arranged very easily and quickly into Cartesian, cylindrical, spherical, and other coordinate systems. It can handle very complicated boundaries, source and sink conditions, and transients.

12. *A review of the* 1954 *Oregon Summer Conference*, by Professor F. M. Carpenter, Colorado School of Mines.

F. M. CARPENTER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA, South Dakota State College, Brookings, October 15, 1955.

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA

OHIO, April, 1956.

OKLAHOMA, Oklahoma City University, October 28, 1955.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

ROCKY MOUNTAIN

SOUTHEASTERN, UNIVERSITY OF GEORGIA, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.

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Contributions to Geometry

Number 4
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

AUGUST-SEPTEMBER 1955

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CONTRIBUTIONS TO GEOMETRY

The Fourth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

This publication has been financed by the Arnold Buffum Chace Fund
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Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 62 Number 7

August–September, 1955

EDITORIAL NOTE

This number of the *Slaught Papers* consists of a collection of articles in the field of geometry which were submitted independently to the MONTHLY. Because of the close connection among several of these it appeared desirable to publish them together in a single issue. Since enough space was not available in a regular issue of the MONTHLY for this purpose, they are being published together in this *Slaught Paper*.

The first article by Bruck is a summary of the present status of work on Euclidean plane geometry since the time of Hilbert. It discusses the effects of modifications of the axioms, and relates the various geometries so obtained to appropriate algebraic systems. The succeeding articles by Hall, Ryser, and Wesson elaborate on special aspects of this theme.

The remaining articles are less directly related to this central theme, but illustrate other features of recent work in geometry. Stockton writes within the framework of classical projective geometry. Rosenfeld presents an axiomatic treatment of analytic trilinear geometry. Court gives new results in the classical field of "modern geometry." Finally Gans discusses the foundations of Elliptic Geometry.

It is hoped that the simultaneous publication of these papers will serve to arouse the interest of mathematicians in the currently under-emphasized field of geometry.

RECENT ADVANCES IN THE FOUNDATIONS OF EUCLIDEAN PLANE GEOMETRY*

R. H. BRUCK, University of Wisconsin

1. Introduction. A program of axiomatizing Euclidean plane geometry in a manner consistent with present standards of rigour was beautifully carried out by Hilbert [1].§ Upon reading Hilbert's book in its entirety one sees what is not at first evident—that Hilbert is alive to the interesting questions which arise when some of his geometric axioms are dropped or modified. And many mathematicians, both before and since the appearance of Hilbert's book, have investigated such problems.

Unfortunately for the wide audience whose interest in geometry was awakened in high school or university, answers to the deeper questions of the sort I have in mind, if they have been given at all, require a long excursion into abstract algebra. I know of no remedy for this situation. What can be done—or, at any rate, what the present paper attempts to do—is to give a pictorial account of some of the geometric axioms, a simple explanation of the algebraic problems which these pose and a brief account (with references rather than proofs) of the answers.

Think of this paper as an excursion from wherever you are towards a town named Cayley Numbers. As we saunter along, we pass by Planar Ternary Rings, Veblen-Wedderburn Systems, Division Rings with the Right Inverse Property, Right Alternative Division Rings, Alternative Division Rings—and down the hill we see Cayley Numbers.—Strange names they have for towns in these parts, but I believe you'll enjoy the scenery.

2. The axioms of incidence. We shall use geometric language quite informally. In particular we shall assume that everyone feels at home with phrases such as "point is on line," "line is through point" and with adjectives such as "collinear," "concurrent," "parallel." Such carelessness is a little dangerous (since the language may have unintended connotations) but saves a great many words.

A Euclidean (or affine) plane π is a system of undefined objects, called points and lines, subject to the following *axioms of incidence:*

(i) *If P, Q are distinct points of π, there is one and only one line, PQ, of π, through both of P and Q.*

(ii) *If a, b are distinct lines of π, there is at most one (and there may be no) point of π on both of a and b.*

---

* This paper had its origin in 1950 in my seminar at the University of Wisconsin. It has been presented since then by my students or myself in various forms at several universities. The present account was delivered by invitation to the Iowa Section of this Association in Ames, Iowa, April 30, 1954.

§ The book [1a] is included because it is in English and indicates the spirit of Hilbert's later approach but all specific references [1] are to [1b].

(iii) *If the point P of π is not on the line a of π, there is exactly one line of π which passes through P and is parallel to a.*

(iv) *There is at least one set, A, B, C, D, of four distinct points of π, no three of which are collinear.*

The axioms of incidence require so little of a Euclidean plane that very few theorems have been proved. Indeed, the main "theorem"—I like to call it Hall's Theorem—might be stated as follows: Any damn thing can happen. As a relatively respectable example of this we may note that the following system satisfies the axioms: π consists of four distinct points, A, B, C, D, and of six distinct lines, namely the following three pairs of parallels: AB, CD; AC, BD; AD, BC. Each line contains exactly two distinct points and each point lies on exactly three distinct lines.

Even so, the axioms of incidence do ensure a certain amount of regularity. Suppose a is a line of π and P is a point of π not on a. By (iii), there is exactly one line through P, say b, which does not meet a. The remaining lines through P are in one-to-one correspondence with the points of a, for,* by (i), (ii), (iii), each of these lines is a line PQ for exactly one point Q of a, and conversely. Hence we can say that there is "one more" line through P than there are points of a. Consequently, every two lines not through P have the same number of points. Then, considering the points A, B, C, D of (iv), we see that every line not through A has the same number of points as BC or BD or CD, and similarly for B, C, D. Thus, if some line of π has n points (where n is a positive integer or a transfinite cardinal) then every line has n points and every point lies on $n+1$ lines.

We can add: if a, b, c are distinct lines and if a is parallel to b and b parallel to c, then a is parallel to c—else the common point of a, c would have through it two parallels to b, namely a, c. Now consider two lines a, d, meeting in a point P. Each of the $n-1$ points of d, other than P, determines a unique parallel to a, and conversely. Hence the *parallel class* of a, consisting of a and the lines parallel to a, contains precisely n lines. Moreover the parallel class of a is the parallel class of each of the lines contained in it.

In a rigorous treatment of Euclidean planes it is necessary at various points to give special consideration to planes with a small number of points on each line. We shall be content to ignore this difficulty entirely.

3. Hall's planar ternary rings. How can we introduce coordinates into a Euclidean plane π subject only to the axioms of incidence? We cannot talk of rectangular axes—since we have no notion of angle aside from the special case of parallel lines. We cannot talk of lengths—we have no notion of distance. We cannot talk of the slope of a line—but, on the contrary, we shall do just that in a moment.

---

* A few simple diagrams, which we feel compelled to omit, make the following remarks quite obvious.

The method of Marshall Hall [2] seems as good as can be expected. We select an arbitrary point O of π and three distinct lines§ through O which we call the x-axis, the y-axis and the unit line (Fig. 1). On the unit line we select any point I (the unit point) distinct from O. The line through I parallel to the y-axis we call the slope line.

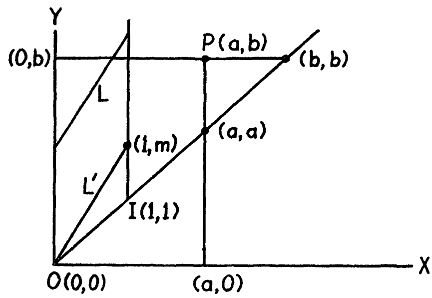

FIG. 1

Now we choose an arbitrary set R of elements or "labels" subject to two restrictions: (a) Among the elements of R are two distinct elements 0, 1. (b) The elements of R can be put into one-to-one correspondence with the points of the unit line OI (and hence with the points of any line of π). To each point of the unit line OI we assign a pair of coordinates (x, y) with $x=y$, where $x=y$ is an element of R. In particular we give the origin O the coordinates $(0, 0)$ and the unit point I the coordinates $(1, 1)$. We arrange that each point of OI has coordinates (a, a) for a uniquely determined element a of R and that, conversely, there is a unique point on OI with coordinates (b, b) for each element b of R.—Despite all this arbitrariness in assigning coordinates to the points of OI, we use the coordinates in such a way that any two ways of assigning them would be equally good or bad.

Next consider any point P of π. The line through P in the parallel class of the y-axis meets OI in a unique point; say the point with coordinates (a, a). And the line through P in the parallel class of the x-axis meets OI in a unique point, say (b, b). Then we assign to P the coordinates (a, b) (see Fig. 1). In particular, points of the x-axis have coordinates of form $(x, 0)$, points of the y-axis have the form $(0, y)$, and the four points $(0, 0)$, $(x, 0)$, $(0, y)$, (x, y) form the vertices of a parallelogram.

The line through the point (a, b) in the parallel class of the y-axis will naturally have the equation $x=a$. Similarly, it is clear what we mean by the line with equation $y=b$. We have yet to assign equations to the other lines, except that the unit line OI should certainly have the equation $y=x$.

Consider any line L. There is a unique line L' in the parallel class of L which

§ Note that, in Figure 1, OX, OY are perpendicular and angle XOY is bisected by the unit line OI. This is meaningless but somehow comforting.

passes through the origin O. If L' is the y-axis, we assign no slope to L or L'. Otherwise, L' must meet the slope line $x=1$ in a unique point, say the point $(1, m)$; in this case we assign to L (and L') the slope m (see Fig. 1). In particular, every line $y=b$ has slope 0, and every line in the parallel class of the unit line OI has slope 1.

At this stage every point of π has a unique pair of coordinates and every line, except for the lines $x=a$, has a unique slope. Now consider a line L which intersects the y-axis in the "y-intercept" $(0, b)$. This line L has a unique slope m. We should *like* to be able to say that L has equation $y=xm+b$; but, at the present stage,* at least, such an equation is meaningless. Instead, we use the plane π and the coordinate system which we have set up to define a *ternary operation* (or function) F on the elements of the label set R, in such a manner that, for each ordered triple a, m, b of elements of R, $F(a, m, b)$ is an element, say d, of R. This is done as follows: for any m, b, consider the line L of slope m through the y-intercept $(0, b)$. The line $x=a$ meets the line L in a unique point P whose coordinates are (a, d) for some definite element d of R. Then we define $F(a, m, b)=d$.

In view of the definition of F it is tautological to say (see Fig. 2) that the equation of the line with slope m, y-intercept $(0, b)$ is $y=F(x, m, b)$. As a particular case, the unit line OI has slope 1 and y-intercept $(0, 0)$. The equation of OI is surely $y=x$ and yet it is also $y=F(x, 1, 0)$. Consequently, $F(x, 1, 0)=x$ for every x in R.—There are other facts about F which arise, like this, directly from the definitions, and there are deeper facts which come by insisting upon the full import of the axioms of incidence.

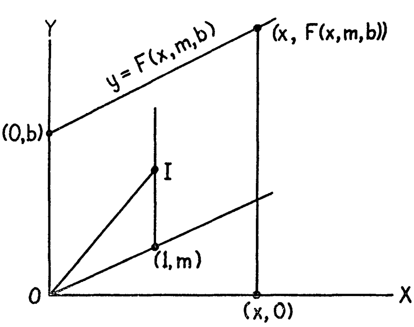

FIG. 2

The system (R, F), consisting of the label set R and the ternary operation F, is known as a *planar ternary ring*. Such systems are of course useless unless we have some way of singling out planar ternary rings from among all systems with ternary operations—for a suitable set of postulates see Appendix I. And even

* After we impose the vector axiom, equations of lines will indeed take this familiar form. (See §§ 6, 7, 8.)

then they are relatively useless until we find some way of handling them more easily than the Euclidean planes themselves.

4. Addition. For any planar ternary ring the operation of addition $(+)$ is defined equationally by

$$a + b = F(a, 1, b), \qquad \text{all } a, b \text{ in } R. \tag{4.1}$$

The algebraic consequence of (4.1) is this: *the equation of the line with slope 1 and y-intercept* $(0, b)$ *can now be written* $y = x + b$.

The geometric counterpart of (4.1) is the notion of addition of points on the unit line OI. Every ordered pair A, B of points $A = (a, a)$, $B = (b, b)$ of OI uniquely determines a sum-point $S = (a+b, a+b)$ of OI as follows (see Fig. 3):

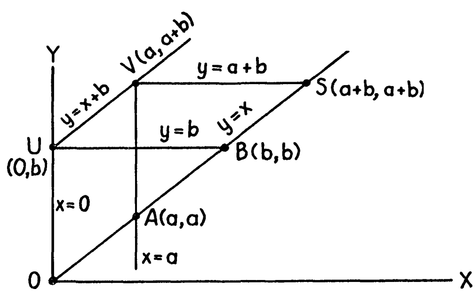

FIG. 3

the line $(y = b)$ of slope 0 through $B = (b, b)$ meets the y-axis in point $U = (0, b)$. The line $(y = x + b)$ of slope 1 through U and that line $(x = a)$ through A which is in the parallel class of the y-axis meet in a point $V = (a, a+b)$. The line $(y = a+b)$ of slope 0 through V meets the unit line $(y = x)$ in $S = (a+b, a+b)$. This geometric operation can be explained, without reference to coordinates or axes, in terms of three parallel classes: the lines in the parallel class of the x-axis (slope 0); the lines in the parallel class of the y-axis (no slope); the lines in the parallel class of the unit line OI (slope 1).

If we write $A + B = S$, it is easily verified from Figure 3 that if any two of A, B, S are arbitrarily assigned as points of OI, the third is uniquely determined by the equation. Moreover $A + 0 = A$ and $0 + A = A$ for every point A of OI. This means (see Appendix I) that *the system* $(R, +)$ *is a loop.*

Now let us suppose, temporarily, that the Euclidean plane π satisfies all the usual axioms of high-school geometry, so that we can make use of line segments.* In Figure 3, from the parallelogram $AOUV$, $\underline{OA} = \underline{UV}$, and, from the parallelogram $UVBS$, $\underline{UV} = \underline{BS}$. Consequently, $\underline{OA} = \underline{BS}$, so that the sum of the line segments $\underline{OA}$, $\underline{OB}$ is equal to $\underline{OS}$. In this special case, then, the somewhat

* In order to distinguish between the line through the two points A, B and the line segment (or vector—see § 6) with initial point A, endpoint B, we underline the latter. Thus: line AB, line segment $\underline{AB}$.

arbitrary equation $A+B=S$ is illuminated by the equation $\underline{OA}+\underline{OB}=\underline{OS}$. The equation (4.1) is thus related to very familiar things indeed. We shall return to this subject in §6.

5. Multiplication. For any planar ternary ring (R, F) the operation of multiplication $(\cdot)$ is defined equationally by

$$ab = F(a, b, 0), \qquad \text{all } a, b \text{ in } R. \tag{5.1}$$

The algebraic consequence of (5.1) is this: *the equation of the line through* $O=(0, 0)$ *with slope* m *can now be written* $y=xm$.

The geometric counterpart of (5.1) is the notion of multiplication of points on the unit line OI. Every ordered pair A, B of points $A=(a, a)$, $B=(b, b)$ of OI uniquely determines a product-point $P=(ab, ab)$ of OI as follows (see Fig. 4): The line $(y=b)$ of slope 0 through B meets the slope line $(x=1)$ in a point $U=(1, b)$. The line $OU(y=xb)$ of slope b meets the line $(x=a)$ of no slope through A in a point $V=(a, ab)$. The line $(y=ab)$ of slope 0 through V meets the unit line $(y=x)$ in the point $P=(ab, ab)$. This geometric operation can be explained, without reference to coordinates or axes, in terms of three classes of lines: the parallel class of the x-axis (slope 0); the parallel class of the y-axis (no slope); the class consisting of all lines through O except the y-axis (one line for every slope).

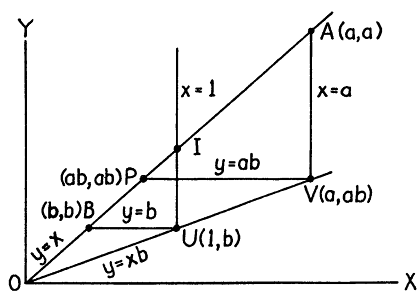

FIG. 4

If we write $A\cdot B=P$, we first note from Figure 4 that $A\cdot 0=0\cdot A=0$ for every point A of OI. Then, if we restrict attention to the set S consisting of the points of OI other than O, we find from Figure 4 that if any two of A, B, P are assigned in S, the equation $A\cdot B=P$ uniquely determines the third as a point in S. Moreover, $I\cdot A=A\cdot I=A$ for every A in S. This means (see Appendix I) that, if R^* denotes the set of elements of R exclusive of 0, *the system* $(R^*, \cdot)$ *is a loop.*

Again let us assume, temporarily, all the axioms of high school geometry. In Figure 4 we take $A\neq O$, $B\neq O$. From two sets of similar triangles,

$$\underline{OA}:\underline{OI} = \underline{OV}:\underline{OU} = \underline{OP}:\underline{OB},$$

so that

$$\underline{OA}\cdot\underline{OB} = \underline{OI}\cdot\underline{OP}.$$

Consequently, if we take $\underline{OI}$ to have unit length, we can parallel the abstract equation $A\cdot B=P$ with the familiar equation $\underline{OA}\cdot\underline{OB}=\underline{OP}$. (Although Hilbert introduces enough axioms to validate these calculations, we shall not quite do so.)

6. The vector axiom. At the end of §4 we temporarily made use of the notion of a line segment. In our usual thinking a line segment $\underline{AB}$ has both an inside and an outside. Such concepts require axioms of order (see Hilbert [**1**]) which we do not wish to introduce. We shall be content with a very mild notion of a *vector*. For present purposes, a vector $\underline{AB}$ consists merely of an ordered pair of distinct points A, B: an *initial point* A and an *endpoint* B. As before AB denotes the *line* through A and B.

We introduce a natural notion of equality of vectors. First of all, any vector is equal to itself: $\underline{AB}=\underline{AB}$. Next, if AB, $A'B'$ are distinct lines, the vectors $\underline{AB}$, $\underline{A'B'}$ will be called equal if and only if the lines AB, $A'B'$ are parallel and the lines AA', BB' are parallel. (Note that, in this case, if $\underline{AB}=\underline{A'B'}$ then, also, $\underline{BA}=\underline{B'A'}$, $\underline{AA'}=\underline{BB'}$, $\underline{A'A}=\underline{B'B}$.) Next suppose that AB, $A'B'$, $A''B''$ are distinct lines and that, according to our definition, $\underline{AB}=\underline{A'B'}$ and $\underline{A'B'}=\underline{A''B''}$. Our notion of equality will be useless unless we can be sure that $\underline{AB}=\underline{A''B''}$. It certainly is true that AB, $A''B''$ are parallel. Moreover, if A, A', A'' are collinear then B, B', B'' are collinear and the line $AA''=AA'$ is parallel to the line $BB''=BB'$; so that, in this case, it is true that $\underline{AB}=\underline{A''B''}$. But if A, A', A'' are not collinear the desired equality need not* hold. Therefore we force equality by imposing the vector axiom (see Fig. 5):

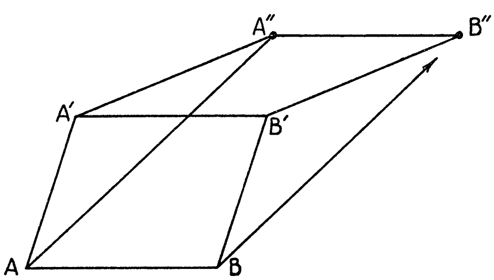

FIG. 5

THE VECTOR AXIOM. *If six distinct points of the Euclidean plane π form two triangles $AA'A''$, $BB'B''$, if the lines AB, $A'B'$, $A''B''$ are parallel and if the pairs of lines (AA', BB'), $(A'A'', B'B'')$ are parallel, then AA'', BB'' are also parallel.*

---

* There exist at least two essentially different Euclidean planes of order 9 (9 points on each line) in which the vector axiom fails. One type can be deduced from Hall [**2**] and another will be found in Carmichael [**3**].

The vector axiom is a special case of the axiom of Desargues and is one of the axioms used by Hilbert [1].

The vector axiom can be reached in still another way. According to the usual definition, the sum of two vectors $\underline{AA'}$, $\underline{A'A''}$ is the vector $\underline{AA''}$. If $\underline{BB'}=\underline{AA'}$ and $\underline{B'B''}=\underline{A'A''}$, we should like the sum $\underline{BB'}+\underline{B'B''}=\underline{BB''}$ to be equal to $\underline{AA''}$. If A, A', A'' are collinear, this is automatic, but in the case of Figure 5 we require the vector axiom.

Before we can go on, there is still one more aspect of vector equality which requires consideration. Suppose we have $\underline{AB}=\underline{CD}$ and $\underline{CD}=\underline{EF}$, where the lines AB, CD are distinct but AB, EF are identical. We would like to say that $\underline{AB}=\underline{EF}$, but in doing so we are in danger of serious trouble. For suppose that also $\underline{AB}=\underline{C'D'}$ and $\underline{C'D'}=\underline{EG}$; we need to be able to assert that $F=G$. Luckily no new axiom is needed. We will indicate how this is so by examining the case that the lines AB, CD, $C'D'$ are distinct. In this case, since $\underline{C'D'}=\underline{AB}$ and $\underline{AB}=\underline{CD}$, the vector axiom gives $\underline{C'D'}=\underline{CD}$. Then, since $\underline{C'D'}=\underline{CD}$ and $\underline{CD}=\underline{EF}$, the vector axiom gives $\underline{C'D'}=\underline{EF}$. However, $\underline{C'D'}=\underline{EG}$. Therefore the line through D' parallel to $C'E$ meets the line $ABEF$ in F and in G, and consequently $F=G$.—For the case that CD, $C'D'$ are the same line we simply introduce a new line $C''D''$ and argue as before.

At this stage, assuming the vector axiom, we can be confident that equality of vectors satisfies the usual reflexive, symmetric and transitive laws. There only remains to introduce zero vectors $\underline{AA}$ (with the same initial and final points) and to define $\underline{AA}=\underline{BB}$ for all points A, B; or, more conveniently, $\underline{AA}=0$ for all points A. We also give a symmetric definition of vector addition as follows: If $\underline{AB}$, $\underline{CD}$ are any two vectors, choose any point P, determine Q so that $\underline{PQ}=\underline{AB}$ and R so that $\underline{QR}=\underline{CD}$, and call $\underline{PR}$ the sum of the ordered pair of vectors $\underline{AB}$, $\underline{CD}$. It is easy to check that (in the sense of vector equality) the sum is independent of the point P.

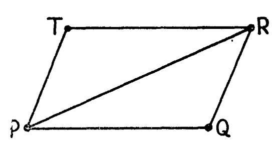

FIG. 6

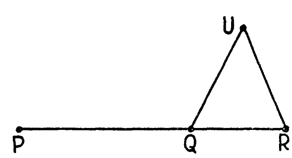

FIG. 7

Now the associative law of vector addition is evident, since $(\underline{PQ}+\underline{QR})+\underline{RS}=\underline{PR}+\underline{RS}=\underline{PS}$ and $\underline{PQ}+(\underline{QR}+\underline{RS})=\underline{PQ}+\underline{QS}=\underline{PS}$. The commutative law of addition is equally evident for vectors $\underline{PQ}$, $\underline{QR}$ if P, Q, R are not collinear, since $\underline{PQ}+\underline{QR}=\underline{PR}$ while (see Fig. 6) $\underline{QR}+\underline{PQ}=\underline{PT}+\underline{TR}=\underline{PR}$. On the other hand, if P, Q, R are collinear, a simple device (see Fig. 7) allows us to use the noncollinear case along with associativity: $\underline{QR}+\underline{PQ}=(\underline{QU}+\underline{UR})+\underline{PQ}=\underline{QU}+(\underline{UR}+\underline{PQ})=\underline{QU}+(\underline{PQ}+\underline{UR})=(\underline{QU}+\underline{PQ})+\underline{UR}=(\underline{PQ}+\underline{QU})+\underline{UR}=\underline{PQ}+(\underline{QU}+\underline{UR})=\underline{PQ}+\underline{QR}$. Consequently, all the usual laws of vector addition are satisfied.

Now we are ready to consider the additive system $(R, +)$ defined by (4.1). More specifically, we consider Figure 3. In Figure 3, $\underline{OA} = \underline{UV}$ and $\underline{UV} = \underline{BS}$, so $\underline{OA} = \underline{BS}$. Therefore $\underline{OS} = \underline{OB} + \underline{BS} = \underline{OB} + \underline{OA} = \underline{OA} + \underline{OB}$. And, inasmuch as $S = (a+b,\ a+b)$, $A = (a,\ a)$, $B = (b,\ b)$, we can assert the following (see Appendix I):

In the presence of the vector axiom, the system $(R, +)$ *is an abelian group isomorphic to the additive group of vectors.*

7. Linearity. If the vector axiom had not already been thrust upon us in connection with equality of vectors, we could urge another reason for its adoption. Namely, we would like every planar ternary ring (R, F) of the Euclidean plane π to have the property of *linearity* embodied in

$$F(a, b, c) = ab + c \qquad \text{all } a, b, c, \text{ in } R. \tag{7.1}$$

where the addition and multiplication on the right hand side are as defined in (4.1), (5.1). The algebraic consequence of (7.1) is this: *the equation of the line with slope* m, *y-intercept* $(0, b)$ *can be written* $y = xm + b$.

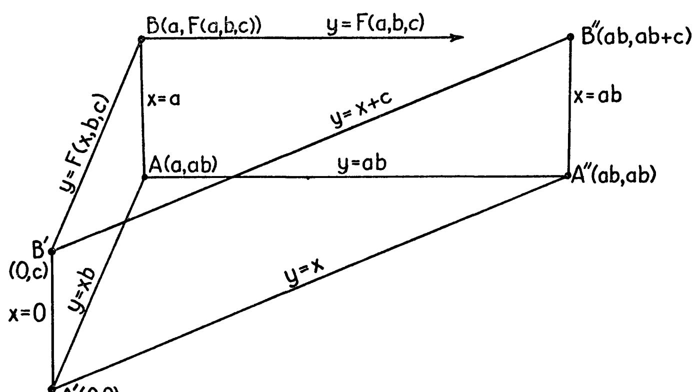

FIG. 8

In examining (7.1) we take $a \neq 0$, $c \neq 0$, $b \neq 0, 1$, since otherwise (7.1) holds trivially. Consider Figure 8, which emphasizes the essential nature of (7.1) by omitting irrelevant points (such as the unit point I) and lines (such as the x-axis). The y-axis appears as the line $x = 0$, and the unit line OI as the line $y = x$. (7.1) will hold if and only if the line $y = F(a, b, c)$ passes through the point $(ab, ab+c)$; that is (in Fig. 8) if and only if the lines AA'', BB'' are parallel. Hence, by comparison of Figure 8 with Figure 5, we see that, in the presence of the vector axiom, every planar ternary ring of π is linear. Now assume conversely that every planar ternary ring of π is linear, and consider Figure 5. With a little care we can construct a coordinate system in which the points and lines of Figure 5 have coordinates and equations of the forms indicated in Figure 8; then, by linearity, we can deduce that BB'' is parallel to AA''. To sum up:

A necessary and sufficient condition that every planar ternary ring of the Euclidean plane π be linear is that π satisfy the vector axiom.

8. Veblen-Wedderburn systems. The vector axiom has still another consequence, namely the right distributive law

$$(a+b)c = ac + bc, \qquad \text{all } a, b, c \text{ in } R, \tag{8.1}$$

of multiplication with respect to addition. We first note that (8.1) holds trivially if any one of a, b, c is zero or if $c=1$. Excluding these cases, consider Figure 9 below. The line $OA'(y=xc)$ meets the line $C'D'(x=a+b)$ in the point

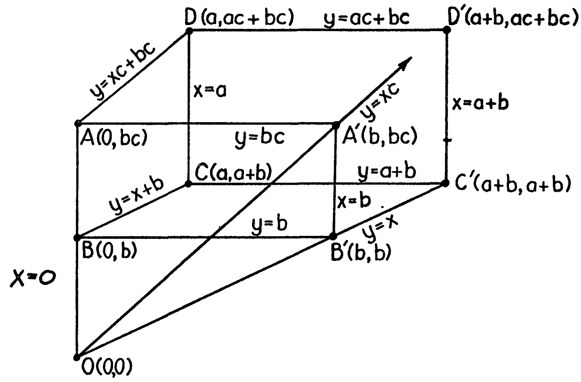

FIG. 9

$D''=(a+b, (a+b)c)$. We want to prove that D'' coincides with $D'=(a+b, ac+bc)$. By use of vectors, $\underline{A'D'}=\underline{A'B'}+\underline{B'C'}+\underline{C'D'}$. By vector equality, $\underline{A'B'}=\underline{AB}$, $\underline{B'C'}=\underline{BC}$, $\underline{C'D'}=\underline{CD}$. Therefore $\underline{A'D'}=\underline{AB}+\underline{BC}+\underline{CD}=\underline{AD}$. In particular, then, the line $A'D'$ has the slope of AD, namely c. However, OA' has slope c, so O, A', D' are collinear. That is, OA' meets $C'D'$ in D', proving that $D''=D'$. This completes the proof of (8.1).

To sum up, in the presence of the vector axiom each planar ternary ring (R, F) is linear, the additive system is an abelian group and the right distributive law (8.1) holds. A planar ternary ring with these properties is known as a Veblen-Wedderburn system, after O. Veblen and J. H. M. Wedderburn, who first studied* such systems in 1907. (For a complete set of postulates, see Appendix II.) It can be shown, conversely, that if one planar ternary ring of π is a Veblen-Wedderburn system then the vector axiom holds. Therefore:

THEOREM 1. *If the vector axiom holds in a Euclidean plane π, then every planar ternary ring of π is a Veblen-Wedderburn system. Conversely, if any one ternary ring of π is a Veblen-Wedderburn system, then the vector axiom holds in π.*

The theory of Veblen-Wedderburn systems is still relatively undeveloped—

* See [4]. The discussion in [2] is better suited to present purposes.

though I would hazard a guess that abstract algebra soon will be able to cope with these systems. In the meantime we are much hampered by the lack of the left distributive law, which must be paid for with an additional geometric axiom.

9. The distributive axiom. Our next axiom (see Fig. 10) is worded§ for ready comparison with one of Hilbert's:

THE DISTRIBUTIVE AXIOM. *Let seven distinct points of the Euclidean plane consist of two triangles ABC, $A'B'C'$ in perspective from a point O, with the pairs $(AB, A'B')$, $(BC, B'C')$ of corresponding sides parallel and with* (*) *BC parallel to OA. Then the third pair of sides, AC, $A'C'$, are also parallel.*

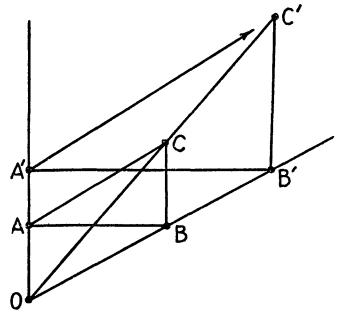

FIG. 10

The corresponding axiom of Hilbert [1] is stronger in that the restrictive hypothesis (*) is omitted. Both of these axioms are special cases of the axiom of Desargues.

Now consider the identity

$$F(a, b, ac) = aF(1, b, c), \qquad \text{all } a, b, c \text{ in } R. \tag{9.1}$$

In the presence of (7.1) (since $1b = b$ for every b) (9.1) is equivalent to the left distributive law

$$ab + ac = a(b + c), \qquad \text{all } a, b, c \text{ in } R. \tag{9.2}$$

Therefore we are interested in the geometric axiom which asserts (9.1) for every planar ternary ring (R, F) of π. Just as we showed that (7.1) was equivalent to the vector axiom, so we can show that (9.1) is equivalent to the distributive axiom. This is indicated by Figure 11 below.

10. Division rings with the right inverse property. Recall the common saying: "You get out of anything just what you put into it." Surely one would have to work hard to justify such a statement in mathematics. For example, if we put into the Euclidean plane π just enough to ensure that every coordinate ring of π is linear (the vector axiom) we get out all the properties of Veblen-

§ It would be neater to say that the three lines OAA', BC, $B'C'$ are parallel.

Wedderburn systems, including the right distributive law. When we insist further on the left distributive law (9.2) for every coordinate ring of π (that is, on the distributive axiom) the harvest is even more remarkable, as we shall see.

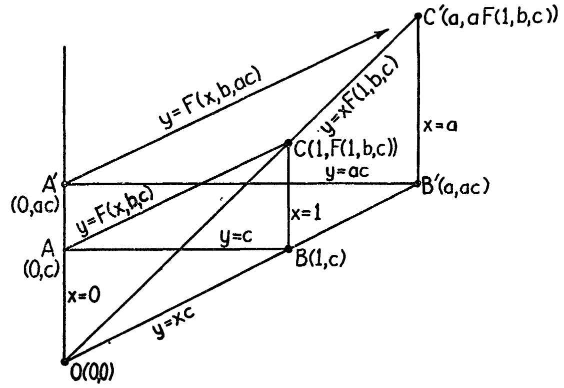

FIG. 11

A Veblen-Wedderburn system which also satisfies the left distributive law is much better known as a *division ring* (with identity element). (For a complete set of postulates see Appendix II.) Thus: *A necessary and sufficient condition that every planar ternary ring of a Euclidean plane* π *be a division ring is that* π *satisfy both the vector axiom and the distributive axiom.* In Theorem 1 it was stated that if any one coordinate ring of a Euclidean plane is a Veblen-Wedderburn system then (the vector axiom holds and) all are. A comparable statement would be false here: there exist Euclidean planes in which some but not all of the coordinate rings are division rings, the rest being merely Veblen-Wedderburn systems. For the correct theorem we need the notion of the right inverse property.

A division ring R (with identity element 1) is said to have the *right inverse property* if each nonzero element a of R has an inverse a^{-1} such that

$$(ba)a^{-1} = b, \qquad \text{all } a, b \text{ in } R,\ a \neq 0. \tag{10.1}$$

(From (10.1) with $b=1$, $aa^{-1}=1$; thus (10.1) yields the weak associative law $(ba)a^{-1}=b(aa^{-1})$.) The correct theorem is as follows:

THEOREM 2. *The following properties are equivalent for a Euclidean plane* π:

(i) π *satisfies the vector axiom and the distributive axiom.*

(ii) *Every planar ternary ring of* π *is a division ring.*

(iii) *Every planar ternary ring of* π *is a division ring with the right inverse property.*

(iv) *Some planar ternary ring of* π *is a division ring with the right inverse property.*

We have already seen that (i) is equivalent to (ii) and we shall be content

to show now that (i) implies (iii). Assuming (i), we have that every planar ternary ring (R, F) of π is a division ring. Consider Figure 12 below. If to Figure 12 we add another triangle $A'B'C'$ in such a manner that the hypotheses of the distributive axiom hold, $A'C'$ will be parallel to AC. We may phrase this more conveniently as follows: if the lines OB, OC remain fixed, the slope of AC is the same for every choice of $A\,(A \neq O)$ on the x-axis. We take $A = (b, 0)$ for any nonzero b. We assume that OB has the fixed slope $1+a$, distinct from 1, 0; so that $a \neq 0, -1$. The equation of OB is then $y = x(1+a)$. Since AB has equation $x = b$, the y-coordinate of B is $b(1+a) = b+ba$. BC, OC have equations $y = b+ba$, $y = x$ respectively, so that $C = (b+ba, b+ba)$. If AC has slope m, the equation of AC is $y = xm+k$ for some k, where, since A and C lie on the line, $0 = bm+k$ and $b+ba = (b+ba)m+k = (bm+k)+(ba)m = (ba)m$. Therefore m satisfies

$$(ba)m = b + ba. \tag{10.2}$$

Since m is independent of b we may set $b = a^{-1}$ where a^{-1} is defined by $a^{-1}a = 1$. Then (10.2) yields $m = a^{-1}+1$. Hence, for all $a \neq 0, -1$ and $b \neq 0$, $(ba)(a^{-1}+1) = b+ba$. From this we get the right inverse property (10.1).—Strictly speaking, we must examine (10.1) for the case $b = 0$ and the case $a = -1$, but these give no trouble.

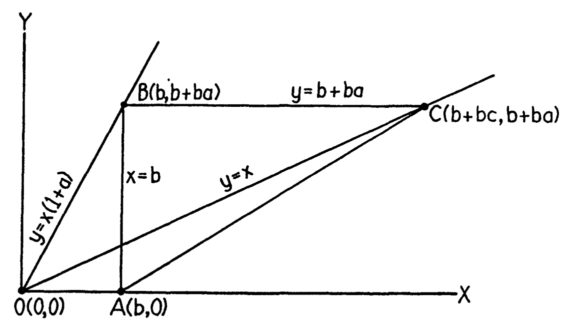

FIG. 12

11. The algebra begins in earnest. At this stage it becomes important to give an algebraic characterization of division rings with the right inverse property. The main theorems are as follows:

THEOREM 3. *Every division ring with the right inverse property is an alternative division ring, and conversely.*

THEOREM 4. *Every alternative division ring is either an associative division ring (that is, a field or skew-field) or a Cayley division algebra over its centre.*

We shall discuss these theorems briefly. It may be shown that a division ring R (with identity element 1) has the right inverse property if and only if it satisfies the identity

$$((ab)c)b = a((bc)b), \qquad \text{all } a, b, c \text{ in } R. \tag{11.1}$$

From (11.1) with $c=1$ we derive

$$(ab)b = a(bb), \qquad \text{all } a, b \text{ in } R. \tag{11.2}$$

A ring satisfying (11.2) is called *right alternative;* and it is called *alternative* if it satisfies both (11.2) and

$$b(ba) = (bb)a, \qquad \text{all } a, b \text{ in } R. \tag{11.3}$$

In 1950, Skornyakov [**5**] proved Theorem 4 for characteristics other than 2, 3. Quite independently, Bruck and Kleinfeld [**6**] proved Theorem 4 for characteristic not 2 and later Kleinfeld [**7**], by combining the methods of the first two papers, removed the restriction as to characteristic. Since then, Kleinfeld [**8**] has given a definitive characterization of simple alternative rings and, incidentally, a new proof of Theorem 4.

In 1951, only a few months after I had become aware of Theorem 2, Skornyakov [**9**] proved that every right alternative division ring of characteristic not 2 is alternative. In the case of characteristic not 2, (11.2) implies (11.1); but this is false for characteristic 2. However, in 1953, San Soucie [**10**] showed that division rings of characteristic 2 which satisfy (11.1) are alternative. Thus Theorem 3 is true.

Instead of elaborating here the properties of Cayley division algebras, we refer the reader to an elementary discussion of these algebras from an entirely different point of view (Dickson [**11**]). It seems of more importance to indicate the geometric significance of Theorems 2, 3, 4. Hilbert [**1**] shows that if the Euclidean plane π satisfies the vector axiom and that strong form of the distributive axiom obtained by omitting (*) (or, in standard language, if π is Desarguesian) then, and only then, every planar ternary ring of π is an associative division ring. Since the class of all division rings with the right inverse property turns out to be very little more extensive than the class of all associative division rings, we draw the following conclusion:

If we intend to insist that every planar ternary ring of the Euclidean plane π be a division ring, we may as well go the whole way and require π to be Desarguesian.

12. Other points of view. It would be unjust to leave the present topic without some brief reference to a great mass of literature entirely neglected here. Most of this literature (including Hall [**2**], Veblen and Wedderburn [**4**]) is written in the language of projective rather than Euclidean planes, which did not suit my aims.* But of course the study of Desarguesian planes, for example, did not originate with Hilbert.

* I shall make no serious attempt to link the present discussion to projective geometry. The reader can discover how to do this for himself by first reading §2 of Hall [2] through the first three lines of p. 232 and then considering Hall's Figure 4 (p. 264) together with the statement (below the figure) of the projective axiom Theorem L. First, in Hall's Figure 4, delete line $AMNB$ and its points and then carefully redraw the figure in the resulting Euclidean plane so that parallel lines

More particularly, the geometric meaning of alternative division rings was first studied by Ruth Moufang (see the references in [2] or [6]) and characterized by the uniqueness of the projective construction of a fourth harmonic point. Later Hall [2] gave an independent characterization in terms of his Theorem L. Theorem L is a projective axiom which, in two of its Euclidean forms, becomes* respectively the vector axiom and the distributive axiom. The characterization of Veblen-Wedderburn systems is originally due to Hall.

One final remark. It is an amusing fact that Theorem 6.4 of Hall [2], although true, was not completely proved until 1953, just ten years after the publication date of Hall's paper. By a trivial slip, the theorem contains the word "two" where "three" would have been appropriate. In 1950 I had a rude awakening in this connection which led to Theorem 2. The record shows that a similar experience led Skornyakov to the study of right alternative division rings.—May there be more such slips!

Appendix I. Planar ternary rings. A planar ternary ring is a system (R, F) consisting of a set R and a ternary operation F subject to the following postulates:

(i) 0 and 1 are two distinct elements of R.

(ii) If a, b, c are in R, $F(a, b, c)$ is a uniquely defined element of R.

(iii) $F(0, b, c) = F(a, 0, c) = c$ for all a, b, c of R.

(iv) $F(a, 1, 0) = F(1, a, 0) = a$ for each a in R.

(v) If b, b', c, c' are in R, with $b \neq b'$, the equation $F(x, b, c) = F(x, b', c')$ has a unique solution x in R.

(vi) If a, a', b, b' are in R, with $a \neq a'$, the system of equations $F(a, x, y) = b$, $F(a', x, y) = b'$ has a unique solution x, y in R.

(vii) If a, b, c are in R, the equation $F(a, b, x) = c$ has a unique solution x in R.

A planar ternary ring (R, F) determines a unique Euclidean plane defined as follows: The points of the plane are the ordered pairs (x, y) of elements x, y of R. Each ordered pair $[m, b]$ of elements m, b of R is a line of the plane which passes through those points (x, y) such that $y = F(x, m, b)$. Each symbol $[a]$, a in R, is a line of the plane which passes through those points (x, y) such that $x = a$.

Addition is defined for a planar ternary ring (R, F) by $a + b = F(a, b, 0)$. The system $(R, +)$ is a loop. That is:

(viii) In the equation $x + y = z$, if any two of x, y, z are assigned as elements of R, the third is uniquely determined as an element of R.

(*e.g.*, RS, XY) appear parallel. You should recognize the figure for the vector axiom (triangles ZXY, TRS). Then note that Theorem L can be interpreted as the vector axiom: ZY is parallel to TS. Now begin afresh with Hall's Figure 4 (or, equivalently, restore the deleted line $AMNB$ and its points). This time delete line ARX and its points and apply the same process. Theorem L, as now interpreted, gives a statement about triangles NZT, MYS which is slightly different from but clearly equivalent to the distributive axiom.—As an alternative suggestion, the reader may prefer to consult a pamphlet by H. G. Forder [12] which (I am told—I have not yet seen it) covers much the same ground as the present paper with more emphasis on the projective formulation.

* See footnote pp. 15–16.

(ix) There exists an element 0 of R such that $0+a=a+0=a$ for every a in R.

A loop $(R, +)$ is a group provided:

(x) $(a+b)+c=a+(b+c)$ for all a, b, c of R, and is an abelian group if also

(xi) $a+b=b+a$ for all a, b of R.

Multiplication is defined for a planar ternary ring (R, F) by $ab=F(a, b, 0)$. In particular,

(xii) $0a=a0=0$ for all a in R.

If R^* consists of R with 0 removed, $(R^*, \cdot)$ is a loop; that is, (viii), (ix) hold with R, $+$, 0 replaced by R^*, $\cdot$, 1 respectively.

Appendix II. Special planar ternary rings. A Veblen-Wedderburn system is a system $(R, +, \cdot)$ consisting of a set R and two binary operations $+$, $\cdot$, subject to the following postulates:

(I) $(R, +)$ is an abelian group with zero 0.

(II.1) $(a+b)c=ac+bc$ for all a, b, c of R.

(III) $(R^*, \cdot)$ is a loop with identity 1.

(IV) $a0=0$ for each a of R.

(V) If a, a', b are in R, with $a\neq a'$, the equation $xa=xa'+b$ has a unique solution x in R.

A Veblen-Wedderburn system $(R, +, \cdot)$ becomes a planar ternary ring (R, F) when F is defined by $F(a, b, c)=ab+c$.

A division ring (with identity element 1) is a system $(R, +, \cdot)$ which satisfies (I), (II.1), (III) and

(II.2) $c(a+b)=ca+cb$ for all a, b, c of R.

Every division ring (with identity) is a Veblen-Wedderburn system, but not conversely.

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FINITE PROJECTIVE PLANES*

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1. Coordinates. We may introduce coordinates [6] in a projective plane in the following way: We take four points X, Y, O, I no three of which are on a line.

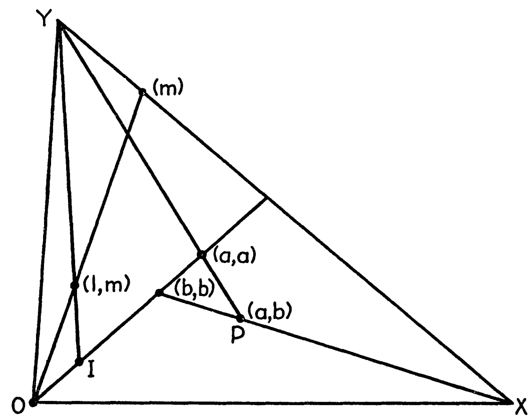

FIG. 1

We call XY the line at infinity, $O=(0, 0)$ the origin, $I=(1, 1)$ the unit point, OX the x-axis and OY the y-axis. For the finite points on OI besides $(0, 0)$ and $(1, 1)$ we assign further coordinates (a, a), (b, b) using a set S of distinct symbols including 0 and 1. If P is any finite point and PY intersects OI in (a, a) and PX intersects OI in (b, b) assign to P the coordinates (a, b). This rule reassigns the same coordinates to the points of OI. So far no coordinates have been assigned to points on the line at infinity. The line through $(0, 0)$ and $(1, m)$ will cut the line at infinity in some point to which we assign the slope coordinate (m). This assigns (0) to X, (1) to the intersection of OI and XY but leaves Y unassigned. We may represent Y by (∞).

A line through Y contains those points (x, y) with $x=c$, a constant. A line through X will be $y=c$. The lines define operations on the elements of S in a natural way. If (x, y) is on the line joining $(0, b)$ and (1) we put $y=x+b$ to define addition. If (x, y) is on the line joining $(0, 0)$ and (m) we put $y=xm$ to define multiplication. More generally, if (x, y) is on the line joining $(0, b)$ and (m), we put $y=x\cdot m \circ b$, defining a ternary operation on the elements of S. The ternary operation includes both addition and multiplication as special cases.

S as a ternary ring satisfies the following laws which are consequences of the axioms for a projective plane:

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T1 $0 \cdot m \circ c = a \cdot 0 \circ c = c$, $1 \cdot m \circ 0 = m$, $a \cdot 1 \circ 0 = a$.
T2 $x \cdot m \circ b = c$ for given $m \neq 0$, b, c, has a unique solution x.
T3 $a \cdot m \circ z = c$ for given a, m, c has a unique solution z.
T4 $a \cdot z \circ b = c$ for given $a \neq 0$, b, c, has a unique solution z.
T5 $x \cdot m_1 \circ b_1 = x \cdot m_2 \circ b_2$ has a unique solution x if $m_1 \neq m_2$.
T6 The pair of equations $x_1 \cdot m \circ b = y_1$, $x_2 \cdot m \circ b = y_2$ has a unique solution for m and b if $x_1 \neq x_2$, $y_1 \neq y_2$.

These rules are dependent (in particular T2 follows from T1 and T5). Every choice of an ordered quadrilateral X, Y, O, I in a plane leads to a ternary ring and conversely a ternary ring satisfying T1, $\cdots$, T6 determines a projective plane.

The different choices for the ordered quadrilateral will in general lead to non-isomorphic ternary rings. Any such isomorphism will determine a collineation of the plane and since there are planes with no collineations these planes have all their ternary rings different.

Any further properties of the ternary ring not given by T1 $\cdots$ T6 will correspond to configuration theorems. An important case is that in which the lines have linear equations: $y = x \cdot m \circ b = xm + b$. This relation corresponds to

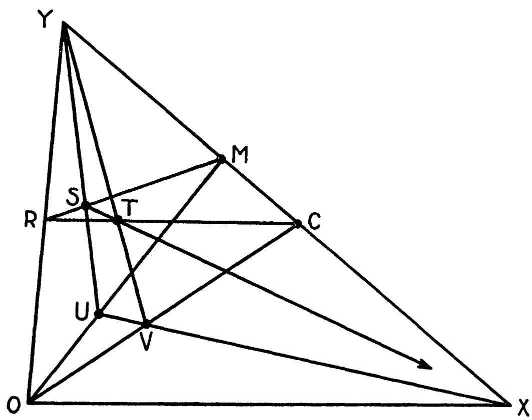

FIG. 2

the minor theorem of Desargues which I call Theorem L. Here Y, the center of perspectivity for the triangles RST and OUV lies on the axis of perspectivity MCX. As given here the line at infinity plays a special role and Theorem L is therefore called an affine theorem. In fact it is even more specialized in that O, X, Y and C all are fixed points.

If Theorem L is taken as a universal configuration theorem then the plane is coordinatized by an alternative division ring, in which multiplication satisfies the weak associative laws: $a(ab) = (aa)b$ and $b(aa) = (ba)a$. This may also be characterized by the weak associative laws: $a^{-1}(ab) = b = (ba)a^{-1}$. For a division ring the one-sided laws $a^{-1}(ab) = b$ and $a(ab) = (aa)b$ are equivalent if the char-

acteristic is not two; but if it is two, the law $a^{-1}(ab)=b$ is stronger. In fact it has been shown recently that a division ring with both distributive laws and the single law $a^{-1}(ab)=b$ is alternative. Geometrically this means that if Theorem L is satisfied for all choices of $XYMC$ on two different lines, then it is satisfied universally. This is a very deep result and has not been shown by any purely geometric means.

In his classical studies on the Foundations of Geometry, Hilbert [**10**] showed that coordinatization by an associative division ring was equivalent to the universal validity of the full theorem of Desargues and was implied by the affine form in which the axis of perspectivity is the line at infinity. Hilbert also showed that if in addition the Theorem of Pappus was valid then equivalently the division ring would be a field. It was shown by Hessenberg [**11**] that the Theorem of Pappus implies the Theorem of Desargues. Geometrically the Theorem of Pappus is equivalent to the assertion that in every ternary ring for a plane the multiplication is commutative, whence this result shows that all further field properties of the ternary rings follow from the commutativity of multiplication. Naturally the commutativity of multiplication in one ternary ring is a relation of a different kind in another.

The uniqueness of harmonic conjugates was shown by Monfang [**17**] to be equivalent, for characteristic not two, to coordinatization from an alternative division ring. Perhaps the most striking recent result in this subject is the discovery by Bruck and Kleinfeld [**5**] and independently by Skornyakov [**25**] that an alternative division ring is either (1) associative or (2) a Cayley-Dickson algebra over its center. Different degrees of specialization of Theorem L have been studied recently by Herbert Naumann [**18**] who has shown these to be equivalent to various properties of addition, multiplication, and distributivity. A ring with addition, a group, and one distributive law is called a *near-ring;* and if there are no divisors of zero, it is called a *near-field.* All finite (associative) near-fields have been determined by Zassenhaus [**27**]. A near-field is a special case of a *quasi-field* or Veblen-Wedderburn system. A quasi-field has abelian addition and one distributive law $(x+y)m=xm+ym$, but no further properties are assumed for multiplication other than that it be a loop. A system opposite to the quasi-field is the neo-field, investigated by Lowell Parge [**20**] and D. R. Hughes [**13**]. Here, multiplication is taken to be a group and the distributive laws are assumed, but no properties are required for the additive loop. Some infinite planar neo-fields are known, but no finite ones are known, except for finite fields.

An interesting unsettled question in the study of coordinates and configurations is the matter of the consequences of the Fano configuration. This says that the diagonal points of every quadrilateral are collinear. This is the configuration for characteristic two and is certainly satisfied in any alternative plane of characteristic two. But it is not known if the Fano planes are a still more general class. Rashevskii [**22**] has made some elementary investigations into this.

2. Finite planes. We may take $y=x'm \circ b=xm+b$ with coordinates from any field, and so, if we wish, from any finite field with $n=p^r$ elements. In such a projective plane every line contains $n+1$ points, and every point is on $n+1$ lines. In fact if for some finite $n\geqq 2$, one line of a plane contains $n+1$ points, it is easy to show that every line contains exactly $n+1$ lines, and the total number of points is $N=n^2+n+1$, and the total number of lines is $N=n^2+n+1$. Since every finite associative division ring or even finite alternative division ring is a finite field, in a finite plane Theorem L, the full Desargues theorem, and the theorem of Pappus are all equivalent if taken as universal theorems. Thus, for every finite Desarguesian plane with $n+1$ points on a line, n is a prime power $n=p^r$. Although several different types of non-Desarguesian planes are known, in every case n is a prime power. Professors Bruck and Ryser [4] have shown that a plane cannot exist if $n\equiv 1, 2 \pmod 4$, $n\neq a^2+b^2$. But the gap between this restriction and the prime powers is considerable. The first critical value of n is $n=10$. A thorough investigation of this case is currently beyond the facilities of computing machines.

For $n=2, 3, 4, 5$ it is easy to test all cases and show that only the Desarguesian plane exists. The case $n=6$ is excluded by Bruck and Ryser. But this case had earlier been eliminated by Tarry [26] and stood for a long time as a solitary curiosity. This study was in terms of orthogonal latin squares. A *latin square* is an n by n square of n^2 cells filled with n letters $a_1, a_2, \cdots, a_n$ so that each row contains each letter exactly once and also each column contains each letter exactly once. Two squares are said to be *orthogonal* if, when superposed, the ordered pairs a_ib_j associated with the cells give each of the n^2 possibilities exactly once. A set of $n-1$ mutually orthogonal squares determines an affine plane (and so a projective plane) in this way. Regard the n^2 cells as points. Then the points are arranged in lines in the following ways: the n points forming each row are on a line, the n points forming each column are on a line, and each square determines n lines, namely for each letter a the n points to which it assigns a. Conversely, an affine plane determines $n-1$ mutually orthogonal squares. Tarry showed that for $n=6$ there were not even two mutually orthogonal squares. Thus *a fortiori* there is no plane. It has been conjectured that for $n\equiv 2 \pmod 4$ there are no orthogonal squares. A proof of this based on topological considerations was attempted by MacNeish but the proof contains a major error. It has been shown by H. B. Mann [15] that for $n\not\equiv 2 \pmod 4$ there are at least two mutually orthogonal squares. More precisely for any $n=p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$ there are at least min $(p_i^{e_i}-1)$, $i=1\cdots r$, mutually orthogonal squares. For $n=10$ extensive searches on the SWAC machine at UCLA have failed to produce any orthogonal squares. But even 100 machine hours will not cover more than a microscopic part of the complete search.

A pair of orthogonal latin squares is associated with a quadratic representation. We associate variables x_i with rows, y_j with columns, z_k with letters of the first square and w_e with letters of the second square. Then if z_k and w_e are in the cell of x_i, y_j, we form $(x_i+y_j+z_k+w_e)^2=Z$. Here

$$Q = n\left(\sum x_i^2 + \sum y_j^2 + \sum z_k^2 + \sum w_e^2 \right) + 2 \sum x_i \sum y_j + \cdots + 2 \sum z_k \sum w_e$$

expresses precisely the conditions for the orthogonality. In this connection an interesting situation has arisen. Here Q is the sum of squares of non-negative linear forms. As such it is surely semi-definite and has non-negative coefficients. I raised the question as to whether in general there were further conditions on a sum of squares of non-negative forms. Professor Alfred Horn showed that further conditions do exist and that in the theory of convex spaces these forms are adjoint to forms which are non-negative for non-negative arguments. On 5 or more variables further conditions exist.

For $n=7$ the only plane is the Desarguesian plane. There are 147 essentially different 7×7 latin squares of which 146 were listed by Norton [**19**] and the missing 147th was found by Sade [**22**]. Finding 6 mutually orthogonal squares gives only the Desarguesian plane. A more direct counting method by Pierce [**21**] and M. Hall [**8**] also establishes this result. Calculations on the SWAC machine show that the Desarguesian plane is the only one for $n=8$.

For $n=9$ there are several non-Desarguesian planes. Albert [**1**] has shown the existence of non-Desarguesian planes for $n=p^r$, p odd, $r\geqq 2$. There are also non-Desarguesian planes for $n=2^{2s}$, $s\geqq 2$. For $n=p$ only the Desarguesian plane is known. Near-fields and Veblen-Wedderburn systems have given many non-Desarguesian planes.

If addition is an elementary abelian group, then $n=p^r$. Thus to find a plane with $n\neq p^r$ we must not restrict the addition too heavily. In a neo-field multiplication is taken as an abelian group and the distributive laws are assumed, but no further direct property of addition. A number of additional properties have been proved for finite neo-fields. Unfortunately no planar neo-field has been found yielding anything but Desarguesian planes in the finite case.

A further attempt to construct finite planes is in terms of a collineation group. Let us assume that a plane has a collineation of order $N=n^2+n+1$, cyclic on the points. It will also be cyclic on the lines of the plane and the points of a line will yield a difference set. Thus 0, 1, 3, 9 are a difference set modulo 13 since every $d\not\equiv 0 \pmod{13}$ is of the form $a_i-a_j\equiv d \pmod{13}$ exactly once with the a's from the set 0, 1, 3, 9. Here $i+0$, $i+1$, $i+3$, $i+9 \pmod{13}$, $i=0, \cdots, 12$ give the lines of the plane with $n=3$. More generally with $N=n^2+n+1$, residues $a_0, a_1, \cdots, a_n \pmod{N}$ form a difference set if $d\not\equiv 0 \pmod{N}$ is expressible exactly once in the form $d\equiv a_i-a_j \pmod{N}$. The construction of difference sets is precisely equivalent to constructing cyclic planes. Singer [**24**] showed that every finite Desarguesian plane is cyclic and unfortunately no others have been found. Evans and Mann [**16**] have shown that for a cyclic plane we must have $n=p^r$ at least for $n\leqq 1600$. The cyclic approach has at least yielded some results of interest for symmetric designs. If $k(k-1)=d(v-1)$ and $a_1, a_2, \cdots, a_k \pmod{v}$ are such that for $d\equiv 0 \pmod{v}$ the congruence $a_i-a_j\equiv d \pmod{v}$ has exactly λ solutions, then the sets $a_1+j, \cdots,$

$a_k+j \pmod v$, $j=0, \cdots v-1$, will form the blocks of a symmetric design.

In constructing the cyclic planes and cyclic designs, a *multiplier* is very useful. The number t is a multiplier if $ta_1, ta_2, \cdots, ta_k \pmod v$ are $a_1+s, \cdots, a_k+s \pmod v$ in some order. It is easy to see that the multipliers are prime to v and form a multiplicative group modulo v. I have proved that if $p \mid k-\lambda=n$, $(p, v)=1$ and $p>\lambda$, then p is a multiplier. For the planes, since $\lambda=1$, the condition $p>\lambda$ is trivial. Actually $p>\lambda$ seems by examples to be an unnecessary condition and also out of place in a purely arithmetical situation, but so far the proofs of the existence of multipliers require this. Here for $K=9$, $v=37$, $\lambda=2$, we have $n=7$ and the solution is 1, 7, 9, 10, 12, 16, 26, 33, 34 (mod 37). Although there have been found only the Desarguesian planes among the finite cyclic planes it is conceivable that there may be others. The cyclic designs are not necessarily unique. There are two cyclic designs for $K=15$, $v=31$, $\lambda=7$, and two for $K=21$, $v=43$, $\lambda=10$.

R. H. Bruck* has generalized to replace the cyclic group by any abelian group and A. Hoffman [12] has considered cyclic affine planes, these being planes with a collineation fixing L and O and cyclic on the remaining n^2-1 points. In both cases the multiplier theorem carries over.

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GEOMETRIES AND INCIDENCE MATRICES*

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1. Introduction. A projective plane π is a mathematical system composed of elements, called "points," and certain sets of points, called "lines," subject to the following postulates:

P (1) *Two distinct points are contained in one and only one line.*

P (2) *Two distinct lines contain one and only one point in common.*

P (3) *There exists at least one set of four points of π, no three of which are contained in one line.*

Other equivalent sets of postulates for a projective plane are available. Those which we have selected appear in a now classical paper by M. Hall [6]. Briefly, postulates P (1) and P (2) are entirely basic to the system. Postulate P (3) serves to exclude certain degenerate systems, which satisfy only P (1) and P (2). The recent expositions of Skornyakov [**21**] and Hall [**7**] lend considerable insight into the geometrical consequences of these postulates. A book by Günter Pickert concerning projective planes is scheduled to appear soon.

Let us now impose a further restriction on π—one that at first glance appears to be quite exotic. Suppose that the total number of points of π is finite. Such planes are called *finite planes* and these systems play an important role in modern combinatorial analysis. If π is finite, then one proves without great difficulty that there exists a positive integer $n \geqq 2$ with the following properties:

(1) *Each line of π contains exactly $n+1$ distinct points.*

(2) *Each point of π is contained in exactly $n+1$ distinct lines.*

(3) *The plane π is made up of exactly n^2+n+1 distinct points and n^2+n+1 distinct lines.*

The integer n is clearly an important invariant of the finite plane. The first and most primitive problem that arises is the determination of the precise range of values of n for which finite planes exist. This problem is largely unsolved at the present time. Consider a finite Galois field of $n=p^\beta$ elements, where p is a prime and where β is a positive integer. By utilizing the properties of these finite fields, one readily constructs a finite plane π, with $n+1$ points per line, in which the Theorem of Desargues is valid. Finite non-Desarguesian planes have also been constructed by various methods, but in all known cases, we have $n=p^\beta$.

Thus it is natural to conjecture that planes π do not exist for values of n other than $n=p^\beta$. The investigations of Tarry imply the correctness of the conjecture for the case $n=6$ [**22**]. Concerning this conjecture, one may establish the following Theorem of Bruck and Ryser on the nonexistence of finite planes [**2**].

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THEOREM 1. *Let* $n \equiv 1$ *or* 2 (mod 4) *and let the squarefree part of n contain at least one prime factor of the form* $4k+3$. *Then* π *with* $n+1$ *points on a line does not exist.*

Evidently Theorem 1 excludes geometries for infinitely many values of n, for example, the values $n=2p$, where p is a prime of the form $4k+3$. Of course, there remain infinitely many undecided values. No values of n other than those covered by Theorem 1 have been excluded up to the present time. This state of affairs has resulted in the development of two opposing points of view. Conservatives maintain π exists if and only if $n=p^{\beta}$, and liberals maintain π exists for all values of n, except those specifically excluded by Theorem 1. The smallest undecided value is $n=10$.

2. The v, k, λ problem. Let it be required to arrange v elements into v sets such that every set contains exactly k distinct elements and such that every pair of sets has exactly λ elements in common, $0<\lambda<k<v$. This problem we refer to as the *v, k, λ problem*, and we call the resulting arrangement a *v, k, λ configuration* [3]. These configurations are actually a generalization of the finite planes π described in the introduction.

Let us first consider an example of a v, k, λ configuration. Denote the elements by x and the sets by T, and let

$$T_1 = \{x_1, x_2, x_4\},$$
$$T_2 = \{x_2, x_3, x_5\}, \quad T_3 = \{x_3, x_4, x_6\}, \quad T_4 = \{x_4, x_5, x_7\},$$
$$T_5 = \{x_5, x_6, x_1\}, \quad T_6 = \{x_6, x_7, x_2\}, \quad T_7 = \{x_7, x_1, x_3\}.$$

It is easy to verify that this is a v, k, λ configuration for the case $v=7$, $k=3$, and $\lambda=1$.

For a v, k, λ configuration, let us list the elements $x_1, \cdots, x_v$ in a row, and the sets $T_1, \cdots, T_v$ in a column. Insert 1 in row i and column j if x_j belongs to set T_i, and 0 otherwise. In this way we obtain a v by v matrix A of 0's and 1's, called the *incidence matrix* of the v, k, λ configuration. This matrix contains all the essential information given by the v, k, λ configuration.

THEOREM 2. *The v, k, λ problem has a solution if and only if there exists a* 0, 1 *matrix A of order v such that*

$$AA^T = B,$$

where A^T denotes the transpose of A and where the matrix B has k in the main diagonal and λ in all other positions.

Theorem 2 follows directly from the definition of a v, k, λ configuration.

THEOREM 3. *For a v, k, λ configuration,*

$$\lambda = \frac{k(k-1)}{v-1}.$$

Theorem 3 appears in [**16**]. A generalization may be found in [**10**]. To prove the result, let s_i denote the sum of column i of the matrix A. Then for $i=1, 2, \cdots, v$,

$$a_{i1}s_1 + \cdots + a_{iv}s_v = k + (v-1)\lambda. \tag{4}$$

The determinant of this system is $|A|$. Since $AA^T=B$, and

$$|B| = (k-\lambda)^{v-1}(k+(v-1)\lambda),$$

we have $|A|\neq 0$. Thus the system (4) has a unique solution, $s_i=c$. But $s_1+\cdots+s_v=kv$, and hence $c=k$. Thus $k^2=k+(v-1)\lambda$.

THEOREM 4. *The incidence matrix of a v, k, λ configuration is normal, that is*

$$AA^T = A^TA.$$

For we may let

$$Q = \begin{bmatrix} -k & \sqrt{-\lambda} \cdots \sqrt{-\lambda} \\ \sqrt{-\lambda} & \\ \vdots & A \\ \sqrt{-\lambda} & \end{bmatrix}.$$

Since $\lambda=k(k-1)/(v-1)$, we have $QQ^T=(k-\lambda)I$, where I is the identity matrix of order $v+1$. But then $Q/\sqrt{k-\lambda}$ is orthogonal, and hence $QQ^T=Q^TQ$. But because of the structure of Q, $AA^T=A^TA$.

In a v, k, λ configuration, let $\lambda=1$, and let $k=n+1$, where $n\geqq 2$. Then $v=n^2+n+1$, and the v, k, λ configuration reduces to a projective plane π with $n+1$ points per line. Indeed, the normality of the incidence matrix A assures us that the principle of duality operates in the finite plane.

Other specializations of the values of v, k, and λ lead to other classical types of configurations. For example, for $v=4m-1$, $k=2m-1$, $\lambda=m-1$, the v, k, λ configuration is equivalent to a Hadamard matrix of order $N=4m$ [**15**]. These are the ± 1 matrices H satisfying $HH^T=NI$. Here H is of order N and I denotes the identity matrix. If H exists, then it is easy to show that the order of H is 1, 2, or $\equiv 0 \pmod 4$. It is conjectured that H exists for each of these orders. The first undecided value is $N=92$. The v, k, λ configurations arise in a natural way also in statistics, where they are referred to as symmetrical block designs. An extensive literature has developed concerning these designs and their generalizations (see, for example [**12**, **11**]).

Finally, it should be mentioned that we have imposed no restrictions on the incidence matrix A. If, for example, we require the incidence matrix to be cyclic, then the v, k, λ configuration reduces to the perfect difference set which arises in the theory of numbers [**20**, **8**]. For geometries, this restriction is roughly equivalent to the validity of the Theorem of Desargues [**5**].

3. Nonexistence theorems. In this section we summarize some of the main results obtained by investigating the matric equation $AA^T = B$, where B has k in the main diagonal and λ elsewhere.

THEOREM 5. *If the v, k, λ problem has a solution and if v is even, then $k-\lambda$ must be a square.*

This result follows easily upon taking determinants in the matric equation $AA^T = B$ [**18**, **3**].

THEOREM 6. *If the v, k, λ problem has a solution and if v is odd, then*

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$$

possesses a nonzero integral solution.

Theorem 6 appears in [**3**] and is derived by methods which are entirely elementary, requiring nothing more than the fact that every positive integer is the sum of four rational squares. For further commentaries concerning [**3**], see [**7**, **8**, **11**]. If $\lambda = 1$, then Theorem 6 reduces to Theorem 1. Indeed, Theorem 6 may be derived along lines entirely analogous to the original proof of Theorem 1 appearing in [**2**], which utilized the Minkowski-Hasse invariants of a quadratic form. In this connection see [**19**].

Theorems 5 and 6 for v, k, λ configurations give us no new information concerning geometries, and they tell us nothing about the state of affairs for Hadamard matrices. They do, however, effectively exclude numerous v, k, λ configurations. Indeed, up to now these are the only v, k, λ configurations excluded, so that one may conjecture their existence in all other cases.

One may show that the values of v, k, λ excluded by Theorems 5 and 6 are precisely those for which there is no rational X such that $XX^T = B$. Thus the nonexistence theorems have not utilized the normality of the incidence matrix, and it is natural to inquire if this additional restriction on A cannot be used to our advantage. Concerning this point, one may prove the following generalization of a Theorem of Albert [**1**, **9**].

THEOREM 7. *Suppose there is a rational X such that $XX^T = B$. Then there is a rational A such that $AA^T = A^TA = B$.*

Finally, Theorem 7 may be extended to yield the following [**9**]:

THEOREM 8. *Suppose there is a rational X such that $XX^T = B$. Let A_1 be an r by v matrix of* 0*'s and* 1*'s, such that*

$$A_1 A_1^T = B_1,$$

where B_1 is of order r, and has k in the main diagonal and λ in all other positions. Then there is a v by v rational A having A_1 as its first r rows such that

$$AA^T = A^TA = B.$$

Thus any consistent 0, 1 "start" for an incidence matrix may always be completed both rationally and normally. These results are in sharp contrast to those of Connor, who has investigated non-symmetrical designs, and obtained restrictions on initial blocks [**4**].

4. An integral approach. It seems reasonably certain that arguments restricted to the rational field cannot by themselves answer all the questions arising in the study of the existence of these configurations. We turn then to a study of the equation $XX^T=B$, where we suppose that X has *integral* elements. The results obtained up to now concerning integral investigations are a good deal more sketchy. One may easily derive the following [**17**].

THEOREM 9. *Suppose that $AA^T=A^TA=B$, where A is integral. Then A is a* 0, 1 *matrix, or a* 0, -1 *matrix, and hence yields a v, k, λ configuration.*

The requirement of the normality of A in Theorem 9 is annoying, and may be removed for some values of v, k, λ [**17**].

THEOREM 10. *Let $(k, k-\lambda)=1$ and let $k-\lambda$ be odd. Suppose that $AA^T=B$, where A is integral. Upon multiplication of the columns of A by ± 1, A is transformed into a* 0, 1 *incidence matrix.*

Thus, at least for the case of $(k, k-\lambda)=1$ and $k-\lambda$ odd, the v, k, λ problem is equivalent to the problem of representing B integrally by means of the identity. Unfortunately, this problem in the theory of quadratic forms is an unsolved one.

Theorem 10 is not valid for the case in which $k-\lambda$ is even. Consider the projective plane, with n an even integer. For this situation, it appears that there will exist integral solutions of $XX^T=B$, provided only that B is rationally congruent to the identity. To substantiate this conjecture, integral solutions have been exhibited for n equal to the order of a Hadamard matrix. Also, an interesting integral solution has been obtained for the case $n=10$ [**9**]. These integral solutions, however, are far removed from 0, 1 incidence matrices.

5. Future research. The incidence matrix has shown itself to be a powerful tool in the study of geometries. Arithmetical properties of the matrix A yield new insight into the geometry of π. These arithmetical properties may have little or no geometrical significance, and this constitutes an inherent weakness of the approach. However, it may also serve to explain in part why no purely geometrical derivation of Theorem 1 has as yet been obtained.

The problems remaining are both difficult and numerous. The most immediate center around an extension of Theorem 1. Other vital problems in geometry, however, also have an incidence matrix interpretation. For example, it has been conjectured that every finite plane possesses a nontrivial collineation. Relative to incidence matrices, this means that we are to find a permutation matrix $P\neq I$ and a permutation matrix Q such that $PAQ=A$. Suppose that the finite plane π has additional requirements imposed upon it of one kind or another.

These requirements must be reflected in the structure of the matrix A. In this way we encounter a large class of problems dealing with configuration theorems and the like. Doubtless some of these can be effectively handled by matric methods. Finally, one may attempt an enumeration of the distinct geometries for specified values of n. This problem, however, appears to be entirely beyond the range of our present machinery, except for small values of n. Each of the problems in question may be generalized to v, k, λ configurations.

These problems are subject to a variety of attacks. If possible, the integral solutions of $XX^T=B$ should be more fully explored. Equally interesting would be a continuation of Pall's study of the elementary divisors of A [**14**]. Concerning elementary divisors, we remark that the inverse of A is equal to $[1/(k-\lambda)][A^T-(\lambda/k)S]$, where S is the matrix of all 1's. This relationship allows one to compute the elementary divisors of A, under the restriction that $(k, k-\lambda)=1$ and that $k-\lambda$ is a squarefree integer.

An arbitrary incidence matrix A may be written in the form $A=L_1+\cdots+L_k$, where the L_i are permutation matrices. This is a consequence of the theory of systems of distinct representatives [**13**]. The relationship has not yet been utilized in the study of v, k, λ configurations. The eigenvalues of A are certainly not invariant under permutations of the rows and columns of A. Nevertheless, they may have a deep combinatorial significance. One eigenvalue of A is k, and the remaining $v-1$ eigenvalues are algebraic integers, each equal to $\sqrt{k-\lambda}$ in absolute value. That the eigenvalues remain invariant in absolute value under arbitrary permutations of the rows and columns is in itself a remarkable property of A. Note also that in the cyclic case the eigenvalues equal the algebraic integers $\theta(\epsilon^i)$, and these integers are basic to the study of difference sets [**8**].

Yet another approach might center around the matric equation $XX^T=B$, where the elements of X are nonnegative reals. Finally, one must mention modern computational procedures. For $n=10$, there are something like $2^{10,000}$ possible candidates for an incidence matrix. Machines cannot cope with numbers of this magnitude. But the right combination of computation and theory will continue to produce worthwhile results.

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FINITE PLANE PROJECTIVE GEOMETRIES*

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1. Introduction. Only finite plane projective geometries will be discussed in this paper, and, for the sake of brevity, they will be called "geometries." A geometry with n points on a line is easily constructed provided $n-1$ is a power of a prime. It is not known if other geometries exist. There is only one geometry with n points on a line if $n=3$, 4, 5, 6, or S, but there are at least four geometries for $n=10$, one desarguesian and the other three non-desarguesian [**2**, p. 273, **3**, p. 912, **1**, p. 411]. (There may be more.) There are no geometries with seven points on a line. It is not known if there exists more than one geometry for the case $n=9$.

Some elementary properties of geometries follow easily from the postulates, and others are obtained when the geometry is constructed by means of an algebraic system. Any geometry may be constructed from a ternary system satisfying certain conditions and it is therefore of interest to examine the algebra of such a system.

2. Definitions and elementary properties of a finite plane projective geometry [**1**, p. 323]. A class of elements or *points* in which there is determined a class of subsets called *lines* satisfying the following conditions or axioms A is called a *finite plane projective geometry.*

A1. *S contains only a finite number of points.*

A2. *There exist a point and a line not incident.*

A3. *Each line is incident with at least three points.*

A4. *Any two distinct points are incident with one and only one line.*

A5. *Any two lines are incident with at least one point.*

The above set of postulates is equivalent to the dual set; therefore, the principle of duality holds. Also, for a geometry with n points on a line there are n lines containing any point, and there are n^2-n+1 points (or lines) in the geometry.

3. Rules for the construction of any geometry [**4**, p. 9]. Let the first n^2-n+1 positive integers be the points of a geometry with n points on a line. Let each line of the geometry be named by the n integers which are points incident with the line.

Below is a skeleton which may be used as a starting point in attempting to construct a geometry.

Each row will represent a line, the elements of the row representing points incident with the line. There will be n lines containing each integer.

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$$
\begin{array}{c}
\overline{n}\\
n\left\{\begin{array}{ccccc}
1 & 2 & 3 & \cdots & n\\
1 & n+1 & n+2 & \cdots & 2n-1\\
1 & 2n & 2n+1 & \cdots & 3n-2\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
1 & n^2-2n+3 & \cdot & \cdots & n^2-n+1
\end{array}\right.
\end{array}
$$

$$
n-1\left\{\begin{array}{cc}
2 & n+1\\
2 & n+3\\
2 & \cdot\\
\vdots & \vdots\\
2 & 2n-1
\end{array}\right.
\quad
\boxed{\begin{array}{ccc}
2n & \cdots & n^2-2n+3\\
2n+1 & \cdots & \cdot\\
\vdots & \cdots & \vdots\\
3n-2 & \cdots & n^2-n+1
\end{array}}
$$

$$
n-1\left\{\begin{array}{cc}
3 & n+1\\
3 & n+3\\
3 & \cdot\\
\vdots & \vdots\\
3 & 2n-1
\end{array}\right.
\quad
\boxed{\begin{array}{ccc}
 & &
\end{array}}
$$

$$
n-1\left\{\begin{array}{cc}
4 & n+1\\
4 & n+2\\
4 & \cdot\\
\vdots & \vdots\\
4 & 2n-1
\end{array}\right.
\quad
\boxed{\begin{array}{ccc}
 & &
\end{array}}
$$

$$\vdots$$

Notice that the arrays inside the rectangles involve only the $(n-1)(n-2)$ consecutive integers $2n, \cdots, n^2-n+1$. Let these $n-1$ arrays, each consisting of $n-1$ rows and $n-2$ columns, be filled in with the same integers $2n$, $2n+1, \cdots, n^2-n+1$, such that the following rules are fulfilled:

I. *The j-th column of each array contains the same integers as the j-th column of the first array.*

II. *Each integer is in a different row in each array.*

III. *There is one row containing any two integers from different columns.*

Rule I is merely an agreement which does not affect the construction of the geometry materially, since the order of the points of a line is immaterial.

It is not difficult to show that the above collection of lines and points form a geometry if and only if rules I, II, III are obeyed.

If we include the column to the left of each array we have $n-1$ arrays, each $n-1$ by $n-1$. Let the arrays be numbered $0, 1, 2, \cdots, n-2$, and let the rows (columns) of each be numbered $0, 1, \cdots, n-2$. Let $\overset{k}{a}_{ij}$ be the element in the i-th row and j-th column of the k-th array. Rule I permits the elements in each column to be composed of the integers $0, 1, \cdots, n-2$ without loss of generality. Also the 0-th array may be considered to consist of $n-1$ identical columns, each containing $0, 1, \cdots, n-2$ in that order. That is,

$$\overset{0}{a}_{xj} = x. \tag{1}$$

We have also that

$$\overset{k}{a}_{x0} = x. \tag{2}$$

The element in the 0-th row and 1-st column of the x-th array may be taken as x. That is,

$$\overset{x}{a}_{01} = x. \tag{3}$$

And, the element in the 0-th row and x-th column of the 1-st array may be taken as x, or

$$\overset{1}{a}_{0x} = x. \tag{4}$$

Rule I gives the following rule:
(5) If $b \in \{0, 1, \cdots, n-2\}$,

$$\overset{k}{a}_{xj} = b \text{ has a solution } x.$$

From rule II we have:
(6) For $j \neq 0$ and $k \neq k'$,

$$\overset{k}{a}_{ij} \neq \overset{k'}{a}_{ij}.$$

Rule III gives the following:
(7) For $j \neq j'$, there exists a pair x, y such that

$$\overset{x}{a}_{yj} = \overset{k}{a}_{ij} \quad \text{and} \quad \overset{x}{a}_{yj'} = \overset{k'}{a}_{i'j'}.$$

Now let $\overset{k}{a}_{ij}$ be considered as a ternary operation jki on the elements of the set $0, 1, \cdots, n-2$, such that jki is in the set. Equations (1) through (7) are then written
(8) $0ax = a0x = x$, $x10 = 1x0 = x$,

(9) $abx=c$ has a unique solution x,
(10) $axb=c$ has a unique solution x if $a\neq 0$,
(11) For $a\neq c$, there exists a unique solution x, y to the equations

$$\iota xy = b,$$
$$cxy = d.$$

Equations (8) and (9) follow from (1), (2), (3), (4), (5). Equation (6) is equivalent to the following:

$$\text{For } a \neq 0 \quad \text{and} \quad c \neq d, \qquad adb \neq acb.$$

This condition is equivalent to (10). Equation (11) follows from (7).

One of the above postulates is redundant, and consequently this *H-system* may be defined as one with a finite number of elements $0, 1, a, b, \cdots$ such that:

H1. $a0b=0ab=b$, $1a0=a10=a$.

H2. *$abx=c$ has a unique solution x.*

H3. *For $a\neq c$, the equations $axy=b$, $cxy=d$ have a unique solution x, y.*

Theorem 3.1. *If $c\neq 0$, then equation $cxb=d$ has a unique solution x.*

Proof. In H3 if $a=0$ and $c\neq 0$, the postulate gives the desired result.

Lemma. *If $a\neq c$, $x_1\neq x_2$, and $x_1ab=x_1cd$, then $x_2ab\neq x_2cd$.*

Proof. Suppose the contrary and let

$$x_1ab = x_1cd = f,$$
$$x_2ab = x_2cd = g.$$

Then the equations

$$x_1xy = f,$$
$$x_2xy = g,$$

have two solutions, namely $x=a$, $y=b$, and $x=c$, $y=d$. This violates H3.

Let the following postulate be called H3′.

H3′. *For $a\neq c$, $xab=xcd$ has a unique solution x.*

Theorem 3.2. *The postulates* H1, H2, H3 *are equivalent to the set* H1, H2, H3′.

Proof. First it is shown that H1, H2, H3 imply H3′. If $a\neq c$, and $b=d$, the only solution of $xab=xcd$ is $x=0$. If $a\neq c$ and $b\neq d$, let d_i be such that $x_iab=x_icd_i$, $(i=1, 2, 3, \cdots)$, where $x_1, x_2, x_3, \cdots$ are distinct. It follows from the lemma that $d_1, d_2, \cdots$ are distinct. Hence, for any d the equation $xab=xcd$ has a unique solution.

Next it is proved that H1, H2, H3′ imply H3. Consider the equations

$axy=b$, $cxy=d$, where $a\neq c$. For any x_i there exists a y_i such that $ax_iy_i=b$. (See H2). This gives a correspondence $x_i\rightarrow y_i$. For $x_i\neq x_j$, $cx_iy_i\neq cx_jy_j$, since the equation $xx_iy_i=xx_jy_j$ has but one solution $x=a$. (See H3′). Hence cx_iy_i takes on the value b for some unique i.

THEOREM 3.3. *In an H-system the equation $xab=c$ has a unique solution x if $a\neq 0$.*

Proof. In H3′ let $0=c\neq a$. Then the theorem follows.

We sum up these results by the following theorem.

THEOREM 3.4. *Any H-system may be characterized by* H1, H2, H3, *or* H1, H2, H3′. *From each such system a geometry may be constructed. Conversely, any geometry leads to an H-system.*

Marshall Hall develops the postulates for a ternary system which characterizes projective geometries (finite or infinite) [**2**, p. 247]. In the finite case, his system is equivalent to the H-system, but the methods of construction of the ternary system from the postulates of the geometry differ in point of view and procedure.

It is of great interest to examine these ternary H-systems.

4. Analysis of Hall's ternary system. In this section are derived some theorems that are fundamental in a treatment of the H-system. All of the properties and theorems discussed correspond closely to some of the elementary properties and theorems related to binary systems. Some of the questions of interest are these:

What additional conditions must be fulfilled in order that the ternary system be abstractly equivalent to a double binary system? Does there exist an H-system with r elements, where r is not a prime-power? What are the characteristics of an H-system which is "distributive" or "associative," or both?

In an H-system the ternary operation abc should not be assumed to be the result of two binary operations. But the following properties are easily shown to be true:

$$ab0 = 0 \rightleftharpoons \text{either } a = 0, \text{ or } b = 0. \tag{1}$$

$$abd = acd,\ a \neq 0 \rightarrow b = c. \tag{2}$$

$$bad = cad,\ a \neq 0 \rightarrow b = c. \tag{3}$$

$$abc = abd \rightarrow c = d. \tag{4}$$

Define for each element x in an H-system

$$x_0 = 0 \quad \text{and} \quad x_i = 1x_{i-1}x$$

for $i=1, 2, \cdots$, and let the *A-order* of x be the first $i\neq 0$ for which $x_i=0$. It is clear that if $x\neq 0$, $x_i\neq x_{i-1}$. Also, if $x_k=x_j$ (k and j positive) it follows that $x_{k-1}=x_{j-1}$. For $1x_{k-1}x=1x_{j-1}x$ implies $x_{k-1}=x_{j-1}$. (See property 2.)

THEOREM 4.1. *Each element of H has an A-order.*

Proof. The A-order of 0 is equal to 1. For $x \neq 0$, let x_k be the first x_i which is a repetition of some preceding one, say x_j. Then $x_0, x_1, x_2 \cdots, x_{k-1}$ are distinct and $x_k \neq x_1, x_2, \cdots, x_{k-1}$, or else x_{k-1} is equal to one of the set $x_0, x_1, \cdots, x_{k-2}$. Hence $x_k = 0 = x_0$.

It is interesting to note that an M-order can be defined in such a way as to be analogous to the multiplicative order of an element in a field, just as the A-order already defined is analogous to the additive order in a field. We have merely to define $x^0 = 1$, $x^i = x^{i-1}x0$, for any $x \neq 0$ and every $x \neq 0$ will have an M-order.

THEOREM 4.2. *For an H-system any one of the following conditions is equivalent to each of the other two.*

$$1x_ix_j = x_{i+j}, \tag{1}$$

$$1(1x_ix_j)x_k = 1x_i(1x_jx_k), \tag{2}$$

$$1(1x_ix_j)x = 1x_i(1x_jx). \tag{3}$$

(Unless otherwise specified, operations among subscripts are modulo m, the A-order of the element.)

Proof. First it is shown that condition (1) implies condition (2). If $1x_ix_j = x_{i+j}$, then

$$1(1x_ix_j)x_k = 1(x_{i+j})x_k = x_{i+j+k} = 1x_ix_{j+k} = 1x_i(1x_jx_k).$$

It is obvious that condition (2) implies condition (3). For if k is set equal to 1 in condition (2), condition (3) is obtained.

Next it is proved that condition (3) implies condition (1). An induction on j is used. For $j=1$, $1x_ix_j = x_{i+j}$ by definition. If k is such that $1x_ix_k = x_{i+k}$, then

$$1x_ix_{k+1} = 1x_i(1x_kx) = 1(1x_ix_k)x = 1x_{i+k}x = x_{i+k+1},$$

and the induction is complete.

Since (1) implies (2), (2) implies (3), and (3) implies (1), the theorem is true.

THEOREM 4.3. *If one of the conditions in Theorem 4.2 is satisfied, then it follows that $(x_i)_j = x_{ij}$.*

Proof. An induction on j is used. Notice that $(x_i)_1 = x_i = x_{i(1)}$, and let k be such that $(x_i)_k = x_{ik}$. Then

$$(x_i)_{k+1} = 1(x_i)_kx_i = 1x_{ik}x_i = x_{ik+i} = x_{i(k+1)}.$$

THEOREM 4.4. *If either of the "distributive" laws*

$$\text{(a)} \quad ax_i0 = (ax0)_i$$

$$\text{(b)} \quad x_ia0 = (xa0)_i$$

holds, then all non-zero elements of H have the same A-order. (The hypothesis could be weakened by letting $x=1$ in (a) and (b)).

Proof. Suppose the contrary, that is, suppose there are elements x and y with A-orders j and k respectively, where $0<j<k$. Then if (a) holds, there exists an element $a\neq 0$ such that $ay0=x$ and

$$0 = x_j = (ay0)_j = ay_j0.$$

On the other hand, if (b) holds, there exists an element $a\neq 0$ such that $ya0=x$ and

$$0 = x_j = (ya0)_j = y_ja0.$$

In either case a postulate has been violated, for $ab0=0$ implies that either a or b must be 0.

The element x in H is said to generate the *m-subset* $X=\{x_1, x_2, \cdots\}$. The element y generates the m-subset Y, *etc.*

THEOREM 4.5. *Let H_1 be any H-system such that one of the conditions of Theorem* 4.2 *and one of the "distributive" laws of Theorem* 4.4 *are satisfied. Then*

(a) *Any two m-subsets of H_1 are either identical or without common elements other than* 0;

(b) *the number N of elements in H_1 is*

$$N = n(p - 1) + 1,$$

where n is the number of m-subsets and p is a prime equal to the A-order of each non-zero element.

Proof. Since $(x_i)_j=x_{ij}$, any non-zero element of X generates X, and any non-zero element of Y generates Y. For (by Theorem 4.4) all non-zero elements have equal A-orders. If $x_i=y_j\neq 0$, then as k varies $(x_i)_k=(y_j)_k$ generates X and also Y. Hence set X is the same as set Y. This completes the proof of (a).

Since $x_i\neq 0$ is a generator of the m-subset X and $(x_i)_j=x_{ij}$, no product ij (modulo m, the order of x) gives 0 unless either $i=0$ or $j=0$. Otherwise the A-order of x_i would not be the same as m, the A-order of x. Hence m is a prime p. It is seen that each of the n m-subsets contains the element 0 and $p-1$ other elements. Since 0 is the only element common to two distinct m-subsets, $N=n(p-1)+1$.

THEOREM 4.6. *Let an H-system be given such that $(x_i)_j=(x_j)_i$ and such that the equation $1_x=a$ has a solution x. Then if one of the "distributive" laws $ab_i0=(ab0)_i$, $a_jc0=(ac0)_j$, holds, the other holds also.*

Proof. Suppose $ab_i0=(ab0)_i$ for all a, b, i, and $1_i=b$. Then

$$a_jb0 = a_j1_i0 = (a_j10)_i = (a_j)_i = (a_i)_j = (a1_i0)_j = (ab0)_j.$$

Next, suppose $a_ib0=(ab0)_i$ for all a, b, i, and let $a=1_i$. Then

$$ab_j0 = 1_ib_j0 = (1b_j0)_i = (b_j)_i = (b_i)_j = (1_ib0)_j = (ab0)_j.$$

Next consider an H-system $H(A)$ in which the "associative" law

$$1(1xy)z = 1x(1yz) \tag{5}$$

holds for all elements x, y, z.

Define a new operation $+$ by

$$x + y = 1xy.$$

It is easily seen that $H(A)$ forms a group under $+$. Also the order of any element x under $+$ is the same as the A-order of x. The following theorems follow at once.

THEOREM 4.7. *N, the number of elements in $H(A)$ is a multiple of the order of each element.*

THEOREM 4.8. *In order that there exist a finite plane geometry with $r+1$ points on a line, where r is not a prime-power, it is necessary that the corresponding H-system be either non-"associative" (5) or non-"distributive."*

Let $H(A, D)$ be an H-system for which the associative law (5) and the distributive law (a) of Theorem 4.4 hold. From Theorems 4.2 and 4.3 we have for an $H(A, D)$ that $1x_ix_j=x_{i+j}=1x_jx_i$ and $(x_i)_j=x_{ij}$.

THEOREM 4.9. *For an $H(A, D)$ the law $1_i1_j0=1_j1_i0$ holds.*

Proof. $1_i1_j0=(1_i10)_j=(1_i)_j=1_{ij}=1_{ji}=1_j1_i0$.

THEOREM 4.10. *For an $H(A, D)$ the law $1_i(1_j1_k0)0=(1_i1_j0)1_k0$ holds.*

Proof. $1_i(1_j1_k0)0=1_i(1_{jk})0=1_{ijk}=1_{ij}1_k0=(1_i1_j0)1_k0$.

THEOREM 4.11. *Let an $H(A, D)$ be given such that for any element a the equation $1_x=a$ has a solution x. Then the system forms a field under the two binary operations defined by $a+b=1ab$, $ab=ab0$.*

Proof. That the system forms a group under addition has been proved. That this group is commutative follows since $11_i1_j=1_{i+j}=11_j1_i$ from Theorem 4.2. Multiplication ab is commutative from Theorem 4.9. That multiplication is associative follows from Theorem 4.10. Finally $a(b+c)=ab+ac$ since for $a=1_i$, $b=1_j$, $c=1_k$ we have

$$\begin{aligned} a(b + c) &= a(1bc)0 = 1_i(11_j1_k)0 = 1_i(1_{j+k})0 = 1_{i(j+k)} \\ &= 1_{ij+ik} = 1(1_{ij})(1_{ik}) = 1(1_i1_j0)(1_i1_k0) \\ &= 1(ab0)(ac0) = ab + ac. \end{aligned}$$

THEOREM 4.12. *In order that an H-system with a prime number p of elements define a field under the binary operations of Theorem* 4.11 *it is necessary and sufficient that the H-system be an $H(A, D)$.*

Proof. (Necessity). If the two operations define a field the associative law $1(1xy)z = 1x(1yz)$ holds since $(x+y)+z = x+(y+z)$. Since $x_i = 1x_{i-1}x = x_{i-1}+x$, we have

$$x_i = \overbrace{x + \cdots + x}^{i}$$

and $ax_i0 = ax_i = (ax)_i = (ax0)_i$ and the H-system is distributive.

(Sufficiency). If the H-system is an $H(A, D)$, then all elements including 1 have orders equal to p, and by Theorem 4.11 a field is obtained.

It is of interest to note that if an H-system with a prime number of elements defines the double binary system above such that all of the field postulates except the two commutative laws are satisfied then these two laws follow from the others.

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A SET OF TRIPLY PERSPECTIVE TRIANGLES ASSOCIATED WITH PROJECTIVE TRIADS

F. G. STOCKTON, Shell Development Company

1. Notation. The letters A, B, and C will refer to triangles, upper case being used for general reference or to direct attention to the points of a triangle, lower case when the lines are of primary interest. Subscripts refer to individual points or lines. Thus the points of triangle A are A_1, A_2, A_3, and the lines of a, the same triangle, are a_1, a_2, a_3; a_1 is the line not on A_1, *etc.* When homogenous coordinates are used the coordinates of A_2 will be (A_{21}, A_{22}, A_{23}), *etc.*

The letters l, m, n, with subscripts, will denote certain other lines needed in the development. The letters r, s, t, u, with subscripts, will represent numbers.

2. Labelling the configuration of Pappus. Take two distinct points A_1 and A_2 in the projective plane, and assume six distinct lines; l_1, m_1, n_1 on A_1 but not on A_2; l_2, m_2, n_2 on A_2 but not on A_1. Then the projectivity $l_1m_1n_1 \sim l_2m_2n_2$ determines a configuration of Pappus whose nine points and nine lines may be enumerated and labelled as follows (Fig. 1):

A_1 = given	l_1 = given
A_2 = given	l_2 = given
$B_1 = l_1n_2$	$l_3 = B_3C_3$
$B_2 = l_2m_1$	m_1 = given
$B_3 = m_2n_1$	m_2 = given
$C_1 = l_1m_2$	$m_3 = B_1C_2$
$C_2 = l_2n_1$	n_1 = given
$C_3 = m_1n_2$	n_2 = given
	$n_3 = B_2C_1$

$A_3 = l_3m_3n_3 =$ the Pappus Point of the projectivity.

If the projectivity is not a perspectivity, then A_1, A_2, and A_3 are not collinear, that is, they are the points of a non-singular triangle. In this case it may happen that B_1, B_2, B_3 are collinear, or that C_1, C_2, C_3 are collinear, but not both. It is assumed in much of the following work that A, B, and C are non-singular triangles.

It has been remarked,* and reference to the construction will demonstrate, that A is perspective with B from three centers, namely the three points of C. This is a circular relation; A and C are triply perspective from the three points of B, and B and C are triply perspective from the three points of A.

* O. Veblen and J. W. Young, Projective Geometry, Vol. 1, 1910, p. 100.

More explicitly, with the labelling chosen, any pair of the three triangles are cyclically centrally perspective in *reverse order*. That is:

$$A_1A_2A_3 \overset{C_2}{\barwedge} B_3B_2B_1$$

$$A_1A_2A_3 \overset{C_3}{\barwedge} B_2B_1B_3$$

$$A_1A_2A_3 \overset{C_1}{\barwedge} B_1B_3B_2$$

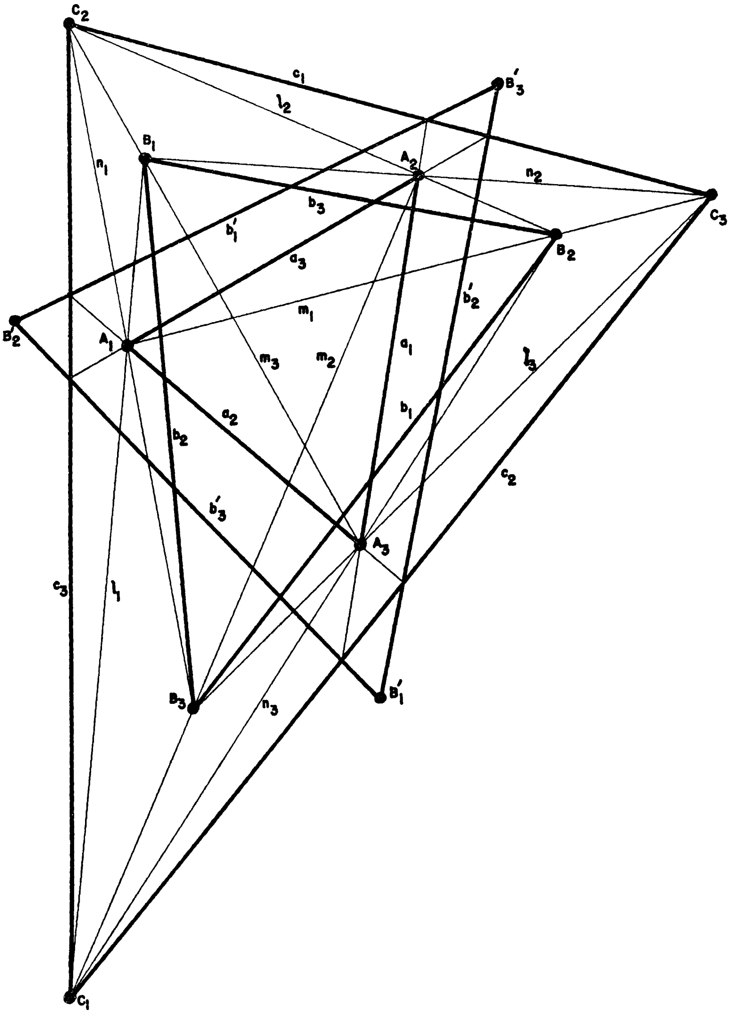

FIG. 1. Labelling system for the Central Cell $\begin{array}{ccc} & C & \\ B & & A \end{array}$ (A Configuration of Pappus) and for the Axial Cell $\begin{array}{ccc} C & & B' \\ & A & \end{array}$

and similarly for the other pairs. The center of perspective bears the subscript of the two points which correspond in the perspectivity and have the same subscript.

3. Definitions.

Cell. Three triangles which are, by pairs, triply perspective in reverse order from the points of the third triangle will be called a *central cell of triply perspective triangles*. The dual figure will be called an *axial cell.* A cell is *non-singular* if all three triangles of the cell are non-singular. If one of the triangles of a central cell is singular (three distinct but collinear points) the cell is a *singular central cell.* The dual figure is a *singular axial cell.*

Resultant. Any triangle of a central cell is the *central resultant* of the other two. Any triangle of an axial cell is the *axial resultant* of the other two.

Lattice. Consider a central cell of triply perspective triangles A, B, and C (Fig. 1). By Desargues' triangle theorem triangles a and c are triply axially perspective on three lines. The lines constitute a new§ triangle, b', and b', c, and a are the triangles of an axial cell. In the same way the axial resultant of b and a, and the axial resultant of b and c can be constructed. The central resultant of C and B' might be constructed next, and the process can be indefinitely continued.

The totality of triangles derivable from two triply perspective triangles by such repeated use of Desargues' triangle theorem and its dual will be called the *lattice of triangles.*

Cycle. The set of triangles derivable from two triply perspective triangles, A and B, by alternate formation of central and axial resultants, A being always paired with the last triangle formed, will be called a *cycle of triangles around* A. The cycle around A is a subset of the lattice. Each triangle in the cycle is triply perspective with A.

4. Relations among triangles of a cycle.

LEMMA 1. *The central resultant of two triangles of a non-singular axial cell is a non-singular triangle.*

Assume it is singular. Then the points of the paired triangles lie on a conic by the converse of Pascal's theorem; the lines of the axial resultant of the two triangles are the polars of the collinear points of the central resultant, hence concurrent, contrary to hypothesis.

LEMMA 2. *The homogeneous coordinates of the points of any triangle B, triply centrally perspective in reverse order with the reference triangle, and not singularly placed with respect to it, may be written in the form:*

$$\begin{aligned} B_1 &= (\ 1, & r, && s) \\ B_2 &= (\ t, & 1, && u) \\ B_3 &= (rt/s, & rt/u, && 1), \end{aligned}$$

§ The conditions allowing $b \equiv b'$ are examined in a later paragraph.

where r, s, t, and u are in general any four numbers, but subject to certain restrictions to insure non-singular placement.

The vanishing of any coordinate would indicate singular placement. Therefore the coordinates of B_1 and B_2 can be reduced to the form given, and B_{33} can be made unity. Solution of the reverse triple perspectivity condition:

$$B_{31}s = B_{32}u = rt$$

for B_{31} and B_{32} completes the proof.

THEOREM 1. *Let A and B be non-singular triangles, triply perspective in reverse order, and let C be their non-singular central resultant. It is assumed, in addition, that A and B are not singularly related, that is, no point of either is on a line of the other. Construct the cycle of triangles around A, beginning with B (B, C, B′, C′, B″, C″, B‴, C‴, ⋯). Then B‴≡B, and the cycle contains only six distinct triangles.*

Proof. Take A as the reference triangle. In view of Lemma 2 take:

$$\begin{aligned} B_1 &= (\ 1, \quad r_0, \quad s_0) \\ B_2 &= (\ t_0, \quad 1, \quad u_0) \\ B_3 &= (r_0t_0/s_0, \quad r_0t_0/u_0, \quad 1). \end{aligned}$$

Then

$$\begin{aligned} C_1 &= \begin{pmatrix} A_1B_1 \\ A_3B_2 \end{pmatrix} = (r_0t_0, \quad r_0, \quad s_0\ ) \\ C_2 &= \begin{pmatrix} A_2B_2 \\ A_3B_1 \end{pmatrix} = (\ t_0, \quad r_0t_0, \quad u_0\ ) \\ C_3 &= \begin{pmatrix} A_1B_2 \\ A_2B_1 \end{pmatrix} = (\ u_0, \quad s_0, \quad s_0u_0). \end{aligned}$$

The restrictions imposed on the numbers r_0, s_0, t_0, u_0, by the assumptions are now explicitly set forth.*

1. No point of B is on a line of A.
   Therefore $r_0 \neq 0$, $s_0 \neq 0$, $t_0 \neq 0$, $u_0 \neq 0$.
2. No point of A is on a line of B.
   Therefore $s_0 - r_0u_0 \neq 0$, $u_0 - s_0t_0 \neq 0$, $r_0t_0 - 1 \neq 0$.

* The single statement, "Neither the matrices of triangles B and C nor any of their minors of any order are singular," is equivalent to the restrictions given.

3. B is non-singular.
Therefore

$$1+\frac{r_0^2t_0u_0}{s_0}+\frac{r_0s_0t_0^2}{u_0}-3r_0t_0\neq 0,$$

and in view of restriction 1, this may be reduced to:

$$\frac{1}{r_0t_0}+\frac{r_0u_0}{s_0}+\frac{s_0t_0}{u_0}\neq 3.$$

4. C is non-singular.
Therefore $r_0^2s_0t_0^2u_0+r_0u_0^2+s_0^2t_0-3r_0s_0t_0u_0\neq 0$, and in view of restriction 1, this may be reduced to:

$$\frac{r_0t_0}{1}+\frac{s_0}{r_0u_0}+\frac{u_0}{s_0t_0}\neq 3.$$

The lines of triangle c can be expressed in the canonical form of the dual of Lemma 2 in terms of four numbers r_1, s_1, t_1, u_1. By expressing them also in terms of r_0, s_0, t_0, u_0, it can be shown that

$$r_1=\frac{u_0-s_0t_0}{s_0(r_0t_0-1)}$$

$$s_1=\frac{t_0(s_0-r_0u_0)}{s_0u_0(r_0t_0-1)}$$

$$t_1=\frac{s_0-r_0u_0}{u_0(r_0t_0-1)}$$

$$u_1=\frac{r_0(u_0-s_0t_0)}{s_0u_0(r_0t_0-1)}.$$

The new numbers r_1, s_1, t_1, u_1 satisfy restriction 1 since the first set satisfies restrictions 1 and 2.

They satisfy restriction 2 since the first set satisfies restriction 4.

They satisfy restriction 3 since triangle c is non-singular.

They satisfy restriction 4 since b' is non-singular, which is true by the dual of Lemma 1, A, C, and B being non-singular.

Since B' is triply perspective with A in reverse order, the coordinates of its points can be expressed in the canonical form of Lemma 2 in terms of four numbers r_2, s_2, t_2, u_2. It can then be shown that

$$r_2 = \frac{u_1 - s_1 t_1}{s_1(r_1 t_1 - 1)}$$

$$s_2 = \frac{t_1(s_1 - r_1 u_1)}{s_1 u_1(r_1 t_1 - 1)}$$

$$t_2 = \frac{s_1 - r_1 u_1}{u_1(r_1 t_1 - 1)}$$

$$u_2 = \frac{r_1(u_1 - s_1 t_1)}{s_1 u_1(r_1 t_1 - 1)}$$

and it can be demonstrated as before that the new number set satisfies the four restrictions.

Since B'' is related to B', and B''' to B'', in precisely the same way that B' is related to B:

$$\begin{aligned} B_1''' &= (\ 1, & r_6, && s_6) \\ B_2''' &= (\ t_6, & 1, && u_6) \\ B_3''' &= (r_6 t_6/s_6, & r_6 t_6/u_6, && 1\ ), \end{aligned}$$

where r_6, s_6, t_6, u_6 are derived from r_0, s_0, t_0, u_0 by the recursion formulas:

$$r_{i+1} = \frac{u_i - s_i t_i}{s_i(r_i t_i - 1)}$$

$$s_{i+1} = \frac{t_i(s_i - r_i u_i)}{s_i u_i(r_i t_i - 1)}$$

$$t_{i+1} = \frac{s_i - r_i u_i}{u_i(r_i t_i - 1)}$$

$$u_{i+1} = \frac{r_i(u_i - s_i t_i)}{s_i u_i(r_i t_i - 1)}\ .$$

Expansion of the recursion formulas for $i+2$ gives:

$$r_{i+2} = 1/s_{i+1}u_i$$

$$s_{i+2} = 1/u_{i+1}t_i$$

$$t_{i+2} = 1/u_{i+1}s_i$$

$$u_{i+2} = 1/s_{i+1}r_i.$$

Armed with these last formulas, the following table can be constructed from left to right:

i	0	1	2	3	4	5	6
r	r_0	r_1	$1/s_1u_0$	t_0	t_1	$1/u_1s_0$	r_0
s	s_0	s_1	$1/u_1t_0$	s_1r_0/t_1	r_1s_0/t_0	$1/t_1u_0$	s_0
t	t_0	t_1	$1/u_1s_0$	r_0	r_1	$1/s_1u_0$	t_0
u	u_0	u_1	$1/s_1r_0$	u_1t_0/r_1	t_1u_0/r_0	$1/r_1s_0$	u_0

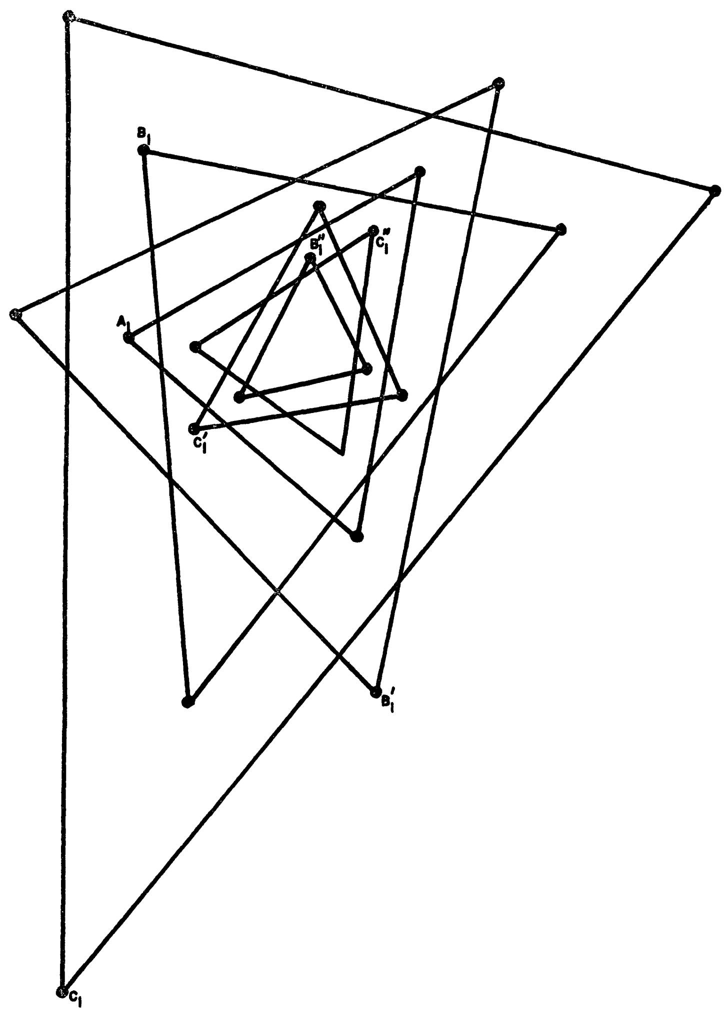

FIG. 2. The Cycle Around A.

Therefore $B''' \equiv B$, since the points of each have the same coordinates.

Discussion. Theorem 1 is a theorem of closure, establishing the cycle of triangles around A as a closed set of six triangles. Figure 2 shows the cycle around A derived from triangles A and B of Fig. 1. To simplify the figure only the first point of each triangle is labelled. The other points are in clockwise order, and the lines are to be labelled as described under "Notation" above.

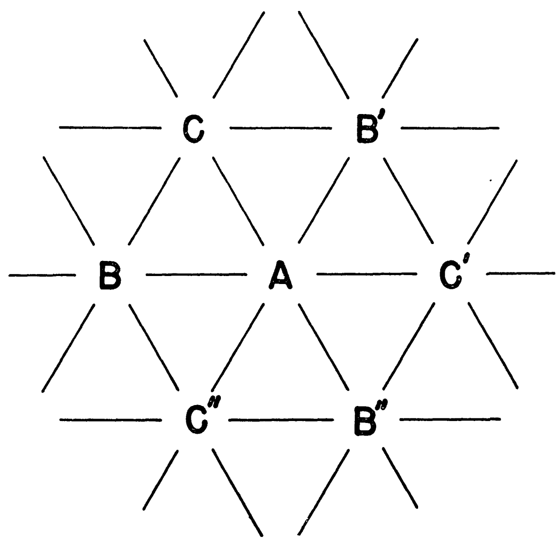

FIG. 3. Isomorphism between Triangles of a Cycle and Vertices of a Triangular Tesselation.

Theorem 1 implies a striking and useful isomorphism in which each triangle of the cycle around A corresponds to one of the six points adjacent to a given point in a plane triangular tesselation (Figure 3). It will be noted that paired triangles are adjacent in the diagram, that a triangular unit of the diagram corresponds to a cell of triangles, that central cells point upwards and axial cells point downwards. Repeated recourse to Theorem 1 during the construction of a lattice of triangles yields the

COROLLARY. *The triangles of the lattice derivable from two triply perspective triangles are isomorphic with the vertices of an unbounded plane triangular tesselation.*

Singular cycles. It has not been shown that the six triangles of the cycle around A are all distinct. The question can be readily examined using point coordinates obtained from the table of Theorem 1, and it develops that, under restrictions 1 and 2, none of the triangles can be identical with any of the six permuted forms of another unless:

$$\frac{r_0t_0}{1} = \frac{s_0}{r_0u_0} = \frac{u_0}{s_0t_0} = \sqrt[3]{1} = \omega \text{ or } \omega^2 \neq 1.$$

Under this constraint $B \equiv B' \equiv B''$ and $C \equiv C' \equiv C''$ so that the cycle around A contains only triangles B and C, and the lattice generated from A and B contains only A, B, and C. A, B, and C are *all three* triply perspective from the points of a fourth non-singular triangle D. The twelve points of the four triangles form a configuration $\begin{matrix}12 & 3\\ 4 & 9\end{matrix}$. Any two of the four triangles are in sextuple perspective from the six points of the other two.*

5. Pairing unpaired triangles.

THEOREM 2. *Each two triangles in a cycle are triply perspective in reverse order.*

It is sufficient to prove the theorem for the pairs B, B' and b, c'. From the table of Theorem 1:

$$B' = \begin{Bmatrix} 1 & \dfrac{1}{s_1u_0} & \dfrac{1}{u_1t_0} \\ \dfrac{1}{u_1s_0} & 1 & \dfrac{1}{s_1r_0} \\ \dfrac{t_0}{s_0s_1u_0} & \dfrac{r_0}{u_1s_0u_0} & 1 \end{Bmatrix} = \begin{Bmatrix} s_1t_0u_0u_1 & t_0u_1 & s_1u_0 \\ r_0s_1 & r_0s_0s_1u_1 & s_0u_1 \\ t_0u_1 & r_0s_1 & s_0s_1u_0u_1 \end{Bmatrix}.$$

Since

$$B = \begin{Bmatrix} 1 & r_0 & s_0 \\ t_0 & 1 & u_0 \\ \dfrac{r_0t_0}{s_0} & \dfrac{r_0t_0}{u_0} & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & r_0 & s_0 \\ t_0 & 1 & u_0 \\ r_0 & \dfrac{r_0s_0}{u_0} & \dfrac{s_0}{t_0} \end{Bmatrix},$$

$$\begin{vmatrix} [B_1B_3'] \\ [B_2B_2'] \\ [B_3B_1'] \end{vmatrix} = \begin{vmatrix} r_0s_0s_1u_0u_1 - r_0s_0s_1 & s_0t_0u_1 - s_0s_1u_0u_1 & r_0s_1 - r_0t_0u_1 \\ s_0u_1 - r_0s_0s_1u_0u_1 & r_0s_1u_0 - s_0t_0u_1 & r_0s_0s_1t_0u_1 - r_0s_1 \\ r_0s_0s_1 - s_0u_1 & s_0s_1u_0u_1 - r_0s_1u_0 & r_0t_0u_1 - r_0s_0s_1t_0u_1 \end{vmatrix}$$

is singular since the sum of the rows is zero. The determinant of the three lines

$$\begin{vmatrix} [B_1B_2'] \\ [B_2B_1'] \\ [B_3B_3'] \end{vmatrix}$$

* W. R. Andress and W. Saddler, Perspective triads. With a note by W. W. Sawyer. Math. Gaz. vol. 37, 1953, pp. 247–255.

is also singular. Therefore

$$B_1B_2B_3 \;\bar{\wedge}\; B_3'B_2'B_1'$$

and

$$B_1B_2B_3 \;\bar{\wedge}\; B_2'B_1'B_3',$$

so that triangles B and B' are doubly (and hence triply)* perspective in reverse order.

The reverse triple perspectivity of b with c' is demonstrable in the same way.

6. Recursive identities. Alternative proofs of Theorem 1 exist, each depending on a permutation of the coordinates of B_1 and B_2 in Lemma 2. Each such proof requires the development of a recursive identity in four numbers. In this way six distinct identities are encountered; they are recorded in Table 1 below, together with their inverses. The identities may be interpreted as transformations of period six in a four dimensional vector space.

Any of the identities may be written:

$$V_6 \equiv V_0$$

where V_i is the vector (r_i, s_i, t_i, u_i),

$$r_{i+1} = f_r(r_i, s_i, t_i, u_i),$$

$$s_{i+1} = f_s(r_i, s_i, t_i, u_i),$$

$$t_{i+1} = f_t(r_i, s_i, t_i, u_i),$$

$$u_{i+1} = f_u(r_i, s_i, t_i, u_i),$$

and f_r, f_s, f_t, f_u are defined for each identity as tabulated below (Table 1).

The identities hold for any initial vector V_0 satisfying four mild restrictions. There are two such sets of restrictions, I and II. Restrictions I are those given in Theorem 1; restrictions II are listed below. Each identity is governed by one set or the other, as noted in Table 1.

Restrictions II

1. $r_0 \neq 0,\ s_0 \neq 0,\ t_0 \neq 0,\ u_0 \neq 0$

2. $r_0u_0 - s_0t_0 \neq 0,\ t_0 - r_0 \neq 0,\ s_0 - u_0 \neq 0$

3. $\dfrac{s_0t_0}{r_0u_0} + \dfrac{r_0}{t_0} + \dfrac{u_0}{s_0} \neq 3$

4. $\dfrac{r_0u_0}{s_0t_0} + \dfrac{t_0}{r_0} + \dfrac{s_0}{u_0} \neq 3.$

* O. Veblen and J. W. Young, *op. cit.*, p. 247.

Illustration. The following small table gives the successive values of r_i, s_i, t_i, u_i for the identity J_1 and the initial vector $V_0=(1, 3, 2, 9)$. The J_1' identity gives the same intermediates in the reverse order.

i	0	1	2	3	4	5	6
r	1	1	$-1/4$	2	$-2/3$	3	1
s	3	$-4/9$	$9/2$	$2/3$	$3/2$	$-1/6$	3
t	2	$-2/3$	3	1	1	$-1/4$	2
u	9	$1/9$	$-9/4$	$2/9$	-6	$1/3$	9

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TABLE 1. $f_r\, f_s\, f_t$, AND f_u FOR THE SIX IDENTITIES AND THEIR INVERSES

Identity	f_r	f_s	f_t	f_u	Restrictions
J_1	$\frac{u-st}{s(rt-1)}$	$\frac{t(s-ru)}{su(rt-1)}$	$\frac{s-ru}{u(rt-1)}$	$\frac{r(u-st)}{su(rt-1)}$	I
J_1'	$\frac{u(rt-1)}{r(u-st)}$	$\frac{rt-1}{s-ru}$	$\frac{s(rt-1)}{t(s-ru)}$	$\frac{rt-1}{u-st}$	I
J_2	$\frac{t(s-ru)}{r(u-st)}$	$\frac{rt-1}{r(u-st)}$	$\frac{r(s-ru)}{s(rt-1)}$	$\frac{r(u-st)}{su(rt-1)}$	I
J_2'	$\frac{u-st}{s-ru}$	$\frac{u-st}{su(rt-1)}$	$\frac{ru(rt-1)}{t(s-ru)}$	$\frac{rt-1}{u-st}$	I
J_3	$\frac{u(rt-1)}{s-ru}$	$\frac{r(u-st)}{s(s-ru)}$	$\frac{r(u-st)}{t(s-ru)}$	$\frac{r(rt-1)}{s-ru}$	I
J_3'	$\frac{t(s-ru)}{s(rt-1)}$	$\frac{t(s-ru)}{s(u-st)}$	$\frac{s-ru}{u-st}$	$\frac{rt(s-ru)}{su(rt-1)}$	I
J_4	$\frac{s(rt-1)}{r(s-ru)}$	$\frac{u-st}{u(s-ru)}$	$\frac{u(rt-1)}{t(u-st)}$	$\frac{s-ru}{s(u-st)}$	I
J_4'	$\frac{t(s-ru)}{ru(rt-1)}$	$\frac{t(s-ru)}{ru(u-st)}$	$\frac{r(u-st)}{st(rt-1)}$	$\frac{r(u-st)}{st(s-ru)}$	I
J_5	$\frac{s-u}{s(t-r)}$	$\frac{ru-st}{su(t-r)}$	$\frac{u(t-r)}{t(ru-st)}$	$\frac{r(s-u)}{s(ru-st)}$	II
J_5'	$\frac{u(t-r)}{rt(s-u)}$	$\frac{t-r}{ru-st}$	$\frac{ru-st}{st(t-r)}$	$\frac{ru-st}{st(s-u)}$	II
J_6	$\frac{ru-st}{rt(s-u)}$	$\frac{t-r}{t(s-u)}$	$\frac{s-u}{s(t-r)}$	$\frac{ru-st}{su(t-r)}$	II
J_6'	$\frac{s-u}{ru-st}$	$\frac{r(s-u)}{su(t-r)}$	$\frac{u(t-r)}{rt(s-u)}$	$\frac{t-r}{ru-st}$	II

AN AXIOMATIC TRIANGULAR GEOMETRY

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1. Introduction. The analytic trilinear geometry of the plane, in which the (three) coordinates of a point are defined as proportional to its distances from the sides of a fixed triangle, or to the areas it determines with these sides, has accumulated quite a literature, especially in the decades following 1860 [1]. These geometries, however, do not seem to have received any systematic axiomatic treatment. It is the purpose of this paper to present a set of axioms for such a geometry in the plane (readily generalizable to n dimensions), indicating in the process some interesting definitions and concepts and illustrating the power of the axioms by developing some specific branches of the theory.

2. The axioms. Suppose given a set of (undefined) elements, which we shall call "points," and denote by capital letters. Let there further be given a function mapping the set of ordered triples of points into the reals, denoted by $\Delta(A, B, C)$, read "area (of the triangle determined by) A, B, C."

We now assume that Δ satisfies the following axioms:

(S) Symmetry axiom:

$$\Delta(\pi(A, B, C)) = (\mathrm{sg}(\pi))\cdot\Delta(A, B, C)$$

(where π is a permutation).

(I) Identity axiom: *If*

$$\sum_{i=1}^{n} \alpha_i\cdot\Delta(X, A_i, B_i) = \alpha_0 \qquad (\alpha_i \textit{ reals})$$

is satisfied by X_1, X_2, X_3 with $\Delta(X_1, X_2, X_3)\neq 0$, then it is an identity in X.

(E) Existence axiom: *If A, B are distinct points, there exist points C, D such that*

$$\Delta(A, B, C) \neq \Delta(A, B, D).$$

Indeed, given any such A, B, C, D, and any pair a, b of reals, there exists a point X such that

$$\Delta(X, A, B) = a; \qquad \Delta(X, C, D) = b.$$

3. Collinearity. We proceed to investigate the structure given to our set of points by these axioms, beginning by defining the line.

DEFINITION: *"A, B, C are collinear" means $\Delta(A, B, C)=0$.*

DEFINITION: *The line determined by A, B is the set of X such that $\Delta(A, B, X)=0$.*

We may now show collinearity to have the expected properties:

(C1) Reflexivity: *By* (S), *$\Delta(A, B, C)=0$ if not all of A, B, C are distinct.*

(C2) Symmetry: *By* (S), $\Delta(A, B, C)=0$ *implies* $\Delta(\pi(A, B, C))=0$.

(C3) Transitivity: *If* A, B, C *are distinct, and* $\Delta(A, B, C)=\Delta(B, C, D)=0$, *then* (*e.g.*) $\Delta(A, C, D)=0$; for, if not, since A, C, D by hypothesis all satisfy $\Delta(X, B, C)=0$, by (I) so would any point X, contrary to (E).

(We remark in passing that the properties (C1–C3) of the ternary relation "collinearity" are quite analogous to those defining the familiar (binary) *equivalence relation.* The generalization to n-ary relations is quite straightforward.)

By means of suitable definitions it is now not difficult to establish most of the *Euclidean* axioms on united position of points and lines; the order of points on a line, *etc.*, whence to define *ray, line segment, etc.* However, we shall not have occasion to make much use of such results in the sequel.

4. Metric properties of Δ. It may further be observed that our Δ-function (in common with its natural realization in the Euclidean plane—the area of a triangle) has properties somewhat analogous to those which define a *distance*-function or *metric.* In particular, we have the

(T) Triangle equality:

$$\Delta(A, B, C) = \Delta(X, B, C) + \Delta(A, X, C) + \Delta(A, B, X).$$

Proof: If $\Delta(A, B, C)\neq 0$, then since $X=A, B, C$ satisfy (T), so does any X, by (I). If $\Delta(A, B, C)=0$, either all of $\Delta(X, B, C)$, $\Delta(A, X, C)$, $\Delta(A, B, X)$ are zero (Q.E.D.), or (by C3) none are zero. (The case where A, B, C are not all distinct may be handled by using (C1) and (S).) But if (*e.g.*) $\Delta(A, B, X)\neq 0$, our first argument proves

$$\Delta(A, B, X) = \Delta(C, B, X) + \Delta(A, C, X) + \Delta(A, B, C)$$

which by (S) is readily equivalent to (T).

(The similarity between (C1, S, T) and the defining properties of a *metric* is strengthened by the following observations: In our *Euclidean* model, if we pass to 3-space, (T) does indeed take on the more familiar form of an *inequality*, expressible in words as: *The area of any face of a tetrahedron does not exceed the sum of the areas of the other three faces.* The plane case for (signed) "area" is thus the analogue of the *line* case for (signed) "distance," as may be immediately seen.)

5. Coordinates. Axiom (I), together with the results already established, enables us to set up formal *triangular coordinates* on our set of points. Indeed, we could actually define general *two-segment* coordinates (a system in which the coordinates of X with respect to the ordered point-pairs A, B; C, D are defined as the reals of axiom (E)); but we choose for simplicity to let the two segments have an end-point in common, making them (A, B) and (A, C), say. For this case it is readily seen that the nondegeneracy hypothesis of (E) reduces to the requirement that A, B, C be noncollinear. For purposes of homogeneity we further choose to define as our coordinates the *ratios*

$$X_B \equiv \frac{\Delta(A, X, C)}{\Delta(A, B, C)}, \qquad X_C \equiv \frac{\Delta(A, B, X)}{\Delta(A, B, C)}$$

(permissible since we are assuming $\Delta(A, B, C) \neq 0$). In the Euclidean realization of our system these may now easily be recognized as two of the familiar areal triangular coordinates of X. For symmetry we may, finally, think of our coordinates as related to the "basic *triple*" of points A, B, C, introducing the third coordinate

$$X_A = \frac{\Delta(X, B, C)}{\Delta(A, B, C)}.$$

By (T) we have evidently

$$X_A + X_B + X_C = 1.$$

It remains only to complete the one-to-one correspondence between points and pairs of coordinates by proving the following

THEOREM. (U) Uniqueness: *With respect to a given basic triple A, B, C, a point is determined uniquely by two of its coordinates, i.e., $X_B = Y_B$ and $X_C = Y_C$ imply $X = Y$.*

Proof: If $X \neq Y$, then since $\Delta(A, B, C) \neq 0$, by (C3) at least one of $\Delta(A, X, Y)$' $\Delta(C, X, Y)$ is $\neq 0$; say $\Delta(A, X, Y)$. Similarly not both $\Delta(X, A, B)$, $\Delta(X, A, C)$ are zero; say $\Delta(X, A, B) \neq 0$. Suppose now that $\Delta(X, A, B) = \Delta(Y, A, B)$ and $\Delta(X, A, C) = \Delta(Y, A, C)$ (obviously equivalent to the hypothesis). Let $\Delta(X, A, C) = \lambda \cdot \Delta(X, A, B)$; then $\Delta(Y, A, C) = \lambda \cdot \Delta(Y, A, B)$. But also $0 = \lambda \cdot \Delta(A, A, B) = \Delta(A, A, C)$; and these three roots X, Y, A are not collinear. Hence by (I), $\Delta(Z, A, C) = \lambda \cdot \Delta(Z, A, B)$ identically in Z, and must be satisfied *e.g.*, by B, so that $\Delta(B, A, C) = \lambda \cdot \Delta(B, A, B) = 0$, contradicting $\Delta(A, B, C) \neq 0$. Thus $X \neq Y$ and the hypotheses are incompatible.

6. Transformations. If we now change over from the basic triple A, B, C to another, say P, R, S (with $\Delta(P, R, S) \neq 0$), it is natural to inquire whether our system as thus far developed enables us to write X_P, X_R, X_S in terms of X_A, X_B, X_C. We shall show that this is indeed the case, first proving:

(H) Homogeneity: *If* $\Delta(A, O, X) = \lambda \cdot \Delta(A, O, Y)$, *and* $\Delta(B, O, X) = \lambda \cdot \Delta(B, O, Y)$, *where* $\Delta(A, B, O) \neq 0$, *then for all* P, $\Delta(P, O, X) = \lambda \cdot \Delta(P, O, Y)$.

(A) Additivity: *If* $\Delta(A, O, X) = \Delta(A, O, Y) + \Delta(A, O, Z)$ *and* $\Delta(B, O, X) = \Delta(B, O, Y) + \Delta(B, O, Z)$, *where* $\Delta(A, B, O) \neq 0$, *then for all* P, $\Delta(P, O, X) = \Delta(P, O, Y) + \Delta(P, O, Z)$.

(These are immediate specializations of (I).)

(D) Determinant Theorem:

$$\frac{\Delta(X,Y,Z)}{\Delta(A,B,C)} = \begin{vmatrix} X_A & X_B & X_C \\ Y_A & Y_B & Y_C \\ Z_A & Z_B & Z_C \end{vmatrix}.$$

Proof: By (T), this determinant is equal to (*e.g.*)

$$\begin{vmatrix} 1 & X_B & X_C \\ 1 & Y_B & Y_C \\ 1 & Z_B & Z_C \end{vmatrix}.$$

Consider its expansion by minors of its first column. If we can show that

$$\begin{vmatrix} Y_B & Y_C \\ Z_B & Z_C \end{vmatrix} = \frac{\Delta(A,Y,Z)}{\Delta(A,B,C)},$$

and similarly that

$$\begin{vmatrix} Z_B & Z_C \\ X_B & X_C \end{vmatrix} = \frac{\Delta(A,Z,X)}{\Delta(A,B,C)}; \qquad \begin{vmatrix} X_B & X_C \\ Y_B & Y_C \end{vmatrix} = \frac{\Delta(A,X,Y)}{\Delta(A,B,C)},$$

our theorem will follow by (S) and (T). We proceed to do so.

Since by (U) the points Y, Z are uniquely determined by any two of the coordinates of each, evidently

$$f(Y,Z) \equiv \frac{\Delta(A,Y,Z)}{\Delta(A,B,C)}$$

is a function of Y_B, Y_C; Z_B, Z_C. This function has the following properties:

1. If $Y_B = Z_B$, $Y_C = Z_C$, then by (U), $Y = Z$, whence (by (C1)) $f(Y, Z) = 0$.
2. If $(Y_B, Y_C) = (1, 0)$; $(Z_B, Z_C) = (0, 1)$, by (U) evidently $Y = B$, $Z = C$, so that
$$f(Y,Z) = f(B,C) \equiv \frac{\Delta(A,B,C)}{\Delta(A,B,C)} = 1.$$
3. If $Y_B = \lambda \cdot Y_B'$, $Y_C = \lambda \cdot Y_C'$, by (H) we have $f(Y, Z) = \lambda \cdot f(Y', Z)$.
4. If $Y_B = Y_B' + Y_B''$, $Y_C = Y_C' + Y_C''$, by (A) we have $f(Y, Z) = f(Y', Z) + f(Y'', Z)$.

But (see, *e.g.*, [2]) a function of the two pairs (Y_B, Y_C); (Z_B, Z_C) which satisfies (1–4) must be their determinant.

(*N.B.*: The technique used in this proof is of interest in its own right, but its inherent limitations are illustrated by the remark that it could not have been used to prove the theorem directly in the 3-by-3 case, since the linearity requirements (3, 4) are meaningless when the sum of the entries in each row is necessarily 1.)

We may now apply (D) to answer the question posed at the beginning of this section, for we have

$$X_P \equiv \frac{\Delta(X, R, S)}{\Delta(P, R, S)} = \begin{vmatrix} X_A & X_B & X_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix} \Bigg/ \begin{vmatrix} P_A & P_B & P_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix},$$

and similarly for X_R and X_S. In other words, X_P, X_R, X_S are precisely the solutions of the system of linear equations

$$\begin{aligned} P_A X_P + R_A X_R + S_A X_S &= X_A \\ P_B X_P + R_B X_R + S_B X_S &= X_B \\ P_C X_P + R_C X_R + S_C X_S &= X_C, \end{aligned}$$

which solutions do indeed exist uniquely by virtue of

$$\Delta(P, R, S) = \Delta(A, B, C) \begin{vmatrix} P_A & P_B & P_C \\ R_A & R_B & R_C \\ S_A & S_B & S_C \end{vmatrix} \neq 0.$$

This discussion gives us an analytic picture of our geometry. We may, indeed, now view P, R, S, not as a new basic triple, but as the image of A, B, C under the coordinate transformation $(X_A, X_B, X_C) \rightarrow (X_P, X_R, X_S)$, which we have just seen to be linear. Indeed, since the property that the sum of the entries in each row is 1 (*barycentricity*) evidently passes to the matrix product and inverse, our geometry is actually isomorphic to the geometry defined by the group of (three-dimensional) linear transformations with barycentric matrices. And (D) incidentally tells us that such a transformation is "area"-preserving if and only if its matrix has determinant $+1$. Our system now has a well-defined place in the set of (numerical) "projective geometries."

7. Further results. A comparison of the "Euclidean" realization of our system with ordinary Euclidean geometry enables us to determine those concepts of the latter which may be generalized to the former. For example, since the medians of a triangle bisect its area, our coordinate system enables us to define "median" (for the basic triangle), whence (guided by the theorem on the line joining the midpoints of two sides of a triangle) the concept of congruence (of line segments) by collinear or parallel translation (see definition of "parallel" just below). However, "congruence" cannot be defined in this system for nonparallel segments. Indeed, if it could, we would then have in our model a general definition of "length" in terms of "area," so that any (*e.g.*, linear) transformation preserving area would also preserve length—which is of course not the case. In consequence of this, concepts like "circle," "isosceles triangle," *etc.*, whence "perpendicular," *etc.*, are undefinable within our system.

As an illustration of what *can* be done within our system as so far developed, we give the theory of parallels.

DEFINITION: *The point-pair A, B "is parallel to" the point-pair C, D (where $A \neq B$, $C \neq D$) if and only if $\Delta(A, C, D) = \Delta(B, C, D)$.*

We proceed to establish the following properties of "parallel":

(P1) *"Parallel" is well-defined for lines, indeed we have also $\Delta(X, C, D) = \Delta(A, C, D)$ if and only if $\Delta(X, A, B) = 0$.*

(N.B.: The "only if" part of this assertion is Playfair's Axiom.)

Proof: Take C, D as two of the points of a basic triple (P, C, D). By (D), we have

$$\frac{\Delta(X,A,B)}{\Delta(P,C,D)} = (e.g.) \begin{vmatrix} X_P & X_C & 1 \\ A_P & A_C & 1 \\ B_P & B_C & 1 \end{vmatrix} = \begin{vmatrix} X_P & 1 & X_D \\ A_P & 1 & A_D \\ B_P & 1 & B_D \end{vmatrix}.$$

But if $X_P = A_P = B_P$, *i.e.*,

$$\frac{\Delta(X,C,D)}{\Delta(P,C,D)} = \frac{\Delta(A,C,D)}{\Delta(P,C,D)} = \frac{\Delta(B,C,D)}{\Delta(P,C,D)},$$

this determinant is evidently zero. Conversely, if the determinant is zero, multiplying the last column by $A_P(=B_P)$, subtracting it from the first, and expanding by minors of the first column, gives

$$0 = \begin{vmatrix} X_P - A_P & X_C & 1 \\ 0 & A_C & 1 \\ 0 & B_C & 1 \end{vmatrix} = (X_P - A_P)(A_C - B_C).$$

Similar treatment of

$$\begin{vmatrix} X_P & 1 & X_D \\ A_P & 1 & A_D \\ B_P & 1 & B_D \end{vmatrix}$$

gives $0 = (X_P - A_P)(A_D - B_D)$. Hence if $X_P \neq A_P$, we would have $A_C = B_C$, $A_D = B_D$, or (by (U)) $A = B$, a contradiction. Thus $X_P = A_P$, which is obviously equivalent to our assertion.

(P2) *"Parallel" is equivalent to "identical or nonintersecting," i.e. if C, D do not lie on the line determined by A, B, then this line is parallel to the line of (C, D) if and only if it has no point in common with it.*

Proof: If these lines had the point X in common, we would have (by (P1)) $0 = \Delta(X, C, D) = \Delta(A, C, D) = \Delta(B, C, D)$, making the lines identical by (C3), a contradiction. Conversely, if the lines were not parallel, by (E) there would

exist a point X with $\Delta(X, A, B)=0$, $\Delta(X, C, D)=0$, *i.e.*, a point common to them, contrary to hypothesis.

(P3) *"Parallel" is reflexive*—by (C1).

(P4) *"Parallel" is symmetric*—by (T).

(P5) *"Parallel" is transitive, i.e. two lines parallel to a third line are parallel to each other*; for, if not, by (P2) they intersect (the case where they're identical is of course trivial), and an application of (P1) to their intersection-point shows that they must be identical.

We may proceed without much difficulty from this point to develop the theory of parallelograms, a theory which incidentally might also serve us as an alternative guide to the definition of congruence of parallel segments. The details are left to the interested reader.

References

1. Coolidge, A History of Geometrical Methods, Oxford, 1940, Book II, Chapter II, Section 2.
2. Artin, Galois Theory, Notre Dame, 1948, Chapter I, Section F.

THREE MUTUALLY ORTHOGONAL REAL CIRCLES

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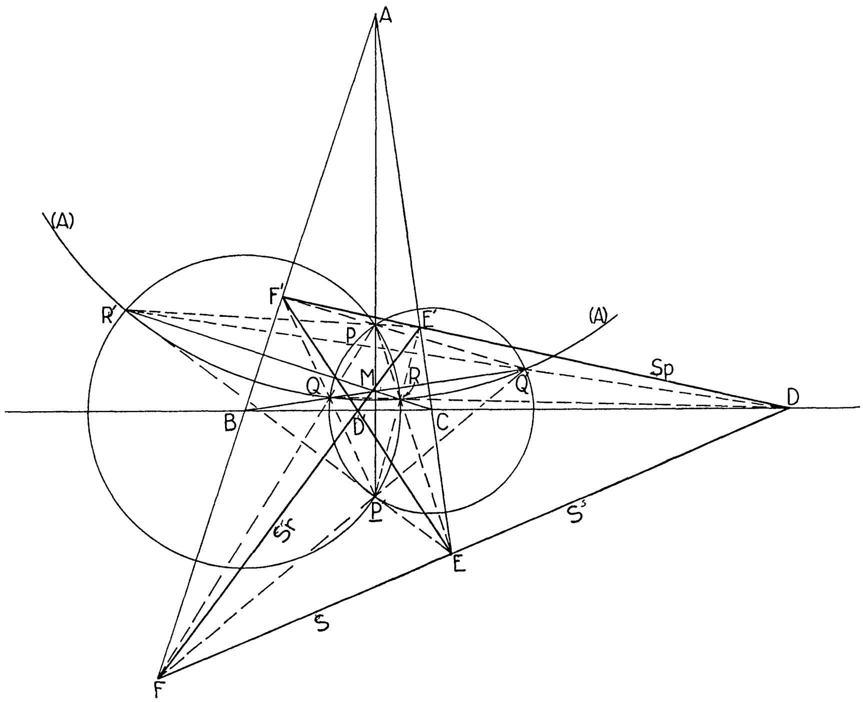
1. Preliminaries. The object of this paper is to consider the six points of intersection of three mutually orthogonal real circles (A), (B), (C), chiefly in relation to the centers and axes of similitude, and to the circles of antisimilitude of those circles.

The triangle ABC formed by the centers A, B, C of the given circles will be said to be their central triangle.

The orthogonal center M of the three circles considered necessarily lies inside those circles, hence their orthogonal circle (M) is imaginary. We consider no points on the circumference of (M), but we make some use of the polarity with respect to that circle [3].



2. Triangles of intersection. Let

$$P,\ P';\ Q,\ Q';\ R,\ R' \tag{a}$$

be the three pairs of points of intersection of the three pairs of circles (B), (C); (C), (A); (A), (B), respectively. The two points of the same pair may be said to be complementary points of intersection.

The points located on the same circle are given by the following:

TABLE A

(A)		Q, Q';	R, R'
(B)	P, P';		R, R'
(C)	P, P';	Q, Q';	

Taking one, and only one, point from each of the three pairs of points (a) we form eight triangles which may be referred to as the triangles of intersection of the three given circles. They are listed in the following:

TABLE B

	I	II	III	IV
Triangles of intersection	$PQR, P'Q'R'$	$PQ'R', P'QR$	$P'QR', PQ'R$	$P'Q'R, PQR'$
Axes of similitude	$s=DEF$	$s_p=DE'F'$	$s_q=EF'D'$	$s_r=FD'E'$
Circles of intersection	$(I), (I')$	$(I_p), (I_p')$	$(I_q), (I_q')$	$(I_r), (I_r')$
Circumcenters	I, I'	I_p, I_p'	I_q, I_q'	I_r, I_r'

The eight triangles of intersection are grouped into four pairs I, II, III, IV of complementary triangles of intersection. Two such triangles have no vertex in common, and thus involve all six points of intersection.

3. Axes of perspectivity. a. Any two complementary triangles of intersection, say, PQR, $P'Q'R'$ are obviously perspective from the orthogonal center M of the given circles, hence their three pairs of corresponding sides intersect in three points $D=(QR, Q'R')$, $E=(RP, R'P')$, $F=(PQ, P'Q')$ which lie on their axis of perspectivity s.

On the other hand, the two points Q, R of the two circles (C), (B) also lie on the circle (A) orthogonal to both (C) and (B), hence Q, R are antihomologous points on the latter two circles [1, p. 198, art. 435]; the same holds for the points Q', R', for analogous reasons, hence the point D is a center of similitude of the circles (B), (C).

Similarly the points E, F are centers of similitude of the pairs of circles (C), (A); (A), (B), respectively. Hence the line $s=DEF$ is an axis of similitude of the circles (A), (B), (C).

Likewise for the other pairs of complementary triangles of intersection. Thus: *The four axes of perspectivity of the four pairs of complementary triangles of intersection of three mutually orthogonal circles coincide with the four axes of similitude of these circles.*

The points $D'=(QR', Q'R)$, $E'=(RP', R'P)$, $F'=(PQ', P'Q)$ are the other three centers of similitude of the three pairs of circles (B), (C); (C), (A); (A), (B), respectively. The lines $s_p=DE'F'$, $s_q=EF'D'$, $s_r=FD'E'$ are the axes of perspectivity of the pairs of triangles II, III, IV, respectively (table B), and also the other three axes of similitude of the circles (A), (B), (C).

b. The circumcircles of the triangles of intersection may conveniently be referred to as the circles of intersection of the circles (A), (B), (C). Two circles of intersection will be said to be complementary if they are the circumcircles of two complementary triangles of intersection.

c. Let (I), (I') denote the circumcircles of the triangles PQR, $P'Q'R'$ (table B). The lines DQR, $DQ'R'$ (section 3a) are the radical axes of the pairs of circles (I), (A); (I'), (A), respectively, hence the radical axis of (I), (I') passes through the point D. The points E, F lie on that radical axis for analogous reasons. Thus the radical axis of (I), (I') coincides with the line $s=DEF$ (section 3a).

By a like argument it may be shown that the lines s_p, s_q, s_r are the radical axes of the pairs of circles (I_p), (I_p'); (I_q), (I_q'); (I_r), (I_r') (table B). Thus: *The radical axis of two complementary circles of intersection* (section 3b) *of three mutually orthogonal circles coincides with the axis of perspectivity of the corresponding triangles of intersection.*

d. As an immediate consequence of sections 3a, 3c we have: *The radical axes of the four pairs of complementary circles of intersection of three mutually orthogonal circles coincide with the four axes of similitude of the latter three circles.*

4. Circles of antisimilitude. a. The point D common to the radical axes of the three circles (A), (I), (I') taken in pairs (section 3c) is the center of the orthogonal circle (D) of those three circles.

On the other hand, the point D is a center of similitude of the circles (B), (C) (section 3a), hence D lies on the line BC. Moreover, the circle (D) being orthogonal to (A), the three circles (B), (C), (D) are coaxial [**1**; p. 207, art. 458]. The circle (D) thus passes through the points P, P' and is a circle of antisimilitude of the given circles (B), (C).

It may be shown in a like manner that the point D' is the center of the second circle of antisimilitude (D') of the circles (B), (C).

Analogous considerations yield the circles of antisimilitude (E), (E'); (F), (F') of the pairs of given circles (C), (A); (A), (B), respectively.

The two circles of antisimilitude (D), (D') are orthogonal to each other and both pass through the vertex P of the triangle PQR; furthermore, (D) is orthogonal to the circumcircle (I) of PQR, as was shown at the beginning of the present paragraph, at the point P, and has its center D on the side QR opposite P (section 3a). Hence the circle (D') is tangent to (I) at P, while the circle (D) coincides with the Apollonian circle of the triangle PQR passing through the vertex P.

For analogous reasons the circles of antisimilitude (E), (F) whose centers E, F are collinear with D, on the axis s, are the Apollonian circles of the triangle PQR passing through the vertices Q, R, respectively, and the circles (E'), (F') are tangent to the circle (I) at the vertices Q, R, respectively.

The arguments presented in connection with the triangle PQR and its circumcircle (I) are equally valid for the complementary triangle of intersection

$P'Q'R'$ and its circumcircle (I'), that is, the circles of antisimilitude (D), (E), (F) are the Apollonian circles of $P'Q'R'$, and the circles (D'), (E'), (F') touch the circle (I') at the vertices P', Q', R', respectively.

The rôle played by the axis of similitude $s = DEF$ with respect to the two complementary triangles of intersection of group I (table B) may be played by the axes of similitude s_p, s_q, s_r with respect to the pairs of complementary triangles of intersection of the groups II, III, IV, respectively.

To sum up: *Given three mutually orthogonal real circles,* i. *Their three circles of antisimilitude, whose centers lie on any axis of similitude, are the Apollonian circles of each of the two complementary triangles of intersection whose axis of perspectivity coincides with the axis of similitude considered.*

ii. *Each of the remaining three circles of antisimilitude of the three given circles is tangent to the circumcircles of the two triangles of intersection considered, the point of contact being a vertex of the respective triangle.*

b. The preceding proposition (section 4a) gives rise to a considerable number of corollaries. Here are some:

i. *The axis of perspectivity of two complementary triangles of intersection is the Lemoine axis of each of the two triangles.*

ii. *Two complementary triangles of intersection have the same Brocard diameter and the same isodynamic points.*

iii. *The four Brocard diameters of the four pairs of complementary triangles of intersection are concurrent.*

Indeed, they have in common the orthogonal center M of the three given circles.

iv. *The circumcircles of two complementary triangles of intersection have no points in common.*

Indeed, the two isodynamic points are inverse with respect to each of the two circumcircles (I), (I') [**1**; p. 262, art. 603], they are therefore the limiting points of the coaxial pencil determined by (I), (I') [**1**; p. 207, art. 457], hence the proposition.

c. The circle (D) has its center $D = ss_p$ on the two axes of similitude s, s_p, hence (D) is a circle of Apollonius for two, and only two, pairs of complementary triangles of intersection, namely, the triangles of the groups I and II, while the circle (D') is tangent to the circumcircles of those two pairs of triangles. The rôles of the two circles (D), (D') are reversed for the triangles of the groups III and IV (section 4a).

Similarly for the other two pairs of circles of antisimilitude (E), (E'); (F), (F'). Thus: *Given three mutually orthogonal circles, each of their circles of antisimilitude is a circle of Apollonius of four triangles of intersection, and is tangent to the circumcircles of the remaining four triangles of intersection.*

5. Circles of intersection (section 3b). a. The point P is a common vertex of the two pairs of triangles PQR, $P'Q'R'$:PQR', $PQ'R$, and the center D of the

circle (D) lies on the four lines s, s_p, QR, $Q'R'$, hence the circle (D) is an Apollonian circle of each of the first two triangles and thus orthogonal to their circumcircles (I), (I_p) at P; and (D) is tangent at P to the circumcircles (I_r'), (I_q') of the second pair of triangles (section 4). The circle (D'), on the other hand, is tangent to the circles (I), (I_p) and orthogonal to the circles (I_r'), (I_q'), at the point P. Similarly for the other points of intersection of the circles (A), (B), (C) taken in pairs. Now the two circles of antisimilitude (D), (D') of the two given circles (B), (C) are orthogonal, hence: *The four circumcircles of four triangles of intersection having a common vertex may be grouped into two pairs such that the two circles of each group are tangent to each other, and the circles of one group are orthogonal to those of the other group.*

Observe that of the two circles of antisimilitude passing through the point of intersection considered one belongs to one of the two groups of circumcircles, and the other circle of antisimilitude belongs to the other group.

b. Consider now one of the triangles of intersection, say, PQR. Its circumcircle (I) is tangent to three other circles of intersection, one at each of the points P, Q, R (section 5a). The circle tangent to (I), say, at Q is the circumcircle (I_q) of the triangle $P'QR'$ obtained by replacing in PQR the other two vertices P, R by their complementary points of intersection (section 2) P', R'.

Through each vertex of PQR pass two circles of intersection orthogonal to (I). But the two circles orthogonal to (I), say, at P cut (I) again in the points Q, R, respectively, hence (I) is orthogonal to three other circles of intersection. The circle orthogonal to (I) at the points, say, P, R is the circumcircle (I_q') of the triangle $PQ'R$ obtained from PQR by replacing the third vertex Q by its complementary point of intersection Q'.

Summing up: *Given three mutually orthogonal circles, each of their eight circles of intersection touches three of the remaining seven circles, at the vertices of its inscribed triangle of intersection; it cuts orthogonally three other circles of intersection, at the vertices of its inscribed triangle taken two-by-two, and has no points in common with the eighth circle of intersection* (section 4b iv).

6. Circumcenters. a. The two triads of circles (I), (I_p), (D'); (I_q'), (I_r'), (D) (section 5a) pass through the point P, and the circles of each triad are tangent to each other at that point, hence we have the two tetrads of collinear points P, I, I_p, D'; P, I_p', I_r', D.

Considering the point P' we obtain, in a like fashion, the two tetrads of collinear points P', I', I_p', D; P', I_q, I', D'. Thus the circumcenters of the eight triangles of intersection lie by twos on the four lines DP, DP', $D'P$, $D'P'$.

Similarly for the other two pairs of centers of similitude E, E'; F, F'. Thus: *Given three mutually orthogonal circles, the circumcenters of their eight triangles of intersection lie by twos on the four lines joining the points of intersection of any two of the three given circles to the two centers of similitude of those two circles.*

b. COROLLARY. *If of three mutually orthogonal real circles, two circles are fixed*

and the third circle varies, the circumcenters of their eight triangles of intersection describe four fixed straight lines.

c. The four chords QR, $Q'R'$; QR', $Q'R$ of the circle (A) (see table B) are each the base of two triangles of intersection, namely, PQR, $P'QR$; $PQ'R'$, $P'Q'R'$; PQR', $P'QR'$; $PQ'R$, $P'Q'R$.

Now the perpendicular from the center A of the circle (A) upon QR passes through the circumcenters I, I_p' of the two triangles of intersection having the chord QR for base. We have thus the triad of collinear points A, I, I_p'.

We obtain in a like manner the triads of collinear points A, I', I_p; A, I_r', I_q; A, I_q', I_r.

Similarly for the centers B, C of the given circles (B), (C). Thus: *The eight circumcenters of the triangles of intersection of three given mutually orthogonal circles lie in pairs on three tetrads of lines passing, respectively, through the three centers of the given circles.*

Observe that we have accounted for the $(8\times 7)\div 2=28$ lines which join the eight circumcenters two-by-two (sections 6a, 6c, and 4b iv).

7. More on the circles of antisimilitude. We have established a number of properties of the circles of antisimilitude of three mutually orthogonal circles, in their relation to the triangles of intersection of the three given circles. These properties permit, in turn, to find some relations of the circles of antisimilitude among themselves. Here are two examples.

a. The three coaxial circles (D), (E), (F) are the circles of Apollonius of a triangle; the same holds for the circles (D), (E'), (F') (section 4). Hence the circle (D) cuts each of the circles (E), (F), (E'), (F') at an angle of 120° [**1**; p. 266, art. 624].

Similarly for each of the other five circles of antisimilitude. Thus: *Each of the six circles of antisimilitude of three mutually orthogonal circles cuts four of the remaining five circles at an angle of* 120°, *and is orthogonal to the fifth circle.*

b. The triad of circles (D), (E'), (F') having their centers on the axis of similitude s_p are the Apollonian circles of a triangle (section 4), hence the center D of (D) is a center of similitude of the two circles (E'), (F') [**1**; p. 263, art. 612].

Considering the axes of similitude s_q, s_r we may show in a like manner that the points E, F are centers of similitude of the pairs of circles (F'), (D'); (D'), (E'), respectively. Hence the line $s=DEF$ is an axis of similitude of the three circles (D'), (E'), (F').

Similarly for the axes of similitude s_p, s_q, s_r of the three given circles. Thus: *Each axis of similitude of three mutually orthogonal circles is also an axis of similitude of those three circles of antisimilitude of the three given circles whose centers lie outside the axis considered.*

8. Centers of similitude of the circles of intersection. Given three mutually orthogonal real circles: a. Their orthogonal center is a center of similitude, and their orthogonal circle is a circle of antisimilitude of each of their four pairs of complementary circles of intersection.

b. The remaining four centers of similitude of the four pairs of circles of intersection coincide with the poles of the four axes of similitude of the three given circles with respect to their orthogonal circle.

c. The same four points are also the trilinear poles of the axes of similitude considered with respect to the central triangle of the three given circles.

d. The complete quadrangle formed by those four points has the central triangle of the three given circles for its diagonal triangle.

e. The six sides of the complete quadrangle just mentioned pass respectively through the six centers of similitude of the three given circles.

The proofs of these propositions are omitted, to save space.

9. Paraphrases. The orthogonal center M of the three mutually orthogonal circles (A), (B), (C) is the orthocenter of the triangle ABC, and their orthogonal circle (M) is the polar circle of ABC (**4**).

The points P, P'; Q, Q'; R, R' (section 2) are the points of intersection of the altitudes of the triangle ABC with the circles having for diameters the respective sides of ABC. Some of the properties obtained for the three circles (A), (B), (C), may be restated in terms of these elements of the triangle ABC. The paraphrasing of those propositions is left for the entertainment of the reader.

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AN INTRODUCTION TO ELLIPTIC GEOMETRY

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1. The problem. Anyone familiar with the intuitive presentations of elliptic geometry in American and British books, even the most recent, must admit that their handling of the foundations of this subject is less than fair to the student. Once these books manage to get past ("prove" would be the wrong word) basic propositions such as *all perpendiculars to a straight line meet in a point* and *the length of a straight line is a multiple of its polar distance* the sailing is comparatively smooth, but until then the waters are, indeed, rough and muddy. For, the stated assumptions on which these theorems are based are vague, the proofs involve an unduly large number of unwarranted procedures even for an intuitive treatment, and far too little care is taken in the proofs to distinguish between the single and double elliptic situations. It is commonly assumed, for example, that a straight line is *boundless*, but this term is not defined. While all straight lines are assumed to meet, there is usually no clear statement as to the number of times they are supposed to meet, nor why they cannot meet more than twice. Nor is it made clear how much of Euclidean theory, particularly that concerning angles and the congruence of triangles, is being retained, nor under what conditions.

This article attempts to remedy the situation by offering an intuitive *introduction* in which more than traditional care is taken in preparing the student's mind for the acceptance of the new ideas, in stating basic assumptions, in carrying out proofs, and in keeping the single and double elliptic geometries separate. Although this is only intended to be a beginning, and hence does not go beyond theorems like those mentioned above, the axioms we have adopted are suitable for developing the subject much further. Only two-dimensional geometry is considered.

2. History. Saccheri seems to have been the first to take the revolutionary step of considering, at least tentatively, the logical possibility that there might be systems of geometry other than Euclidean, one associated with his hypothesis of the acute angle, the other with his hypothesis of the obtuse angle. He ruled out the first system as untenable by reaching contradictions where none existed, and hence failed to discover hyperbolic geometry. The hypothesis of the obtuse angle also led him to a contradiction, a result of his tacit assumption that a straight line is infinite, and so this system was ruled out too. But since he had set himself the task of developing this hypothesis on the basis of Euclid's axioms and postulates, excluding the parallel postulate, and since the infinitude of a straight line is not explicitly included in this basis, it may be claimed that Saccheri had not found a genuine contradiction in his second system.

For a century after Saccheri published his *Euclides Vindicatus* (1733) the geometers who occupied themselves with the problem of Euclid's parallel

postulate took it for granted that a straight line is infinite, and hence it seems not to have occurred to them to ask, let alone try to answer, the question: Can the hypothesis of the obtuse angle lead to a consistent system of geometry if it is not assumed that a straight line is infinite? The question would not have been at all academic when it is realized that Saccheri had already deduced some interesting consequences of this hypothesis, for example, that the angle-sum of every triangle exceeds two right angles, that the locus of points equidistant from a straight line l is not a straight line but a curve convex toward l, and that two straight lines intersect if the sum of the interior angles on the same side of a transversal is less than two right angles.

In 1854, not long after the solution of the parallel problem had culminated in the discovery of hyperbolic geometry, the above question became unavoidable, with the pronouncement by Riemann that space need not be regarded as infinite just because it seems boundless. Attention was thus directed to the consideration of possible systems of geometry in which a straight line, although endless, is nevertheless finite in length. Such a line can be visualized as a closed path which, like a circle or ellipse, does not intersect itself. Adopting this viewpoint mathematicians before long discovered, not one, but two consistent systems of geometry in which Saccheri's hypothesis of the obtuse angle holds. When it is recalled that in Euclidean and hyperbolic geometry the existence of parallel lines is established with the aid of the assumption that a straight line is infinite, it comes as no surprise that there are no parallel lines in the two new, *elliptic geometries*.

3. Geometry on a surface. To make Riemann's ideas and the new geometries intuitively acceptable requires only a simple broadening of our outlook. When the physical world is viewed in the Euclidean manner, a flat surface like that of a blackboard, table top, or swimming pool is regarded as a small, somewhat imperfect portion of an ideal flat surface of infinite extent. Hence, although the straight paths on such physical surfaces, *e.g.* rulings on a blackboard, are necessarily of limited extent, we imagine them to be parts of infinite straight paths, and hence to obey the propositions of Euclidean geometry. When things are viewed in the manner of hyperbolic geometry, however, the very same blackboard can be regarded, with equal correctness, as a tiny part of an enormous pseudosphere, and the very same rulings as obeying hyperbolic geometry, for within limited regions both systems are equally correct.

But these two systems do not exhaust the ways in which we can view the physical world. Our blackboard can be regarded as a small part of many other surfaces, and its rulings as subject to the various geometries of the shortest paths on these surfaces. Among such surfaces are some, the spheres being the simplest, on which the shortest paths are closed curves of great extent. Associated with these surfaces, then, are geometries, including the elliptic types, in which the straight line is endless yet finite, in accord with Riemann's ideas. Thus, in addition to the Euclidean and hyperbolic ways of rationalizing the

blackboard and its rulings (or any flat physical surface and its straight paths) we also have the Riemannian. Indeed, the latter makes more sense than the other two if the universe is of limited extent and not "roomy" enough for their straight lines.

4. Double elliptic geometry. Our problem of choosing axioms for this geometry is something like what would have confronted Euclid in laying the basis for 2-dimensional geometry had he possessed Riemann's ideas concerning straight lines and used a large curved surface, with closed shortest paths, as his model, rather than an infinite flat one. And like Euclid we shall proceed in a strongly intuitive manner.

The terms "point," "line," and "length" are not defined, and it is taken for granted that a line has a finite length and contains points. A "closed line," also undefined, is to be regarded as a special kind of line. (The reader will find it useful to visualize a line as a continuous curve and a closed line as a closed continuous curve.) In turn, a "straight line" is a special kind of closed line. Thus "line" and "straight line" are not synonymous.

AXIOM 1. *A straight line is a closed line (of finite length) not intersecting itself.*

In the geometry we are building there shall be no parallel straight lines. How many times, then, shall two straight lines meet? If two closed paths drawn on a piece of paper cut each other once they will cut each other at least once more. In the simplest situation they cut each other just twice. Guided by this we assume

AXIOM 2. *Each pair of straight lines meet in exactly two points.*

Among the various paths joining two points on a physical surface there is usually at least one which is shortest. Hence we assume

AXIOM 3. *Among the lines joining two points there is one (or more) whose length is least.*

A line with this property is called a *segment*, and its length is called the *distance* between the two points. If there is exactly one segment joining two points, the segment is called *unique*. (It must be repeated that "line" and "straight line" are not synonymous.)

Axiom 1 suggests that each two points A and B of a straight line divide it into two parts, each a line joining A and B. Since a straight line should be a "repository" of shortest paths, we shall want at least one of these two lines joining A and B to be a segment. This leads to Axiom 4. The converse leads to Axiom 5.

AXIOM 4. *A straight line is divided into two lines by each two of its points, and at least one of these lines is a segment.*

AXIOM 5. *Any given segment joining two points is contained in some straight line through the points.*

A simple consequence of Axiom 4 is:

THEOREM 1. *If the two lines into which a straight line is divided by two of its points are unequal, the lesser is a segment joining the points; if equal, both are segments.*

Two points that divide a straight line into equal segments are called *antipodal points* of the line, each being *antipodal to the other* on the line.
From Axioms 3 and 5 we infer:

THEOREM 2. *There is a straight line through each two points.*

When will a unique straight line go through two points? A partial answer is given by Theorem 3, which follows from Axioms 2, 4, and 5.

THEOREM 3. *A unique straight line goes through two points if there is a unique segment joining them.*

The rest of the answer to the above question is given as follows:

AXIOM 6. *A straight line g through two points is the only straight line through them if, and only if, the points are non-antipodal on g.*

From this we deduce Theorem 4 (the converse of Theorem 3) and Theorem 5:

THEOREM 4. *There is a unique segment joining two points if there is a unique straight line through them.*

THEOREM 5. *Two points cannot be antipodal on one straight line and non-antipodal on another.*

If a point B is antipodal to a point A on one straight line through A it is therefore antipodal to A on every other straight line through A. Hence we may simply say *B is antipodal to A* without specifying a particular straight line. It follows that to any given point there corresponds exactly one other which is antipodal to it.
Let a and b be any two straight lines. By Axioms 2 and 4 each divides the other into two lines, at least one of which is a segment. By Axiom 6 the two lines into which a is divided are equal, and hence both are segments. Likewise, the two lines into which b is divided are also segments. By the definition of a segment each segment on a equals each one on b. Thus we have

THEOREM 6. *All straight lines bisect each other and have the same length.*

From Theorem 6 we see that all segments formed by the intersection of straight lines in pairs are equal. Let us take the common length of these segments as unity. The length of a straight line is then 2 units. Also the distance

between two points cannot exceed 1. For, let A and B be any two points. Then, by Axiom 3 and the definition of distance, there is a segment joining them whose length is the distance between them. By Axiom 5 this segment is contained in some straight line g through A and B. If A and B divide g equally, in which case they are antipodal, this segment is one of the two thus formed and its length is 1, as agreed above. If A and B divide g into unequal parts, the mentioned segment must be the lesser and its length less than 1. Thus we have

THEOREM 7. *The distance between two points is maximum if, and only if, the points are antipodal. This maximum value is* 1, *and the length of a straight line is* 2.

It is easy to verify the truth of the following theorem:

THEOREM 8. *There is a unique segment, and hence a unique straight line, joining two points if, and only if, the distance between them is less than* 1.

Since a small portion of a curved surface may be nearly flat, it is clear that the surface may have certain *local* properties in common with a Euclidean plane. For example, it would not be unreasonable to expect that the angular theory at a point is one such common property. By this is meant that the total angle at a point is the same for all points, namely 360°, that the vertical angles formed by two straight lines are equal, that there is a unique perpendicular at any point of a straight line, that all right angles equal 90°, that all straight angles equal 180°, and so forth. For brevity let us assume all these properties at one stroke.

AXIOM 7. *The angular theory at a point is the same as in Euclidean plane geometry.*

Similarly, since our geometry is to be as applicable to the flat physical surfaces of experience as Euclidean geometry, we should expect to find in it some duplication, at least locally, of the Euclidean theory on the congruence of triangles. To attain this local restriction we shall understand a *triangle* to mean only a figure consisting of three non-collinear points and three segments joining them in pairs. (It easily follows that each such segment is unique.) These segments are called the *sides of the triangle*, and the three angles which they determine, each less than 180°, are the *angles of the triangle*. Calling two triangles, as usual, *congruent* if their sides and angles are equal, respectively, we then adopt

AXIOM 8. *Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.*

Our final assumption, which expresses another local property of many familiar surfaces, is:

AXIOM 9. *A straight line which subdivides an angle of a triangle meets the opposite side.*

We now prove two basic theorems dealing with the perpendiculars to a straight line.

Let b be any straight line, and M any point on b (Fig. 1). Then c, the straight line perpendicular to b at M meets b again in a point N. Let P be the midpoint of one of the two segments into which b divides c, and denote this segment by MPN. Let β be one of the two segments into which c divides b. The straight line d, perpendicular to c at P, meets b in two points. By Theorem 6 neither of these points coincides with M or N, and exactly one lies on β. Call this latter point R.

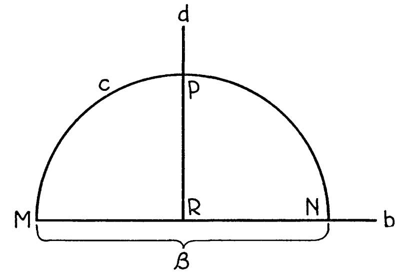

FIG. 1

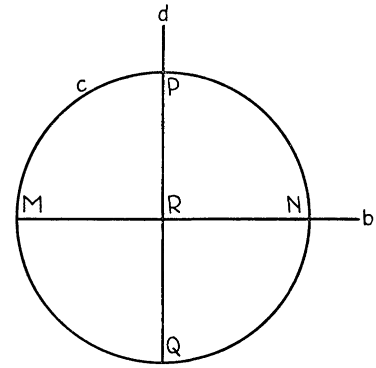

FIG. 2

If d is continued beyond R it will meet c in a point Q antipodal to P by Theorem 6 (Fig. 2). Continued beyond Q it will meet b in a second point S, and continued beyond S it must return to P (Fig. 3). Since c bisects d by Theorem 6, and b does also, we obtain, respectively,

$$PR + RQ = QS + SP = 1$$

and

$$RQ + QS = SP + PR = 1.$$

From this we infer that

$$PR = RQ = QS = SP = 1/2.$$

By Theorem 8 this means that PR, RQ, QS, and SP are unique segments. Also MR, NR, MS, NS are unique segments, each being clearly less than 1. Hence we obtain triangles MPR and NPR, and they are congruent since $MP = NP$, $PR = PR$, $\sphericalangle MPR = \sphericalangle NPR$. Hence $\sphericalangle MRP$ is a right angle. Similarly, we obtain triangles MPS and NPS, and they too are congruent by side-angle-side. Hence $\sphericalangle MSP$ is a right angle.

Thus d, like c, goes through P and Q, and is perpendicular to b at points each $\frac{1}{2}$ unit from P and $\frac{1}{2}$ unit from Q. The same is true of every straight line e through P (Fig. 4). Thus if e goes through P so as to subdivide $\sphericalangle MPR$ of triangle MPR, it must meet side MR in a point T by Axiom 9, and cannot meet sides MP or RP by Theorem 6, since it goes through Q. Since e, when extended beyond T, goes through Q, it must subdivide one of the two pairs of vertical

angles at Q formed by c and d. If it subdivided $\sphericalangle RQN$ of triangle RQN it would meet side RN in a point by Axiom 9, but then b and e would not bisect each other, contradicting Theorem 6. Hence e subdivides $\sphericalangle MQR$.

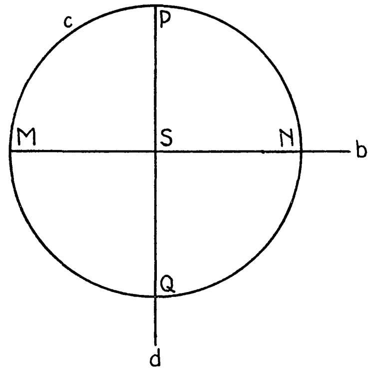

FIG. 3

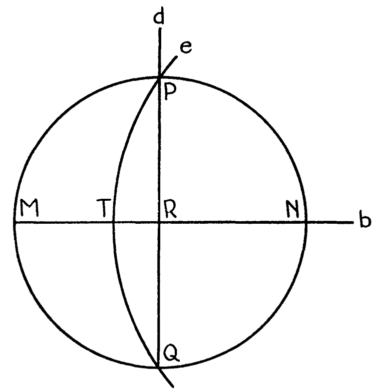

FIG. 4

We can now extend e beyond Q, proceed just as we did for d, and prove that b and c divide e into four equal parts, each a unique segment. If PT and QT in Figure 4 are two of the four parts then we obtain triangles RTP and RTQ, and they are congruent since $RP=RQ$, $\sphericalangle PRT=\sphericalangle QRT$, $RT=RT$. Hence $\sphericalangle RTP=\sphericalangle RTQ=$ a right angle. Likewise for the two other parts. Finally, let X be any point of b, and x the straight line perpendicular to b at X. Since X is not antipodal to P there is only one straight line PX according to Theorems 7 and 8, and since this straight line goes through P it is perpendicular to b at X, in accordance with the above discussion. There being only one straight line perpendicular to b at X, straight lines x and PX are identical. Hence we have

THEOREM 9. *If b is any straight line, each straight line c perpendicular to b goes through the same two points P and Q, and is divided by these points and b into four equal segments. The points in which c meets b are each $\frac{1}{2}$ unit from P and Q.*

Let P be any point. If M is a point $\frac{1}{2}$ unit from P there is a unique straight line c through M and P by Theorem 8. Let b be the straight line perpendicular to c at M. If X is any point $\frac{1}{2}$ unit from P, it follows from the proof of Theorem 9 that straight line PX is perpendicular to b. Hence X is on b by Theorem 9, and we have

THEOREM 10. *The locus of points $\frac{1}{2}$ unit from any point P is a straight line which is perpendicular to each straight line through P.*

It was not intended to develop the subject beyond this point.

5. Single elliptic geometry. The remarks made in the opening paragraph of

article 4 apply here word for word and are omitted in the interest of brevity. It is worth repeating, however, that "line" and "straight line" are not synonymous. Some of the motivations are also the same as those previously given for double elliptic geometry and will be omitted.

AXIOM 1. *A straight line is a closed line (of finite length) not intersecting itself.*

AXIOM 2. *Each pair of straight lines meet in exactly one point.*

The preceding axiom is simple in the sense that it provides us with a familiar state of affairs, *i.e.*, that prevailing in Euclidean geometry. On the other hand it shows that the surface on which single elliptic geometry holds will have to possess some unusual features in order to contain closed lines cutting each other exactly once.

AXIOM 3. *Among the lines joining two points there is one (or more) whose length is least.*

The terms *segment*, *unique segment*, and *distance* mean the same as in double elliptic geometry.

AXIOM 4. *A straight line is divided into two lines by each two of its points, and at least one of these lines is a segment.*

AXIOM 5. *Any given segment joining two points is contained in some straight line through the points.*

From Axiom 4 we infer

THEOREM 1. *If the two lines into which a straight line is divided by two of its points are unequal, the lesser is a segment joining the points; if equal, both are segments.*

It is convenient to define *antipodal points* on a straight line as in double elliptic geometry. From Axioms 2 and 5 we obtain

THEOREM 2. *Exactly one straight line goes through each two points.*

In view of this theorem, if B is antipodal to A on one straight line through A, and C is antipodal to A on a second straight line through A, then B and C are distinct, unlike the case in double elliptic geometry.

AXIOM 6. *The angular theory at a point is the same as in Euclidean plane geometry.*

We change the definition of a triangle given for double elliptic geometry to the extent of permitting its sides to be *any* portions of straight lines, not necessarily segments. Retaining the previous definition for congruence of triangles, we make the following assumption:

AXIOM 7. *Two triangles are congruent if two angles and the included side of one are equal, respectively, to two angles and the included side of the other.*

Two straight lines which are perpendicular to the same straight line will meet by Axiom 2. Hence there are triangles containing two right angles. Concerning such triangles we make the following reasonable assumption, our last:

AXIOM 8. *If two angles of a triangle are right angles, the opposite sides are equal.*

This axiom may call for some comment. In Euclidean geometry the theorem that the sides opposite equal angles are equal results from the fact that there is a unique perpendicular from a point to a straight line. In single elliptic geometry the latter property does not hold, as we shall see, and yet it is true that sides opposite equal angles are equal. Hence we have assumed the latter property, though in a restricted form. In double elliptic geometry, however, we were able to prove the content of Axiom 8.

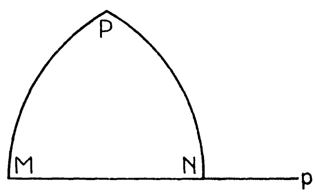

FIG. 5

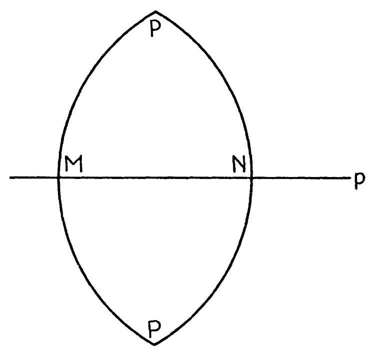

FIG. 6

Now let p be any straight line, M and N any two of its points, and MN one of the two lines into which M and N divide p. The straight lines perpendicular to p at M and N meet in a point P. Since straight lines are closed lines there is not just one triangle MPN. Hence let us fix our attention on a triangle MPN of which the chosen part MN is a side (Fig. 5). If we extend sides PM and PN beyond M and N, respectively, until we return to P, and represent things conveniently as in Figure 6, a second triangle MPN is obtained having MN in common with the first. These triangles are congruent by Axiom 7, so that the two sides PM are equal, and likewise for the two sides PN. Hence P is antipodal to M on straight line PM, and to N on straight line PN. Since M and N are any two points of p we may think of M as fixed and N as variable. From what was just shown it then follows that regardless of the position of N the straight line perpendicular to p at N must meet straight line MP in the point of the latter which is antipodal to M, namely, in P. Conversely, using this fact, together with Axiom 2 and Theorem 2, we infer that each straight line through P is perpendicular to p. We have thus proved:

THEOREM 3. *All the perpendiculars to a straight line p go through the same*

point P, and each straight line through P is perpendicular to p, meeting the latter in a point which is antipodal to P.

We call P the *pole* of p, and p the *polar* of P. In view of Theorem 3 not only does any given straight line have a pole, but any given point is the pole of some straight line. It follows that the locus of points antipodal to a given point is a straight line, the polar of the latter. In double elliptic geometry, on the other hand, this locus is a single point.

If we apply Axiom 8 to the triangle of Figure 5 we infer that sides PM and PN are equal. Since these sides are segments by Theorems 1 and 3, and M and N are any points of p, we see that P is at a fixed distance from the points of p. We call this distance the *polar distance for* p and denote it by α. Using what was shown in connection with Figure 6 we infer that all straight lines perpendicular to p have the same length, namely, 2α. Now let q and r be two straight lines through P which are perpendicular to each other. By Theorem 3 they are also perpendicular to p and hence have the same length 2α. Now r, like p, has the property that all straight lines perpendicular to it have a common length. Hence p and q, being perpendicular to r, have the same length. Thus p, as well as each straight line perpendicular to it, has the length 2α. Finally, let p' be any straight line distinct from p, let P' be the pole of p', and let $2\alpha'$ be the common length of p' and of each straight line perpendicular to it. If we choose equal segments MN and $M'N'$ on p and p', respectively, then triangles MNP and $M'N'P'$ are congruent by Theorem 3 and Axiom 7. Hence sides MP and $M'P'$ are equal, *i.e.*, $\alpha=\alpha'$. Thus p and p' have equal polar distances and, consequently, equal lengths.

THEOREM 4. *All straight lines have the same polar distance α and the same length 2α.*

If A and B are any two points, they are either antipodal or non-antipodal on the straight line determined by them, and hence the distance between them is either α or less than α. The maximum distance between any two points is then α. We thus have

THEOREM 5. *The distance between two points is maximum if, and only if, the points are antipodal. This maximum equals α, the common polar distance for all straight lines.*

It was not intended to develop the subject any further.

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NETS AND FILTERS IN TOPOLOGY

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Since the notion of a limit is of utmost importance throughout all of analysis, it is only proper that it has received frequent and varied treatment at the hands of many mathematicians. Of the several theories* available, there are two which appear to be most popular in current use. One is that of *net*, or *directed system*, which was initiated by E. H. Moore and H. L. Smith [7]† and has been discussed and improved by J. L. Kelley [4]; the other is that of *filter* and was originated by H. Cartan [2] [3]. It appears that the former notion is predominant in this country, while the filter theory reigns supreme in France.

It is well-known that these theories are equivalent in the sense that there are no proofs attainable by one that cannot be reached by the other. However, it is undeniable that each of these approaches has advantages—however psychological—not immediately possessed by the other. The use of nets parallels very closely standard constructions involving sequences, and has the pronounced advantage that sequential arguments may quite readily be adapted to them. Nets also render service in defining integrals as limits over the partitions of the interval. For these reasons they are very natural for many "analytical" arguments. On the other hand, the fact that filters are dual ideals in the algebra of sets renders a certain symmetry to their use, and often an "algebraic elegance." In addition, they are admirably suited for certain transfinite arguments, and enjoy a uniqueness not possessed by nets.

It would appear, therefore, that to neglect one notion, or to put undue emphasis on the other, is a shackle which the student assumes at his own peril. The equivalence of these concepts noted before assures that an argument involving filters can always be translated into one involving nets, and *vice versa.* Yet no matter how simple or how routine such a translation may be, it is likely to be shunned by most of us.

It is the purpose of this note to call explicit attention to certain simple relations between filters and nets and to construct each out of the other so that such a translation procedure is not necessary and so that these notions may be used simultaneously. It is felt that in this way the advantages of each will be preserved.

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* We give some references at the end of the paper; for others, see McShane [6]. *Added in proof.* Since this paper was written, a textbook on topology by J. L. Kelley, making systematic use of nets, has appeared. Mention should also be made of a paper by G. Bruns and J. Schmidt [9] which discusses both nets and filters and gives many references. Finally, it has been pointed out (see Math. Reviews, vol. 13, 1952, p. 829) that the notion of a subnet was employed by E. H. Moore long before the publication of [4].

† Numbers in brackets refer to the references at the end.

1. Preliminary definitions. We will assume a casual acquaintance, though not a detailed familiarity, with these theories. We refer the reader to the notes of Kelley [4] and McShane [5] on nets, and the discussion in Bourbaki [1, §§5, 6] concerning filters. However, partly for the convenience of the reader and partly to establish our notation we now give the basic definitions.

A *directed set* $A = \{\alpha\}$ is a set of elements of any kind in which there is an order relation defined between certain pairs of elements, written "$\leqq$", and satisfying the conditions

(i) $\alpha \leqq \alpha$, for all $\alpha \in A$;
(ii) if $\alpha_1 \leqq \alpha_2$ and $\alpha_2 \leqq \alpha_3$, then $\alpha_1 \leqq \alpha_3$;
(iii) for any pair α_1, α_2 in A there is an $\alpha_3 \in A$ such that $\alpha_1 \leqq \alpha_3$ and $\alpha_2 \leqq \alpha_3$.

A *net*, or a *directed system*, in an abstract set X is a function on a directed set with values in X.

In remembrance of sequences it is common to denote a net $\mathfrak{x}$ on A to X by the notation $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$. Some authors, however, adhere to the functional notation.

If E is a subset of X, we say that the net $\mathfrak{x}$ is *ultimately in* E if there is some index α_0 (depending on E) such that if $\alpha \geqq \alpha_0$ then $x_\alpha \in E$. If X is a topological space, we say that the net $\mathfrak{x}$ *converges to an element* $x_0 \in X$ in case that $\mathfrak{x}$ is ultimately in every neighborhood of x_0.

This much is all very standard and natural. Less standard, but of fundamental importance in many applications is the notion of a subnet due to Kelley [4]. (The unacquainted reader will want to refer to McShane [5] for an excellent motivation and discussion of this idea.) We suppose that we are in possession of a net $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$. In this case, a net $\mathfrak{y} = \{y_\beta\}_{\beta \in B}$ is said to be a *subnet* of $\mathfrak{x}$ in case there is a mapping $\pi: B \rightarrow A$ with the properties*

(i) $y_\beta = x_{\pi(\beta)}$, for all $\beta \in B$;
(ii) given any $\alpha_0 \in A$, there is a $\beta_0 \in B$ such that if $\beta \geqq \beta_0$ then $\pi(\beta) \geqq \alpha_0$.

We emphasize that the domains A and B of the nets $\mathfrak{x}$ and $\mathfrak{y}$ may be totally unrelated. What is essential is that the range of $\mathfrak{y}$ is contained in the range of $\mathfrak{x}$ (condition (i)) and that the net $\mathfrak{y}$ "goes as far as" the net $\mathfrak{x}$ in the sense made precise in condition (ii).

As a simple example of a special case of a subnet, we mention the following. Let α_0 be an arbitrary index; then if $A_0 = \{\alpha \in A : \alpha \geqq \alpha_0\}$ is allowed to inherit its ordering as a subset of A, it remains a directed set and the net $\{x_\alpha\}_{\alpha \in A_0}$ is clearly a subnet of $\mathfrak{x}$. Subnets formed in this manner are sometimes called *residual subnets* [8, p. 10]; they are very useful. Unfortunately, however, neither these residual subnets or the related *cofinal subnets* (see [8]) are sufficient for all purposes.

We now review the necessary definitions in the filter theory. Surprisingly

* The reader will observe that the same symbol is being used to indicate the order in both A and B.

enough, at least as regards convergence, the most important concept in this circle of ideas is that of the filter base rather than the filter itself. A non-void collection $\mathfrak{B}$ of non-void subsets of an abstract set X is called a *filter base* in X, provided that the intersection of two sets in $\mathfrak{B}$ contains a set in $\mathfrak{B}$. A *filter* in the set X is a non-void collection $\mathfrak{F}$ of subsets of X with the properties

(i) every set which contains a set in $\mathfrak{F}$ is itself in $\mathfrak{F}$;
(ii) every finite intersection of sets in $\mathfrak{F}$ is in $\mathfrak{F}$;
(iii) the void set $\emptyset$ does not belong to $\mathfrak{F}$.

Clearly a filter is itself a filter base; conversely, if $\mathfrak{B}$ is a filter base in X, then the collection of all subsets of X which contain a set in $\mathfrak{B}$ is seen to be a filter—called the *filter generated by* $\mathfrak{B}$.

We say that the filter base $\mathfrak{B}$ is *ultimately in* a subset E of X if E contains some set from $\mathfrak{B}$. Similarly, if X is a topological space, the filter base $\mathfrak{B}$ *converges to an element* $x_0 \in X$ if it is ultimately in every neighborhood of x_0.

If $\mathfrak{D}$ and $\mathfrak{B}$ are two filter bases, we say that $\mathfrak{D}$ *is a refinement of* $\mathfrak{B}$ if every set in $\mathfrak{B}$ contains some set in $\mathfrak{D}$. If this is true, we write $\mathfrak{D} \supseteq \mathfrak{B}$. It is readily seen that if $\mathfrak{D} \supseteq \mathfrak{B}$ and if $\mathfrak{G}$ and $\mathfrak{F}$ are the filters they generate, then $\mathfrak{G} \supseteq \mathfrak{F}$ both in the sense we have just defined for filter bases and in the sense that every set in $\mathfrak{F}$ is also in $\mathfrak{G}$.

If $\mathfrak{B}$ is a filter base and $E_0 \in \mathfrak{B}$, then it is evident that the set $\mathfrak{B}_0 = \{E \cap E_0 : E \in \mathfrak{B}\}$ is also a filter base and that $\mathfrak{B}_0 \supseteq \mathfrak{B}$. (This process corresponds exactly to taking a residual subnet of a given net.) A somewhat less trivial construction of a refining filter base is indicated in the following proposition.

1.1. PROPOSITION. *Let $\mathfrak{B} = \{E_\lambda\}$ be a filter base in a set X, and let E be any subset of X. Then at least one of the collections $\{E \cap E_\lambda\}$ and $\{E' \cap E_\lambda\}$ forms a filter base in X which is a refinement of $\mathfrak{B}$.*

Proof. If $E = X$, or if $E = \emptyset$, the statement is trivial, so that we may suppose that E and its complement E' are non-void. If each set $E \cap E_\lambda$ is non-void, the statement follows readily. If there exists a λ_0 such that $E \cap E_{\lambda_0} = \emptyset$, then $E_{\lambda_0} \subseteq E'$. Hence, for any λ, we have $E' \cap E_\lambda \supseteq E_{\lambda_0} \cap E_\lambda \neq \emptyset$ so that the sets $\{E' \cap E_\lambda\}$ are non-void and form a filter base.

It may happen that both of these collections are filter bases that refine $\mathfrak{B}$. The important fact is that we can continue to add more and more sets to $\mathfrak{B}$, getting finer and finer filter bases.

2. Relations between nets and filters. We are now sufficiently well-stocked with definitions to make some observations. While most of these are very simple remarks, we state them formally as propositions.

2.1. PROPOSITION. (a) *If $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$ is a net in an abstract set X, and if $E(\alpha) = \{x_\lambda : \lambda \geqq \alpha\}$, then the collection $\mathfrak{B}(\mathfrak{x}) = \{E(\alpha)\}$ is a filter base in X, called* ***the filter base associated with the net*** $\mathfrak{x}$.

(b) *If the net $\mathfrak{x}$ is ultimately in some set E, then $\mathfrak{B}(\mathfrak{x})$ is ultimately in E.*

(c) *If* $\mathfrak{y}=\{y_\beta\}_{\beta\in B}$ *is a subnet of* $\mathfrak{x}$ *and if* $\mathfrak{B}(\mathfrak{y})$ *is the filter base associated with* $\mathfrak{y}$, *then* $\mathfrak{B}(\mathfrak{y})$ *is a refinement of* $\mathfrak{B}(\mathfrak{x})$.

Proof. Let $E(\alpha_1)$ and $E(\alpha_2)$ be arbitrary sets in $\mathfrak{B}(\mathfrak{x})$. Since $A=\{\alpha\}$ is a directed set, there is an α_3 with $\alpha_1\leqq\alpha_3$ and $\alpha_2\leqq\alpha_3$. Clearly $E(\alpha_3)\subseteq E(\alpha_1)\cap E(\alpha_2)$ and (a) is proved. To prove (b), we note that there exists an α_0 such that if $\alpha\geqq\alpha_0$ then $x_\alpha\in E$. Consequently $E(\alpha_0)\subseteq E$, and $\mathfrak{B}(\mathfrak{x})$ is ultimately in E. Finally, let $E(\alpha_0)\in\mathfrak{B}(\mathfrak{x})$. By condition (ii) in the definition of subnet, there exists a β_0 such that if $\beta\geqq\beta_0$, then $\pi(\beta)\geqq\alpha_0$. Since $F(\beta_0)=\{y_\beta:\beta\geqq\beta_0\}=\{x_{\pi(\beta)}:\beta\geqq\beta_0\}$, we conclude that $F(\beta_0)\subseteq E(\alpha_0)$. This shows that $\mathfrak{B}(\mathfrak{y})$ is a refinement of $\mathfrak{B}(\mathfrak{x})$ and (c) is proved.

2.2. COROLLARY. *If X is a topological space and* $\mathfrak{x}$ *is a net in X which converges to* $x_0\in X$, *then the filter base* $\mathfrak{B}(\mathfrak{x})$ *associated with* $\mathfrak{x}$ *converges to* x_0.

Proof. This follows immediately from 2.1(b).

We now show that filter bases give rise in a natural manner to nets.*

2.3. PROPOSITION. (a) *Let* $\mathfrak{B}=\{E_\alpha\}_{\alpha\in A}$ *be a filter base in an abstract set X, where A is some index class. Order* $A=\{\alpha\}$ *by the requirement that* $\alpha_1\leqq\alpha_2$ *means* $E_{\alpha_1}\supseteq E_{\alpha_2}$. *With this ordering A is a directed set, and if from each set* $E_\alpha\in\mathfrak{B}$ *we pick an arbitrary point* $x_\alpha\in E_\alpha$, *then* $\mathfrak{x}(\mathfrak{B})=\{x_\alpha\}_{\alpha\in A}$ *is a net in X. Any net* $\mathfrak{x}(\mathfrak{B})$ *obtained in this manner is called* ***a net associated with the filter base*** $\mathfrak{B}$.

(b) *If the filter base* $\mathfrak{B}$ *is ultimately in some subset E of X, then any net* $\mathfrak{x}(\mathfrak{B})$ *associated with* $\mathfrak{B}$ *is ultimately in E.*

Proof. The fact that A is directed under the indicated ordering is a consequence of the fact that the intersection of any two sets in $\mathfrak{B}$ contains another set in $\mathfrak{B}$. To prove (b), let $E_{\alpha_0}\subseteq E$; then if $\alpha\geqq\alpha_0$ we have $x_\alpha\in E_\alpha\subseteq E_{\alpha_0}$ and so $x_\alpha\in E$ for $\alpha\geqq\alpha_0$.

2.4. COROLLARY. *If X is a topological space and* $\mathfrak{B}$ *a filter base in X which converges to* $x_0\in X$, *then any net* $\mathfrak{x}(\mathfrak{B})$ *associated with* $\mathfrak{B}$ *converges to* x_0.

Proof. This follows immediately from 2.3(b).

The reader will have observed that Proposition 2.3 lacks a part asserting that a filter base which refines $\mathfrak{B}$ gives rise to a subnet of a prescribed associated $\mathfrak{x}(\mathfrak{B})$. This is not always the case, as the reader may readily show. However, if we start with a net $\mathfrak{x}$, and form $\mathfrak{B}(\mathfrak{x})$, then any refinement $\mathfrak{D}$ of $\mathfrak{B}(\mathfrak{x})$ generates a subnet of $\mathfrak{x}$ in a manner similar to that prescribed in 2.3. This statement will now be proved.

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* The referee has pointed out that one can also construct nets from filters in the following manner: if $\mathfrak{F}$ is a filter, let A be the set of all pairs (x, F), where $x\in F\in\mathfrak{F}$. If A is ordered by the requirement $(x, F)\leqq(x', F')$ if and only if $F\supseteq F'$, then the mapping $(x, F)\to x$ gives a net. Further, the set F is the image under this mapping of all the sets (x', F') for which $(x, F)\leqq(x', F')$. The reader will observe the similarity between this comment and the constructions used in Propositions 2.1 and 2.3.

2.5. PROPOSITION. *Let* $\mathfrak{x} = \{x_\alpha\}_{\alpha \in A}$ *be a net in* X *and let* $\mathfrak{B}(\mathfrak{x})$ *be the associated filter base. Let* $\mathfrak{D} = \{F_\beta\}_{\beta \in B}$ *be a refinement of* $\mathfrak{B}(\mathfrak{x})$, *and order* $B = \{\beta\}$ *as in* 2.3(a). *Using the points* x_α, *rename as* y_β *in any manner subject only to the restriction that* x_α *cannot be renamed* y_β *unless it belongs to* F_β. *Then* $\mathfrak{y} = \{y_\beta\}_{\beta \in B}$ *is a subnet of* $\mathfrak{x}$.

Proof. It is not *a priori* evident that a given set in $\mathfrak{D}$ need contain any point x_α. Assume that F_β does not, and take a set $E_{\alpha'} \in \mathfrak{B}(\mathfrak{x})$. Since $\mathfrak{D}$ is a refinement of $\mathfrak{B}(\mathfrak{x})$, $E_{\alpha'}$ contains some set $F_{\beta'} \in \mathfrak{D}$ and $F_\beta \cap F_{\beta'} = \emptyset$, which is a contradiction. Thus the indicated construction can be carried out and it is clear that $\mathfrak{y}$ is a net in X. It remains to show that it is a subnet of $\mathfrak{x}$. Let $\pi: B \to A$ be the function that renames the x_α; evidently π satisfies condition (i) in the definition of subnet. To see that it satisfies (ii) we observe that for each $E(\alpha_0) \in \mathfrak{B}(\mathfrak{x})$ there is a β_0 such that if $\beta \geqq \beta_0$, then $F_\beta \subseteq E(\alpha_0)$. This is true since $\mathfrak{D}$ is a refinement of $\mathfrak{B}(\mathfrak{x})$. Since $x_{\pi(\beta)} = y_\beta \in F_\beta \subseteq E(\alpha_0)$, we conclude that $\pi(\beta) \geqq \alpha_0$ so that $\mathfrak{y}$ is a subnet of $\mathfrak{x}$.

We would now like to indicate briefly how the two theories may be utilized at the same time with advantage.

2.6. *Example.* We say that a point x_0 in a topological space X is a *cluster point*, or an *adherence point*, of the filter base $\mathfrak{B}$ if every neighborhood of x_0 has a non-void intersection with every set in $\mathfrak{B}$; x_0 is a cluster point of the net $\mathfrak{x}$ if it is a cluster point of the filter base $\mathfrak{B}(\mathfrak{x})$ associated with $\mathfrak{x}$. Now it is not difficult to show directly that if x_0 is a cluster point of $\mathfrak{x}$, then some subnet of $\mathfrak{x}$ converges to x_0. The direct demonstration requires a construction very similar to (but somewhat more technical than) the proof of Proposition 2.5. However, from the view-point of filter theory, it is truly obvious that the collection $\mathfrak{D}$ of sets of the type $E \cap U$, where $E \in \mathfrak{B}(\mathfrak{x})$ and U is any neighborhood of x_0, forms a filter base which is a refinement of $\mathfrak{B}(\mathfrak{x})$ and which converges to x_0. Applying this comment and Proposition 2.5, we obtain immediately the existence of a subnet of $\mathfrak{x}$ which converges to x_0.

2.7 *Example.* We now indicate how nets arise quite naturally in certain indirect arguments. Let X and Y be topological spaces and $f: X \to Y$. Then the following statements are equivalent:

(A) f is continuous at x_0;
(B) if $\mathfrak{B} = \{E\}$ is a filter base which converges to x_0, then $\mathfrak{C} = \{f(E)\}$ converges to $f(x_0)$;
(C) if $\mathfrak{x} = \{x_\alpha\}$ is a net which converges to x_0, then $\mathfrak{y} = \{f(x_\alpha)\}$ converges to $f(x_0)$.

The proof is as follows: Let (A) be true, let V be a neighborhood of $f(x_0)$ and let U be a neighborhood of x_0 such that $f(U) \subseteq V$. Then since $\mathfrak{B}$ is ultimately in U, it follows that $\mathfrak{C}$ is ultimately in V, which proves (B). That (B) implies (C) follows immediately on passing to the filter base $\mathfrak{B}(\mathfrak{x})$ associated with $\mathfrak{x}$. Now let (C) hold. If (A) is false, there exists a neighborhood V of $f(x_0)$ such that every neighborhood U of x_0 contains at least one point x_U with $f(x_U) \notin V$. It is obvious that $\{x_U\}$ is a net which converges to x_0, so the contradiction is obtained, and

the equivalence established. (It should be mentioned that it is equally simple to show that (B) implies (A); the point was to illustrate that the negation of a statement concerning a filter base—here the filter base of neighborhoods of x_0—leads in a natural way to a net.)

3. Ultimate notions and compactness. We now come to two corresponding concepts that are of utmost utility in their respective convergence theories.

A net in a set X is said to be a *universal net*, or a *U-net*, if for any subset E of X, the net is either ultimately in E or its complement E'.

A filter base in a set X is said to be an *ultrafilter base*, or a *U-filter base*, if for any subset E of X, the filter base is either ultimately in E or in E'.

If $\mathfrak{B}$ is a U-filter base and if $\mathfrak{U}$ is the filter generated by $\mathfrak{B}$, then it is readily shown that $\mathfrak{U}$ has the following maximal property: if $\mathfrak{F}$ is any other filter in X such that $\mathfrak{F}\supseteq\mathfrak{U}$, then $\mathfrak{F}=\mathfrak{U}$. (This is a manifestation of the uniqueness properties possessed by filters, and not by nets.)

J. L. Kelley has proved the important fact that any net has a subnet which is a U-net. While his proof is short and not difficult, it is by no means obvious. On the other hand, we feel that the corresponding statement for filters is entirely elementary.

3.1. PROPOSITION. *If $\mathfrak{B}$ is a filter base in a set X, then there is a U-filter base $\mathfrak{U}$ which is a refinement of $\mathfrak{B}$.*

Proof. If $\mathfrak{B}=\{E_\alpha\}$ is not ultimately in a subset A of X, then by Proposition 1.1, either $\{A\cap E_\alpha\}$ is a filter base or $\{A'\cap E_\alpha\}$ is. By a standard transfinite argument we may continue this refinement process until we obtain a U-filter base.

We now examine the constructions in the preceding section in the light of these new concepts.

3.2. PROPOSITION. (a) *If $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$ is a U-net in a set X, then the associated filter base $\mathfrak{B}(\mathfrak{x})$ is a U-filter base.*

(b) *If $\mathfrak{B}$ is a U-filter base in X, then any associated net $\mathfrak{x}(\mathfrak{B})$ is a U-net.*

Proof. Let E be an arbitrary set in X; then either $x_\alpha\in E$ or $x_\alpha\in E'$ for $\alpha\geqq\alpha_0$. Consequently, the set $E(\alpha)=\{x_\lambda:\lambda\geqq\alpha\}$ is either in E or E' when $\alpha\geqq\alpha_0$, so (a) is proved. To prove (b), let $\mathfrak{x}(\mathfrak{B})$ be constructed as in 2.3. Since $\mathfrak{B}$ is a U-filter base it is ultimately in a given set E or in E' and the same follows for $\mathfrak{x}(\mathfrak{B})$.

As a consequence, we obtain Kelley's result:

3.3. PROPOSITION. *Every net has a universal subnet.*

Proof. If $\mathfrak{x}$ is a net in X, then by 3.1 the associated filter base $\mathfrak{B}(\mathfrak{x})$ has a refinement which is a U-filter base. The conclusion then follows from 2.5 and 3.2(b).

Although it is not appropriate to develop here the theory of U-nets and U-filter bases, a final application is in order. We recall that a topological space X

is *compact* (=bicompact) if every covering of X by open sets contains a finite subcovering.

3.4. PROPOSITION. *The following statements are equivalent:*

(a) *X is a compact topological space;*

(b) *if $\{F_\alpha\}$ is a class of closed subsets such that every finite collection has a non-void intersection, then the entire class has a non-void intersection;*

(c) *every filter base in X has a cluster point;*

(d) *every filter base in X has a refinement which converges to an element of X;*

(e) *every U-filter base in X converges to an element of X;*

(f) *every U-net in X converges to an element of X;*

(g) *every net in X has a subnet which converges to an element of X;*

(h) *every net in X has a cluster point.*

Proof. The fact that (a) and (b) are equivalent follows by a familiar application of deMorgan's laws and will be omitted. We now prove that each statement (b), $\cdots$, (g) implies the following one and that (h) implies (b). That (b) implies (c) follows immediately from the observation that if $\mathfrak{B}=\{E_\alpha\}$ is a filter base, then every point in $\bigcap_\alpha \overline{E}_\alpha$ is a cluster point. The implication (c)$\rightarrow$(d) was proved in Example 2.6. Proposition 3.1 trivially shows that (d) implies (e). Proposition 3.2(a) and Corollary 2.4 show that (e) implies (f). That (f)$\rightarrow$(g) follows directly from Proposition 3.3. If (g) holds, then (h) must, since a limit point of any subnet is a cluster point of the original net. Finally, let (h) be valid and $\mathfrak{B}=\{F_\alpha\}_{\alpha\in A}$ be a class of closed subsets every finite collection of which has a non-void intersection; then $\mathfrak{B}$ is a filter base. Let $\mathfrak{x}(\mathfrak{B})$ be any net associated with $\mathfrak{B}$. By (h) there is some point x_0 such that $x_0\in F_\alpha$, for every $\alpha\in A$. Thus (b) is true and the proof is complete.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the fifteenth William Lowell Putnam Mathematical Competition held on March 5, 1955, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Everett C. Dade, David Mumford and Kenneth Wilson; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Marcus T. Grisaru, Barry M. Mitchell and Charles B. H. Watson; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Yale University, New Haven, Connecticut. The members of the team were Morton E. Harris, John S. Lew and John G. Thompson; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Kenyon College, Gambier, Ohio. The members of the team were Trevor Barker, Thomas Jenkins and Robert Mosher; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Trevor Barker, Kenyon College; Everett C. Dade, Harvard University; David Mumford, Harvard University; Howard C. Rumsey, California Institute of Technology; and Jack Towber, Brooklyn College. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are John S. Lew, Yale University; R. J. Parikh, Harvard University; Hugh N. Pendleton 3d, Carnegie Institute of Technology; William G. Strang, Massachusetts Institute of Technology; and Kenneth Wilson, Harvard University.

The following teams, named in alphabetical order, won honorable mention: Brooklyn College, Brooklyn, New York, the members of the team being Toby Spiselman, James Thurber and Jack Towber; California Institute of Technology, Pasadena, California, the members of the team being Norman W. Albright, David G. Cantor and Howard C. Rumsey; Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being David C. Larson,

Hugh N. Pendleton 3d and George B. Rybicki; and New York University, New York, New York, the members of the team being Donald Fredkin, Charles Kahane and (Mrs.) Tilla Klotz.

Ten individuals were given honorable mention. The names are listed in alphabetical order: David G. Cantor, California Institute of Technology; William Hanf, University of California at Berkeley; Barry M. Mitchell, University of Toronto; Paul Monsky, Swarthmore College; Paul Payette, Université de Montreal; Donald L. Reinken, Princeton University; John R. Stallings, University of Arkansas; Donald Stevens, Cornell University; John G. Thompson, Yale University; and Charles B. H. Watson, University of Toronto.

A total of 361 individuals from 70 institutions entered the Competition this year. Of this number 105 individuals and seven institutions were unable to compete, due to various reasons. Therefore, a total of 256 undergraduates from 63 institutions took part in the Competition.

The following is a list of all colleges and universities which entered teams in the Competition. The list is arranged in alphabetical order: Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, Arizona State College, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Columbia University, Cornell University, Harvard University, Iowa State College, Kent State University, Kenyon College, Knox College, Knoxville College, Lebanon Valley College, Lewis and Clark College, Massachusetts Institute of Technology, New York University, Polytechnic Institute of Brooklyn, Princeton University, Queens College (Flushing, N. Y.), Queen's University (Kingston, Ont.), Reed College, Ricks College, Rose Polytechnic Institute, Rutgers University, Stanford University, Swarthmore College, Syracuse University, The Cardinal Stritch College, The College of the City of New York, The College of the Holy Cross, The Cooper Union, The Ohio State University, The United States Naval Academy, University of Arizona, University of Arkansas, University of British Columbia, University of California (at Los Angeles), University of Florida, University of Manitoba, Université de Montreal, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Pennsylvania, University of Rochester, University of Santa Clara, University of Toronto, University of Washington, Ursinus College, Virginia Union University, Washington University (St. Louis, Mo.), Wayne University, and Yale University.

The following colleges and universities entered individual contestants only: Barry College, Bethany College, Blackburn College, Boston University, Haverford College, Sacramento State College, Saint Olaf College, State University of New York, College of Forestry (Syracuse, N. Y.), University of California (at Berkeley), University of Nebraska, University of Omaha, and University of Texas.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative

rankings of the first five will not be divulged) by writing to the Director of the Competition, Room 301 Merrill Hall, Kent State University, Kent, Ohio. These rankings may now be given to the individual contestants by their own departments of mathematics. Any other departments of mathematics may obtain the individual rankings of contestants for the purpose of selecting graduate students.

Those participating in the Competition were given the following lists of problems:

MORNING SESSION:

Omit one question.

1. Prove that there is no set of integers m, n, p except 0, 0, 0 for which $m+n\sqrt{2}+p\sqrt{3}=0$.
2. $A_1A_2 \cdots A_n$ is a regular polygon inscribed in a circle of radius r and center O. P is a point on line OA_1 extended beyond A_1. Show that

$$\prod_{i=1}^{n} \overline{PA_i} = \overline{OP}^n - r^n.$$

3. Suppose that $\sum_{i=1}^{\infty} x_i$ is a convergent series of positive terms which monotonically decrease (that is, $x_1 \geqq x_2 \geqq x_3 \geqq \cdots$). Let P denote the set of all numbers which are sums of some (finite or infinite) subseries of $\sum_{i=1}^{\infty} x_i$. Show that P is an interval if and only if

$$x_n = \sum_{i=n+1}^{\infty} x_i \qquad \text{for every integer } n.$$

4. On a circle n points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent (except at the endpoints), how many points of intersection are there?
5. If a parabola is given in the plane, find a geometric construction (ruler and compass) for the focus.
6. Find a necessary and sufficient condition on the positive integer n that the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

have a rational root.

7. Consider the function f defined by the differential equation

$$f''(x) = (x^3 + ax)f(x)$$

and the initial conditions $f(0)=1$, $f'(0)=0$. Prove that the roots of f are bounded above but unbounded below.

AFTERNOON SESSION:

Omit one question.

1. A sphere rolls along two intersecting straight lines. Find the locus of its center.
2. Suppose that f is a function with two continuous derivatives and $f(0)=0$. Prove that the function g, defined by $g(0)=f'(0)$, $g(x)=f(x)/x$ for $x\neq 0$, has a continuous derivative.
3. Prove that there exists no distance-preserving map of a spherical cap into the plane. (Distances on the sphere are to be measured along great circles on the surface.)
4. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?
5. Given an infinite sequence of 0's and 1's and a fixed integer k. Suppose that there are no more than k distinct blocks of k consecutive terms. Show that the sequence is eventually periodic. (For example, the sequence 1101101-0101 followed by alternating 0's and 1's indefinitely, which is periodic beginning with the fifth term.)
6. Prove: If $f(x)>0$ for all x and $f(x)\to 0$ as $x\to\infty$, then there exists at most a finite number of solutions of

$$f(m)+f(n)+f(p)=1$$

in positive integers m, n, and p.
7. Four forces acting on a body are in equilibrium. Prove that, if their lines of action are mutually skew, they are rulings of an hyperboloid.

Solutions of the Problems*

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The presentation here is intended as a brief sketch of the method of proof rather than as a model of a detailed proof such as is expected from the contestants.

PART I

1. By transposing and squaring $3p^2-m^2-2n^2=2mn\sqrt{2}$ and since $\sqrt{2}$ is irrational, $mn=0$. If $n=0$ then $-m=p\sqrt{3}$, and since $\sqrt{3}$ is irrational, $p=m=0$. If $m=0$ then $n\sqrt{2}+p\sqrt{3}=0$ and $2n=-p\sqrt{6}$, and since $\sqrt{6}$ is irrational, $p=n=0$.

2. Consider the plane of the polygon as the complex plane with O as zero and A_1 as r. Using the same notation for the complex coordinate of a point as for the point, we have

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

$$\prod_{i=1}^{n} \overline{AP}_i = \prod_{i=1}^{n} \left| P - r \times \exp(2\pi(i-1)/n) \right| = \left| \prod_{i=1}^{n} (P - r \times \exp(2\pi(i-1)/n)) \right|$$
$$= \left| P^n - r^n \right| = P^n - r^n.$$

3. If $x^k = 2p + \sum_{i=k+1}^{\infty} x_i$ with $p > 0$ and if $x_k - p = p + \sum_{i=k+1}^{\infty} x_i = \sum_{i=1}^{\infty} x_i\theta_i$ with each θ_i either zero or one, then $\theta_i = 0$, $i = 1, 2, \cdots, k$, by the monotoneity of the terms, but then $\sum_{i=1}^{\infty} x_i\theta_i \leqq \sum_{i=k+1}^{\infty} x_i = x_k - 2p$ gives a contradiction. Thus the condition is necessary.

If $x_n \leqq \sum_{i=n+1}^{\infty} x_i$, $n = 1, 2, \cdots$, and if $0 < p < \sum_{i=1}^{\infty} x_i$, take i_1 as the least positive integer n for which $x_n \leqq p$. Assuming inequality occurs, take j_1 as the largest positive integer n such that $S(i_1, j_1) \leqq p < S(i_1, j_1+1)$, where by $S(m, n)$ we mean the sum $\sum_{i=m}^{n} x_i$. Such a choice of j_1 is possible since $S(i_1, \infty) \geqq x_{i_1-1} > p$ if $i_1 > 1$ and $S(1, \infty) > p$ if $i_1 = 1$. If k blocks of consecutive terms have been chosen such that $i_h < j_h < i_{h+1} < j_{h+1}$, $h = 1, 2, \cdots, k-1$, and $0 \leqq p - \sum_{r=1}^{k} S(i_r, j_r) < x_{i_k+1}$ and if equality does not obtain in the last formula then take i_{k+1} as the smallest positive integer n such that $x_n \leqq p - \sum_{k=1}^{k} S(i_r, j_r)$ and j_{k+1} as the largest positive integer n such that $0 \leqq p - \sum_{r=1}^{k+1} S(i_r, j_r) < x_{j_{k+1}+1}$. The existence of i_{k+1} and j_{k+1} with the required properties follows as before. The process either terminates if an equality occurs or produces an infinite subseries each partial sum of which is less than p. Since any partial sum which terminates in the k-th group of terms exceeds $\sum_{r=1}^{k-1} S(i_r, j_r) > p - x_{j_{k-1}+1}$, the series converges to p.

4. Every subset of four distinct points of the given points determines an intersection point within the circle and conversely. Thus the number of intersections within the circle is ${}_nC_4$.

5. The midpoints of any two parallel chords determine a line L parallel to the axis. The axis is then constructed as the perpendicular bisector of a chord which is perpendicular to L. This determines the vertex V. A line through V making a 45° angle with the axis intersects the parabola in a point P. If Q is the orthogonal projection of P on the axis then the focus F is determined by $\overline{VQ} = 4\overline{VF}$.

6. Clearly n must be odd. If expanded, the leading term of the polynomial is x^n and the constant term is 2^{n+1}. Thus every rational root is in the set $\pm 2^k$, $k = 0, 1, \cdots, n+1$. The positive values are excluded since if x is positive $(x+2)^n > (x-2)^n$. Since $k=0$ makes all three terms odd, this case is also excluded. Substituting $x = -2^k$, $1 \leqq k \leqq n+1$, and dividing by 2^n we have $2^{n(k-1)} = (2^{k-1}+1)^n - (2^{k-1}-1)^n = 2\sum_{j=0}^{n-1} (2^{k-1}+1)^j(2^{k-1}-1)^{n-1-j}$. The sum on the right is clearly odd and thus $n(k-1) = 1$, and $n = 1$ and $k = 2$. Substitution verifies the sufficiency.

7. Let $f(x_0) = f'(x_0) = 0$. Then if $|x^3 + ax| \leqq M$ for $|x| = 1 + |x_0|$ and if $K = \max |f(x)|$ for $|x - x_0| \leqq 1/(1+\sqrt{M})$, then in this last interval $|f(x)|$

$=|f''(\theta x)|\,|x-x_0|^2/2$, with $0<\theta<1$. Taking x to maximize the left member in $|x-x_0|\leqq 1/(1+\sqrt{M})$ we have $K\leqq MK/2(1+\sqrt{M})^2<K/2$ and thus $K=0$, and f vanishes identically in an interval about x_0 of half-length $1/(1+\sqrt{M})$. The same argument may then be applied to the end point of this interval which is nearest the origin to extend the interval by the same amount. By finite induction, since the length of the interval does not decrease, $f(0)=0$, which is absurd. Thus f and f' do not vanish together and the zeroes of f are isolated. This results also from standard existence and uniqueness theorems of the theory of differential equations.

If consecutive zeroes of f occurred in a region where x^3+ax was positive, then f positive (negative) in this interval implies the graph of f is concave upward (downward) in the interval, and this is absurd. Thus the zeroes are bounded above.

Now suppose $x^3+ax<-1$ for $x\leqq x_0$ and suppose $f(x_0)=k>0$. Then f' must become positive somewhere to the left of x_0, for to deny this would require $f(x)\geqq k$ for $x<x_0$, and then $f'(x)>f'(x_0)+k(x_0-x)$ which contradicts the denial. But if $f'(x_1)>0$, $x_1<x_0$, then $f'(x)\geqq f'(x_1)$ for all $x<x_1$ until a zero of f is reached. Thus a zero of f exists to the left of x_0. A similar argument applies if $f(x_0)=k<0$, and thus f has an infinite number of negative zeroes.

Part II

1. Take the xy-plane as the plane of the intersecting lines and the positive x-axis as bisector of the angle formed by the rays in contact with the sphere. Let P: $(x, 0, z)$ be the center of the sphere and then Q: $(x, 0, 0)$ is the center of the circle in which the xy-plane cuts the sphere. The points of contact of the rays, T_1 and T_2, are distant from Q by $x \sin\theta$ where 2θ is the angle between the rays. From the right triangle PQT_1 we have $x^2\sin^2\theta+z^2=R^2$ with R the radius of the sphere. The complete locus is thus a pair of ellipses.

2. For $x\neq 0$, $g(x)=f(x)/x=f'(\theta x)$ with $0<\theta<1$, and since f' is continuous $\operatorname{Lim}_{x\to 0} g(x)=f'(0)=g(0)$. Thus g is continuous at the origin and so at all points. $\operatorname{Lim}_{x\to 0}\ [g(x)-g(0)]/x=\operatorname{Lim}_{x\to 0}\ [f(x)-xf'(0)]/x^2=\operatorname{Lim}_{x\to 0}\ [f'(x)-f'(0)]/2x$ $=f''(0)/2$ by L'Hospital's rule. For $x=0$, $g'(x)=[xf'(x)-f(x)]/x^2$ and by L'Hospital's rule the limit as $x\to 0$ is also found to be $f''(0)/2$. Thus g' is continuous at $x=0$ and at other points by standard theorems of calculus.

3. Under a distance preserving map the length of curves is unaltered. A small circle in the spherical cap with spherical distance $R\theta$ from the circumference to the center must transform into a circle in the plane of radius $R\theta$. The perimeter of the circle on the sphere is then $2\pi R\sin\theta$ and for the image circle is $2\pi R\theta$. Since $\sin\theta<\theta$ for $\theta>0$, the contradiction follows.

4. If $j+i$ is divisible by p_i^2, $i=1, 2, \cdots, k$, then so is $j+i+rP_k$ for any integer r and with $P_k=\prod_{i=1}^k p_i^2$. If p_{k+1} is relatively prime to P_k, then there are

integers s and t such that $tp_{k+1}^2+sP_k=1$. By squaring if necessary, we may assume $s>0$. Take a positive integer u such that $v=up_{k+1}^2-j-k-1>0$ and then $vsP_k+j+k+1=(u-vt)p_{k+1}^2$. Thus we see $j+i+vsP_k$ is divisible by p_i^2, $i=1, 2, \cdots, k+1$.

5. Let S_j denote the finite sequence $a_{j+1}, a_{j+2}, \cdots, a_{j+k}$ with a_n the n-th term of the given sequence. Let r be the least positive integer such that for some j, S_j and S_{j+r} are identical. Then the S_i are distinct for $i=j, j+1, \cdots, j+r-1$ and so $0<r\leqq k$. Assume $r<k$. The sequence of r digits $a_{j+1}, a_{j+2}, \cdots, a_{j+r}$, which we denote by R_j, is then identical with the sequence R_{j+r}. By finite induction it follows readily that $a_{j+h}=a_{j+h+r}$, $h=1, 2, \cdots, k$. The minimal property of r requires that the R_i be distinct for $i=j, j+1, \cdots, j+r-1$. Assume now that $a_{j+k+r+1}\neq a_{j+k+1}$. Some two of $S_{j+1}, S_{j+2}, \cdots, S_{j+k+r}$ are identical. But if S_{j+h} and S_{j+k+s} are in this set and identical, then s is a multiple of r, for otherwise the first r digits differ. But if $s=mr$ then $a_{j+k+r+1}$ is the $[k-(m-1)r-h+1]$-st digit of S_{j+h+mr} and the corresponding digit of S_{j+h} is $a_{j+k-(m-1)r+1}$ which is different. But this is a contradiction and thus $a_{j+h}=a_{j+h+r}$, $h=1, 2, \cdots, k+1$, so that the pattern persists. Since the same hypotheses now apply to S_{j+1} and S_{j+r+1}, the pattern must persist another step, and by induction $a_{j+h}=a_{j+h+r}$ for all positive integers h. If $r=k$ then $S_{j+1}, S_{j+2}, \cdots, S_{j+k}$ are distinct by the minimal property of $r=k$. S_{j+k+1} differs from S_{j+1} in the last place if $a_{j+2k+1}\neq a_{j+k+1}$, and differs from the others because of the minimal property of $r=k$. Thus the S_i, $i=j+1, \cdots, j+k+1$, are distinct, and the contradiction proves $a_{j+2k+1}=a_{j+k+1}$ and the proof follows as before.

6. Take i_1 such that $i>i_1$ implies $f(i)<1/3$. Let $k=\text{Max}\, f(i)$ for $i\leqq i_1$ and $f(i)<1$. If the set of such i is empty no solution exists. Otherwise $k<1$ and we take $i_2>i_1$ so that $i>i_2$ implies $f(i)<(1-k)/2$. Let $h=\max\,[f(i)+f(j)]$ $i\leqq i_2$, $j\leqq i_2$, and $f(i)+f(j)<1$. If no such pairs i and j exist then no solution exists. Otherwise $h<1$ and we take $i_3>i_2$ so that $i>i_3$ implies $f(i)<1-h$. Let m, n, p be any solution and we suppose $f(m)\geqq f(n)\geqq f(p)$. Then $f(m)\geqq 1/3$ and thus $m\leqq i_1$. Hence $f(n)+f(p)=1-f(m)\geqq 1-k$ and thus $f(n)\geqq(1-k)/2$ and $n\leqq i_2$. Finally $f(p)=1-f(m)-f(n)\geqq 1-h$ and so $p\leqq i_3$.

7. Let the lines of action of the forces be L_i, $i=1, 2, 3, 4$. Any line L which intersects three of these must also intersect the fourth since the moments about L must vanish. Let L_5, L_6, L_7 be distinct lines each of which intersect all four of the lines of action of the forces. Moreover L_5, L_6, and L_7 must be mutually skew and thus determine a regulus of lines, *i.e.*, all lines which intersect all three, and the lines of this regulus are rulings of an hyperboloid. For the construction of the equation of the hyperboloid and further discussion see Bell, Robert, *Coordinate Geometry of Three Dimensions*, Macmillan, 1954, pp. 163–165.

MATHEMATICAL CONSULTANTS, COMPUTATIONAL MATHEMATICS AND MATHEMATICAL ENGINEERING

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1. Introduction. This note, and a companion (which may eventually appear on *The teaching of concrete mathematics*) are after-effects of the Conference on Training in Applied Mathematics held in New York on October 22–24, 1953.* They represent an attempt to combine, in unknown proportions, some implicit conclusions of the conference, as I sensed them, and some very personal feelings which have developed slowly over a longer period. The present note discusses attitudes which mathematicians and mathematics departments might take toward (1) the training of mathematical consultants, (2) the place of computational mathematics, (3) the place and role of groups working in theoretical mechanics, and (4) the emergence of Departments of Mathematical Engineering and of training in computation engineering.

Before going on, a word about the word "engineering." Some think it is a bad word—they like to be pure. Some think it is a good word—they like to be engineers. The writer associates with both pure mathematicians and engineers regularly and by choice—he thinks of it as a neutral word, and he hopes that his readers will do the same, at least while reading through and mulling over this discussion.

2. Mathematical consultants. Probably the most important single mathematical function in industry and government is that of the mathematical consultant. Today there are few mathematical consultants, tomorrow there will not be very many, but their importance will far outweigh their numbers.

On their training the writer has strong views, and, he believes, strong support for them. First, and foremost, it should be a research training, most commonly in pure mathematics, but not infrequently in physics, a mathematized branch of engineering, or even in applied mathematics. Secondly, it should involve as wide a background as possible.

The basic requirements for a mathematical consultant are threefold. In addition to the research training and the broad mathematical background, the mathematical consultant needs, as do consultants based on any scientific field, *above all else* an interest in the other man's problems—in these problems as wholes, not just in their mathematical aspects.

The future consultant will be trained along the same lines as the future research mathematician, and will be distinguished from him, not by a lack of research ability (for he needs as much as they do), but rather by his broader interests. His training asks only of Departments of Mathematics that:

(1) they encourage their students to browse around, and

* Bull. A.M.S., vol. 60, 1954, pp. 38–44.

(2) they consider it reasonable that some of their best students become mathematical consultants.

For mathematical *consultants* we don't want mathematical engineers, we *do want mathematicians* (or theoretical physicists turned mathematicians, or electrical engineers turned mathematicians, *etc.*)!

3. The place of computational mathematics. The writer makes a sharp and important distinction between the mathematics and the engineering of computation. The mathematics of computation is concerned with what happens as a result of various computational steps or procedures. The engineering of computation is concerned with how best to choose steps or procedure to obtain the desired results with relatively small expenditures of effort or money.

Thus, for example, computational mathematics recognizes *one* interpolating polynomial through a given set of points, and notes in passing that values of this polynomial can be found by Aitken's method, by forward or backward Newton's formula, by Everett's formula, and by many other specific expressions. Computation engineering is concerned with the advantages and disadvantages of the various methods, and with the choice between them in particular situations.

Computational mathematics is a branch of mathematics like any other, and is logically taught by mathematicians. Today many places teach a little at the undergraduate level, mostly mixed with considerable computation engineering. Tomorrow, a few institutions will have graduate programs preparing a few Ph.D.'s in computational mathematics. These Ph.D.'s will need to meet the requirements of mathematical ability and background currently required of Ph.D.'s in pure mathematics, and many of them will go on to do research in computational mathematics as teachers in colleges and universities.

The responsibilities of mathematics departments toward computational mathematics are, then, to be met by:

(1) readiness, nay even eagerness, to teach computational mathematics at the undergraduate level to all who wish to learn (in all departments), and

(2) provision of needed research support, which is needed by both computation engineering and the further development of computational mathematics, through the training of a few good Ph.D.'s in computational mathematics. (The need exists now, and will grow rapidly. The Ph.D.'s should fully meet the standards of pure mathematics. A few institutions could meet the present needs.)

4. Theoretical mechanics. To allow for the mental habits of many readers, we shall use the term "theoretical mechanics" for what would be better called, in the writer's opinion. "(classical) physical mathematics," where the order of the words (just as in "physical chemistry" or in "mathematical physics") implies that the subject is *mathematics* which is oriented toward physics. Besides

all those things which are truly mechanics we include here electromagnetic theory and boundary value problems.

In view of the prevalence of the relation

applied mathematics $\equiv$ theoretical mechanics (modulo unimportant details)

in many minds, no discussion bearing on the applications of mathematics can avoid a discussion of theoretical mechanics sufficient for orientation. (In England the congruence of applied mathematics with theoretical mechanics seems to have been strengthened to an equality.) According to the recent NRC-AMS conference on applied mathematics in New York City, the recipe for theoretical mechanics in the United States seems to be:

> 2 parts mathematics, 1 part physics, 2 parts engineering, mix well and serve labelled "Applied Mathematics."

Why should this recipe be preferred?

The writer's estimate runs about as follows: At the present time, theoretical mechanics tends to be too routine for the science departments (physics and mathematics) and is almost surely too fundamental for the engineering departments (aeronautical, electrical, hydraulic, mechanical). Thus it has to be operated on an intermediate "applied science" level. In most institutions this is most easily done, if it is done at all, through a cooperative arrangement between science and engineering. This is quite likely to be a transient situation. If the engineering departments strengthen their fundamental side, as many of them indicate desires and plans to do, then theoretical mechanics may become a sophisticated branch of engineering. (After all, some engineers assert that, now the physicists have dived into the nucleus, they will have to take over not only theoretical mechanics, but experimental mechanics, optics, acoustics and the rest of classical physics. This seems unlikely in the long run.) Whether it will then be a branch of *mathematical* engineering or not is another matter, but if the ratio of mathematicians to physicists interested in theoretical mechanics continues to have its present high value, this will probably happen.

What, then, is the appropriate attitude of the mathematics department toward theoretical mechanics? As far as an undergraduate program is concerned, the answer is clear. It is best for the mathematics department to teach mathematics, for the physics department to teach physics and for the engineering departments to teach engineering. Unless there is an unusually strong base in the engineering department, a base so strong as to be extremely rare today, such a program is probably best guided by a joint committee. And the recipe: 2 parts mathematics, 1 part physics, 2–3 parts engineering, seems entirely natural for such a committee. This can be considered, without loss of generality for *internal* discussion among mathematicians, a program in mathematical engineering. (For discussion with other departments it will undoubtedly be necessary to emphasize the joint aspects.) As such it poses no new problems beyond those discussed in the next section.

At the graduate and postgraduate level, the situation is different. Here there

must be strong research interests in theoretical mechanics. These research interests are not (with rare exceptions) housed in engineering, in physics or in mathematics. There must be a house for them somewhere. Hence the development of graduate institutes in the area (*e.g.*, Brown, Indiana, Maryland). Toward such a group what should be the attitude of the mathematics department? Clearly those groups function, at those instants when they function mathematically, at a more fundamental level than, at present at least, any of the conventional engineering departments. Thus they are closer relatives. On the other hand, they are not in the immediate family. The appropriate relation would seem to be that of double first cousins—very close, but not quite inside the family. (Hence one must be doubly careful of the Persian proverb "To hate like cousins!")

The most obvious outward implications of such an attitude would seem to be these:

(1) encouragement of joint appointments whenever the appointee would fit into the mathematics department;
(2) acquiescence, for the present, in a title for such a group involving the words "applied mathematics" if that is what the group wishes,
(3) continuous, but gentle, education about the advantages to both parties of a title which expresses the character of the group more clearly and accurately.

In brief: support and cooperation but avoidance of blending.

5. Science, mathematics and engineering. Every mathematician acquainted with the history of United States colleges and universities which have had two mathematics departments, whether the second be called a department of applied mathematics or a department of engineering mathematics, fears such arrangements. For in almost every case, they have led to watering down the quality of mathematics and to ill feeling. There is a natural tendency to carry over this feeling to any administrative arrangement where all the "mathematics" is not in one mathematics department. This natural tendency may not always be wise, as we shall try to show.

The writer believes, indeed, that chemistry departments have gained rather than lost from the presence of chemical engineering departments and that physics departments have gained rather than lost from the existence of mechanical, electrical, and aeronautical engineering departments. He believes, furthermore, that most members of chemistry and physics departments would agree to this. Let us try to inquire into why this is so.

There was then, and surely continues today, a serious need for chemical, mechanical, electrical and aeronautical engineers. Three ways of meeting this need are conceivable (we shall state them in terms of physics alone for convenience):

(1) The physics department could have taught both physics and engineering.
(2) The physics department could have taught physics and the engineers could have taught engineering, or

(3) The engineers could have taught both physics and engineering.
Of these, (2) was generally selected. The writer submits that this was the best choice for the physics department. Why?

If (1) had been selected, what would have happened to physics departments? They would have faced the teaching of courses in alternating current machinery, bridge design and propeller theory, to name some examples. Such courses would have been boring to the permanent staff, but would have required too much background to be safely turned over to fresh Ph.D.'s. Either the morale and research potential of the permanent staff or the reputation of the training would have gone down, down, down.

If (3) had seen selected, what would have happened to physics departments? At first glance, the main penalty would have been a size and support penalty due to a loss of students. But actually the situation would have been far worse in the long run. The wholly engineer-taught engineer would in the long run have lost out on fundamental physics. And in the longer run, this would have been evident in comparison with the increasing demands placed on the engineer. As a consequence, the physics departments would have to shoulder the responsibility of providing more fundamentally trained engineers and the difficulties associated with (1) would not have been avoided.

Either (1) or (3) would have a fate worse than (2) *for physics departments*—and, incidentally, either would, of course, have provided engineers with poorer training. Perhaps this example has a moral for mathematics departments! This moral, if it exists, must be this: "When the demand for mathematical engineers begins, plan to meet it cooperatively—plan to teach the relevant mathematics in the mathematics department—plan to teach the mathematical engineering elsewhere." The writer believes that this moral exists and is important. He hopes that mathematics departments will preserve both their research potential and their student-semesters of classes by adopting this policy when it becomes timely.

6. What is mathematical engineering? Many readers may by now be muttering under their breaths, "But what *is* mathematical engineering?". To this question we have no complete answer, any more than Faraday could explain the uses of newly discovered electromagnetism. But we can and will give partial answers.

Mathematical engineering consists of those branches of engineering where the single most important tool is mathematics. This is the natural definition. Does it help us? Certainly it helps us a little, for it makes it clear that the engineering of computation is certainly a branch of mathematical engineering, for surely in planning computation—in planning mathematics—mathematics is the one most essential tool. In those parts of theoretical mechanics too, where the physics and the differential equations are long known and firmly settled, the most important single tool may well be mathematics, and, where it is, these parts of theoretical mechanics may prove to be mathematical engineering. (In the area now labelled industrial engineering, some further branches are probably

being shaped, but it would be premature to try to identify them here.)

First to be of importance is undoubtedly that which is first above—computation engineering. It is here that the first great needs will come.

7. Computation engineering. Today the need for mathematical engineers has begun to appear. The first need is for computation engineers! The arrival of the first batch of IBM type 701 high-speed calculators (formerly known as Defense Calculators, latterly as Electronic Data Processing Machines) is a substantial indication. The estimate of a staff of 30 persons per machine for economic balance is almost certainly conservative. But this means that over 500 trained persons are required for these machines alone.* What kind of persons?

Today the man with a 701 seeks out A.B.'s or M.A.'s in mathematics for his coders and lower level problem analysts, who will form 90% or more of his staff. The work he has for them is high-level routine and requires a solid grasp of a substantial amount of mathematics. Qualitatively, it seems to the writer entirely comparable to the work of an average mechanical or electrical engineer. As time goes on, are not the increasing numbers of such jobs going to be filled by mathematical engineers, whatever they may be called? How can it be otherwise?

Today the demand for computation engineers is definitely here. Tomorrow it will be larger. Today a few institutions will set up sources of supply. (How many? This is not clear.) Tomorrow, more and more will join them. (How fast? This is far from clear.) What should mathematics departments do, or be prepared to do?

If the moral drawn above is correct, then mathematics departments should, in the writer's judgment, act as follows:

(1) They should prepare to cooperate in the setting up of training for computation engineers under engineering auspices, and
(2) They should prepare to teach the necessary mathematics, *including the mathematics of computation*, in the mathematics department.

8. Summary. In the writer's opinion, then, Departments of Mathematics should:

(1) encourage at least some of their students to "graze around"
(2) look forward to a very few of their best students becoming mathematical consultants in industry or government
(3) stand ready to teach the mathematics of computation to all who wish to learn
(4) in a few cases, start training high-caliber Ph.D.'s in computational mathematics
(5) encourage joint appointments with theoretical mechanics groups whenever this is reasonable
(6) acquiesce, for the present, in a title of "applied mathematics" for such a group

* *Note added in proof.* In the year or year and a half since these words were first written, the predictable demand has increased by a *factor* of at least five, or more nearly ten!

(7) point out the advantages to both parties, gently but steadily, of a more appropriate title
(8) prepare to cooperate in the setting up of training in computation engineering under engineering auspices, and
(9) prepare to teach the necessary mathematics in the mathematics department.

Through such policies can Departments of Mathematics maintain and increase their strength and quality.

MATHEMATICAL NOTES

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NOTES ON MATRIX THEORY—VI

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1. Introduction. In two recent notes, [1], [2], we have shown how various results relating to the determinant of A, a positive definitive matrix, could be derived from the well-known identity

$$\frac{c_n}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dV_n \tag{1}$$

where the integration is over all x, and $c_n = \sqrt{\pi}^n$, a constant depending only upon the dimension n.

If we define, for $k = 1, 2, \cdots, n$,

$$|A|_k = \prod_{i=1}^{k} \lambda_i \tag{2}$$

where λ_i, $i = 1, 2, \cdots, n$, are the characteristic roots of A, a positive definite matrix, arranged in increasing order of magnitude, it was shown by Ky Fan, [3], [4], that

$$|A\lambda + B\mu|_k \geqq |A|_k^\lambda |B|_k^\mu, \qquad \lambda, \mu \geqq 0, \lambda + \mu = 1. \tag{3}$$

This result, together with some additional results, was recently obtained in a different fashion by Oppenheim, [5].

The purpose of the present note is to establish an identity for $|A|_k$ similar to (1). This identity may then be utilized to derive (3) in the same manner that

(1) was used to obtain the particular case $k=n$ of (3) (*cf.* [1]), namely by a direct application of Hölder's inequality.

The identity is:

THEOREM.

$$\frac{\sqrt{\pi}^k}{|A|^{\frac{1}{k}}} = \operatorname*{Max}_L \int_L e^{-(x,Ax)} dV_k \tag{4}$$

where L is a k-dimensional subspace of the n-dimensional space.

Proof. It is easy to see that we may begin with the case where A is already in diagonal form, $(x, Ax)=\lambda_1x_1^2+\lambda_2x_2^2+ \cdots +\lambda_nx_n^2$, since an orthogonal transformation leaves the formulation of the problem invariant. Hence we must show that

$$\frac{\sqrt{\pi}^k}{(\lambda_1\lambda_2 \cdots \lambda_k)^{1/2}} = \operatorname*{Max}_L \int_L e^{-\lambda_1x_1^2-\lambda_2x_2^2-\cdots-\lambda_nx_n^2} dV_k. \tag{5}$$

Define $V_\alpha(\rho)$ to be the volume contained in the k-dimensional region

$$\begin{aligned} \lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_nx_n^2 &\leqq \rho, \\ (x, \alpha_i) &= 0, \qquad i = 1, 2, \cdots, n-k. \end{aligned} \tag{6}$$

Then

$$V_\alpha(\rho) = \rho^{k/2}V_\alpha(1) \tag{7}$$

and we have

$$\int_L e^{-\lambda_1x_1^2-\lambda_2x_2^2-\cdots-\lambda_nx_n^2} dV_k = \int_0^\infty e^{-\rho}dV_\alpha(\rho) = \left[\frac{k}{2}\int_0^\infty e^{-\rho}\rho^{(1/2)k-1}d\rho\right]V_\alpha(1). \tag{8}$$

To complete the proof, we must show that the maximum of $V_\alpha(1)$ is attained when the relations $(x, \alpha_i)=0$ are $x_{k+1}=x_{k+2}= \cdots =x_n=0$. This, however, is an immediate consequence of the formula for the volume of an ellipsoid in terms of the characteristic roots of the associated symmetric matrix and the separation theorem for characteristic roots furnished by the min-max characterization of the characteristic roots.

References

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APPROXIMATION BY ENUMERABLE SETS

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Let $\{r_n\}$ be an enumerable set of real numbers and let $\{d_n\}$ be a sequence of positive real numbers. We say that a set E of real numbers is *approximated by* $\{r_n\}$ *within* $\{d_n\}$ if, whenever ξ is in E, the inequality $|\xi - r_n| < d_n$ holds for infinitely many n.

For example, let the rationals be enumerated in the sequence

$$0,\ 1,\ 1/2,\ 1/3,\ 2/3,\ 1/4,\ 2/4,\ 3/4,\ 1/5,\ 2/5,\ \cdots, \tag{1}$$

so that $p/q = r_n$ with

$$n = p + 2 + (q-1)(q-2)/2.$$

The theorem of Dirichlet, that $|\xi - p/q| < 1/q^2$ for infinitely many q, gives

$$|\xi - r_n| < 1/(2n) \tag{2}$$

for infinitely many n. If duplicates are omitted in (1) then we have, asymptotically,

$$|\xi - r_n| < 3/(\pi^2 n)$$

upon using the fact that the Euler ϕ function satisfies

$$\phi(1) + \cdots + \phi(n) \sim 3(n/\pi)^2.$$

Suppose one replaces the sequence of rationals $\{r_n\}$ by another enumerable sequence constructed expressly for the purpose of getting a good approximation. How good an approximation can be obtained? It is known that (2) is nearly optimum when $\{r_n\}$ is the rational sequence. How nearly is it optimum in this broader sense? Such are the questions with which the present discussion is concerned. It will be seen that a rather complete answer can be given by very elementary methods.

Direct approximation. With regard to approximation of the type just considered, we have the following:

Remark 1. If $\sum d_n = \infty$, then there is an enumerable set $\{r_n\}$ (depending on $\{d_n\}$ but not on E) such that every set E is approximated by $\{r_n\}$ within $\{d_n\}$.

For proof, it suffices to lay off adjacent intervals of lengths $d_0, d_1, d_2, \cdots$ on the real axis. We start at the origin, proceed to the right until the point $+1$ is passed, then turn to the left until -1 is passed, then to the right until $+2$ is passed, then -2, $+3$, -3, and so on. Since $\sum d_n = \infty$, the process is possible; and the end points of the intervals yield the sequence $\{r_n\}$.

That the condition $\sum d_n = \infty$ is necessary is shown by the following:

Remark 2. Suppose $\sum d_n < \infty$. If there is an enumerable set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$, then the measure of E is zero.

For proof, construct an interval of length $2d_n$ centered at r_n. If N is fixed,

each point of E is interior to one of these intervals, with $n>N$. Hence E is covered by a set of intervals whose measure does not exceed

$$2d_N + 2d_{N+1} + 2d_{N+2} + \cdots ;$$

and this is arbitrarily small.

This result shows that (2) is almost optimum; the set for which infinitely often

$$|\xi - r_n| < 1/n^{1+\epsilon}$$

with $\epsilon>0$ has measure zero. Alternatively, the set for which infinitely often

$$|\xi - p/q| < 1/q^{2+\epsilon}$$

has measure zero. Instead of $1/q^{2+\epsilon}$ one can use $f(q)/q$ where $\sum f(q) < \infty$. (Of course these well-known results can be obtained without use of the general theory.)

The following shows that one can deduce no more than $m(E)=0$, from the mere convergence of $\sum d_n$:

Remark 3. If E has measure zero, then there is an enumerable set $\{r_n\}$ and a sequence $\{d_n\}$ with $\sum d_n < \infty$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$.

For proof, cover E by a countable set of intervals $\{I_{n,1}\}$, then by a new set $\{I_{n,2}\}$, and so on. With $d_{n,k} = |I_{n,k}|$ we can carry out the construction in such a way that

$$\sum d_{n,k} < 1/2^k, \qquad k = 1, 2, \cdots .$$

The midpoints of these intervals yield an enumerable set, $\{r_{n,k}\}$; the associated intervals yield a sequence of lengths $\{d_{n,k}\}$. Since

$$\sum \sum d_{n,k} < 1,$$

the construction satisfies the required conditions. By enumerating $\{r_{n,k}\}$ suitably, we can make the corresponding sequence $\{d_{n,k}\}$ monotonic.

The preceding remark suggests the following:

Query 1. If $\{d_n\}$ is given with $\sum d_n < \infty$, is there a set E of measure zero which is not approximated within $\{d_n\}$ by any $\{r_n\}$?

Considering the Liouville numbers shows that the approximation can be arbitrarily good without our having E enumerable:

Remark 4. Given a sequence $\{d_n\}$ with $d_n>0$, there is a nonenumerable set E and an enumerable set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$.

This result suggests a question related to query 1:

Query 2. For each sequence $\{d_n\}$, $d_n>0$, suppose there is a corresponding set $\{r_n\}$ such that E is approximated by $\{r_n\}$ within $\{d_n\}$. Does it follow that E is enumerable?

Approximation in mean. Let $\delta(\xi)$ denote the distance from ξ to the nearest

point of a finite set $r_1, r_2, \cdots, r_n$. We write

$$D(E, r_i, n) = \int_E \delta(\xi) d\xi$$

as a measure of the accuracy with which E is approximated by the finite set $\{r_i\}$. If $\{r_i\}$ is infinite, we consider the behavior as $n \to \infty$. For simplicity, E is taken as the interval $[0, 1]$ henceforward.

Remark 5. We have $D(E, r_i, n) \geqq 1/(4n)$, and there is an enumerable set $\{r_n\}$ for which this bound is attained infinitely often as $n \to \infty$.

For proof, we observe that D is a continuous function of the r_i for fixed n, hence attains a minimum. A short calculation shows that the result is least, for three points, when the second is half-way between the others. Hence the minimum for n points is not attained unless they are equally spaced. Therefore it is attained when they are equally spaced. The spacing is easily found to be given by $r_1 = 1/2n$, $r_2 = 1/2n + 1/n$, $r_3 = 1/2n + 2/n$, $\cdots$, $r_n = 1 - 1/2n$. To construct an infinite set, we bisect the intervals in the set just described to obtain a new set of the same type, and so on.

From time to time nD is as large as $9/32$ in this construction, and we are led to the following:

Query 3. What enumerable set $\{r_n\}$ minimizes $\overline{\lim}\, nD(E, r_i, n)$, and what is the minimum value?

The foregoing result suggests that

$$\underline{\lim}\, nD(E, r_i, n)$$

can be used as a measure of the efficiency with which E is approximated by $\{r_i\}$.

Remark 6. If $\underline{\lim}\, nD(E, r_i, n) = 1/4$, then $\{r_i\}$ is uniformly distributed; but if $\underline{\lim}\, nD(E, r_i, n) > 1/4$, the set need not be uniformly distributed.

If $\{r_i\}$ is not uniformly distributed there is a subinterval (α, β) of $[0, 1]$ which does not obtain its proper quota of points; $N(\alpha, \beta) \leqq \theta n(\beta - \alpha)$, say, with $\theta < 1$. The integral is least when the points in (α, β) are equally spaced, and those in the remainder of $(0, 1)$ are also equally spaced. Even then we find that $\lim nD > 1/4$, as $n \to \infty$, and the result follows. The second part is proved by the same calculation.

If $\{r_n\}$ is dense we have $\lim D(E, r_i, n) = 0$, as is obvious. It is a curious fact that a uniformly distributed set is no better than a dense set:

Remark 7. If $\lim d_n = 0$, there is a uniformly distributed set (depending on $\{d_n\}$) such that $D(E, r_i, n) > d_n$ for all large n.

Without loss of generality we may suppose $\{d_n\}$ decreasing. Take $r_1 = r_2 = \cdots = r_n = 1/2$ with n so determined that $d_n < 1/12$. Then take $r_{n+1} = \cdots = r_m = 1/4$, $1/2$, $3/4$ so that the numbers of points at these three positions are equal within ± 1 for large m. This enumeration should take due account of the n points at $1/2$. We choose m so that $d_m < 1/28$. Similarly, distribute points

$r_{m+1} \cdots r_p$ at $1/8$, $2/8$, $3/8$, $\cdots$, as uniformly as possible, with due regard to those already present. We choose p large, and also such that $d_p < 1/60$. The process yields the desired set. By a slight modification one can avoid coincident points.

Note added in proof: According to Maurice Sion, a result of Morse and Beasley shows that the answer to Query 1 is affirmative; and a result of Besicovitch and Sierpínsky shows that the answer to Query 2 is negative, granted the continuum hypothesis.

A NOTE ON THE CIRCLE THEOREM IN HYDRODYNAMICS

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Milne-Thomson (see Proc. Cambridge Philos. Soc., vol. 36, 1940) found a certain function of a complex variable and showed that it had all the properties required for it to be the perturbation complex potential due to a circular cylinder placed at the origin in an infinite liquid in irrotational motion in two dimensions. The following is a method of obtaining the perturbation velocity potential in the above problem from simpler considerations.

Let $\phi(x, y)$ be the potential of a liquid which is in irrotational motion before the cylinder, $r=a$, is inserted. Then $\phi(x, y)$ is a harmonic function whose only singularities are outside $r=a$. Hence ϕ has an expansion given by $\phi = \sum_0^\infty r^n C_n$ valid in $r \leqq a$, where $C_n = p_n \cos n\theta + q_n \sin n\theta$, using (r, θ) as the plane polar co-ordinates. Hence

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \sum_0^\infty n a^{n-1} C_n.$$

The inverse point of (r, θ) with regard to the circle $r=a$ is (R, θ), where $Rr = a^2$. If $\psi(x, y)$ is the perturbation potential due to the cylinder, then ψ must satisfy the following conditions:

(a) ψ is a harmonic function in two dimensions, regular at every point on and outside of $r=a$.

(b) $\psi = 0$ at $r = \infty$.

(c) $\dfrac{\partial\psi}{\partial r} = -\dfrac{\partial\phi}{\partial r}$ on $r = a$.

Thus ψ may be expanded in the form, $\psi = \sum_0^\infty r^{-n} C_n'$, where $C_n' = p_n' \cos n\theta + q_n' \sin n\theta$.

Condition (c) gives $\sum_0^\infty -na^{-(n+1)} C_n' = \sum_0^\infty n a^{n-1} C_n$ for all θ in $0 \leqq \theta \leqq \pi$. So $p_n' = a^{2n} p_n$, $q_n' = a^{2n} q_n$, and $C_n' = a^{2n} C_n$. Hence $\psi(x, y) = \sum_0^\infty r^{-n} a^{2n} C_n = \sum_0^\infty R^n C_n = \phi(X, Y)$, where (X, Y) is the inverse point of (x, y), *i.e.*,

$$X = \frac{a^2 x}{r^2}, \qquad Y = \frac{a^2 y}{r^2}.$$

I wish to thank Dr. R. Mohanty for drawing my attention to the subject of this note.

CLASSROOM NOTES

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ON AN APPLICATION OF THE MEAN VALUE THEOREM

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In the study of the calculus, a number of important theorems are usually developed, but unfortunately their importance is not fully appreciated by beginning students. A useful application of the mean value theorem of the differential calculus is described below. The author has occasionally utilized this material with some success in indicating the importance and utility of this familiar theorem. Naturally all of the results in this paper are particular cases of Jensen's inequality, but the elementary way in which they may be derived apparently does not appear in the literature.

We establish first a lemma which is a direct consequence of the mean value theorem.

LEMMA. *If $a \leqq b \leqq c$, if $f'(x)$ is monotone increasing on the open interval $a < x < c$, and if $f(x)$ is continuous on the closed interval $a \leqq x \leqq c$, then*

$$(b - a)f(c) + (c - b)f(a) \geqq (c - a)f(b).$$

The inequality is reversed if $f'(x)$ is monotone decreasing. If further f' is strictly monotone and a, b, c are all distinct, equality is excluded.

Proof: Under the hypothesis the mean value theorem is valid so that we may write:

$$\frac{f(c) - f(b)}{c - b} = f'(\beta) \geqq f'(\alpha) = \frac{f(b) - f(a)}{b - a}\cdot$$

Upon performing the indicated algebra, the desired result follows.

It is surprising that such an elementary device should prove rather powerful. To indicate its versatility we consider several examples. Let $f(x) = \ln\,(1+x)$. Then f' is monotone decreasing for $x \geqq 0$. Choose $\alpha > 0$, and let 0, α/n, α/m with $0 < m < n$ correspond to a, b, c of our lemma. Then

$$\frac{\alpha}{n}\ln\left(1 + \frac{\alpha}{m}\right) < \frac{\alpha}{m}\ln\left(1 + \frac{\alpha}{n}\right),$$

and we secure the familiar inequality:

$$(1 + \alpha/m)^m < (1 + \alpha/n)^n, \qquad m < n.$$

As a second example choose $f(x) = a^x$. Then f' is monotone increasing provided a is positive and different from unity. Let 0, $r-1$, and r be chosen as the points with $r > 1$. Then we obtain

$$(r - 1)a^r + 1 > ra^{r-1}.$$

Letting $a = 1/x$ we obtain the more familiar relationship:

$$x^r - 1 > r(x - 1).$$

It is easy to show that these inequalities are reversed if $0 < r < 1$ but unchanged if $r < 0$.

For a third example, let $f(x) = |x|^{1+\alpha}$, and let $\pm a$, $(x+a)/2$, and x be the numbers where $|x| \neq |a|$. Applying the lemma, we have:

$$(x \mp a) \left| (x \pm a)/2 \right|^{1+\alpha} < [(x \mp a)/2] \left| x \right|^{1+\alpha} + [(x \mp a)/2] \left| a \right|^{1+\alpha},$$

or simplifying,

$$\left| x \pm a \right|^{1+\alpha} < 2^\alpha(\left| x \right|^{1+\alpha} + \left| a \right|^{1+\alpha}).$$

From symmetry it is unnecessary to consider the cases $x < a$ and $x > a$ separately. In passing we remark that this inequality can be used to show that if moments of order $1+\alpha$ exist, then moments of order $1+\alpha$ about the mean exist.

As a final example we apply our method to Liapounoff's famous moment inequality. Letting μ_x represent the absolute moment of order x, we seek to show that $\mu_b^{c-a} \leqq \mu_c^{b-a} \cdot \mu_a^{c-b}$. Choose $f(x) = \ln\left(\sum p_i x_i^x\right)$, $p_i > 0$, $x_i > 0$. Then

$$f''(x) = \left[\sum p_i x_i^x \sum p_i x_i^x \ln^2 x_i - \left(\sum p_i x_i^x \ln x_i \right)^2 \right] / \left(\sum p_i x_i^x \right)^2.$$

Choosing $a_i = \sqrt{p_i} x_i^{x/2}$ and $b_i = \sqrt{p_i} x_i^{x/2} \ln x_i$, we find that $f'' > 0$ by the Cauchy-Schwartz inequality provided the x_i's are not all equal. Letting a, b, c be an increasing sequence of numbers, we therefore obtain

$$(b - a) \ln \mu_c + (c - b) \ln \mu_a \geqq (c - a) \ln \mu_b,$$

with equality holding only when all the x's are equal or when equality holds between two or more of the letters. Taking exponentials of both sides we have therefore proved the theorem for the case that X is a random variable assuming the value x_i with probability p_i. The theorem can now be extended to the continuous case as in the reference.

Reference

Uspensky, J. V., Introduction to Mathematical Probability, p. 265.

UNIFORMLY CONTINUOUS LINEAR SETS

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In this note, E will always denote a linear set of points. For E both closed and bounded (compact), it is well known that every continuous, real-valued $f(x)$ defined on E is uniformly continuous on E. The converse of this theorem is not true. For if E is the set of all positive integers, every single-valued, real function $f(x)$ on E is uniformly continuous, but E is not compact.

DEFINITION 1: *A linear set E will be termed a U.C. set if every real-valued' continuous function $f(x)$ on E is uniformly continuous on E.*

It is the purpose of this note to characterize U.C. sets.

THEOREM 1. *Every U.C. set is a closed set.*

Proof: Let E be a U.C. set. If E is not closed, there is a point c which is a limit point for E, but which does not belong to E. Then, without loss of generality, there exists a sequence of points p_j in E such that $p_j < p_{j+1}$ and $\lim p_j = c$. Define:

$$f(x) \equiv \begin{cases} 1 & \text{for} \quad x = p_{2i-1} \\ 0 & \text{for} \quad x = p_{2i} \end{cases} \qquad i = 1, 2, \cdots$$

and linear between p_j and p_{j+1}, and let $f(x) = 1$ for $x > c$, and 1 for $x \leqq p_1$. It is easy to verify that $f(x)$ is continuous on E, but not uniformly continuous on E, contrary to E being a U.C. set.

DEFINITION 2. *A set E is termed uniformly isolated if there exists an $\epsilon > 0$, such that for every $x \neq y$ in E, $|x - y| \geqq \epsilon$.*

THEOREM 2. *Every set E expressible in the form $E_1 + E_2$, where E_1 is compact' and E_2 is uniformly isolated, is a U.C. set.*

Proof: Let $E = E_1 + E_2$, where E_1 is compact and E_2 is uniformly isolated. Let $A = E_2 - E_1$. Then A is uniformly isolated and hence closed, and $E = E_1 + A$. Let $f(x)$ be continuous on E. Then $f(x)$ is uniformly continuous on both E_1 and A. But since A is closed, and E_1 is compact, and A and E_1 are disjoint, A and E_1 are a positive distance apart. It is now clear that $f(x)$ is uniformly continuous on $E_1 + A$ and hence uniformly continuous on E. Therefore E is a U.C. set. Conversely, we have

THEOREM 3. *Every U.C. set E is expressible in the form $E_1 + E_2$, where E_1 is compact and E_2 is uniformly isolated.*

Proof: Let E be any U.C. set. Let $I_j \equiv \{x \mid -j \leqq x \leqq j\}$, $j = 1, 2, \cdots$. Let $E_+ \equiv \{x \mid x \epsilon E \text{ and } x \geqq 0\}$; and $E_- \equiv \{x \mid x \epsilon E \text{ and } x \leqq 0\}$. There exists a j' such that $E_+ - I_{j'}$ is uniformly isolated. For if $E_+ - I_j$ is uniformly isolated for no j, we can construct a sequence of points p_j in E such that $\lim p_j = +\infty$ and $p_j < p_{j+1}$ and

$p_{2i}-p_{2i-1}<1/i$. Then, as in Theorem 1, we can construct a continuous function $f(x)$ on E that is not uniformly continuous, contradicting E being a U.C. set.

Similarly, there is a j'' such that $E_{-}-I_{j''}$ is uniformly isolated. Now, let J be any integer greater than both j' and j''. By Theorem 1, E is closed and therefore $E\cdot I_J$ is closed and bounded, and hence compact. It follows from above that $E-I_J$ is uniformly isolated and $E=E\cdot I_J+(E-I_J)$. This proves the theorem.

The sum of two U.C. sets may not be a U.C. set. For if E_1 is the set $\{1, 2, \cdots, n, \cdots\}$ and E_2 is the set $\{1-1, 2-\frac{1}{2}, \cdots, n-1/n, \cdots\}$, then E_1 and E_2 are uniformly isolated and hence U.C. sets. However, E_1+E_2 is not a U.C. set as may be easily seen.

THEOREM 4. *Every closed subset of a U.C. set is a U.C. set.*

Proof: Let E be a U.C. set, and E_* a closed subset of it. By Theorem 3, $E=E_1+E_2$, where E_1 is compact and E_2 is uniformly isolated. Then $E_*=E_*\cdot E_1+E_*\cdot E_2$ and $E_*\cdot E_1$ is compact since E_* is closed, and $E_*\cdot E_2$ is uniformly isolated. Therefore, E_* is a U.C.set by Theorem 2.

COROLLARY. *The intersection of any family of U.C. sets is a U.C. set.*

Proof: This follows from Theorem 1 and Theorem 4.

BOUNDEDNESS OF A CONTINUOUS FUNCTION

M. D. MARCUS, University of British Columbia

Let f be a continuous real valued function and let M be a compact subset of the domain of f. It is classical that f is bounded on M and assumes its bounds. However, the following elementary proof using the Heine-Borel property may be of interest.

Let n be a positive integer and $I_n=(-n+f(x_0),\ n+f(x_0))$ for some $x_0\in M$. Then $\bigcup_{n=1}^{\infty} f^{-1}(I_n)$ is clearly an open covering of M. Select a finite subcovering

$$\bigcup_{j=1}^{K} f^{-1}(I_{n_j}) = f^{-1}(I_{\max n_j}) \supset M.$$

Then $-\max n_j \leqq f(x)-f(x_0) \leqq \max n_j$ and $f(x)$ is bounded. Now let $m=\mathrm{lub}_{x\in M} f(x)$ and $J_n=(-\infty,\ m-1/n)$. Then if m is not assumed by f, $\bigcup_{n=1}^{\infty} f^{-1}(J_n)$ is an open covering of M and again select a finite subcovering

$$\bigcup_{j=1}^{K} f^{-1}(J_{n_j}) = f^{-1}(J_{\max n_j}) \supset M.$$

Then $f(x) \leqq m-1/\max n_j$, in contradiction to the definition of m.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1181. *Proposed by F. L. Wolf, Carleton College*

Show that if an integer has all its prime factors greater than 5 then it divides some integer whose decimal representation is a string of 1's.

E 1182. *Proposed by I. A. Barnett, University of Cincinnati*

If $d_k = u^k - x^k - y^k - z^k$, for $k = 1, 2, 3, \cdots$, establish the identity

$$d_{n+2} - (x + y + z)d_{n+1} + (xy + yz + zx)d_n - (xyz)d_{n-1} = u^{n-1}(u - x)(u - y)(u - z).$$

E 1183. *Proposed by F. J. Duarte, Caracas, Venezuela*

Prove that $\sum_{n=1}^{\infty}(n-1)/n! = 1$.

E 1184. *Proposed by Viktors Linis, University of Ottawa*

Let $f(x)$ be differentiable at $x = a$ and let $f(a) \neq 0$. Evaluate

$$\lim_{n\to\infty} [f(a + 1/n)/f(a)]^n.$$

E 1185. *Proposed by Frederic Cunningham, Jr., University of New Hampshire*

Prove that every infinite sequence of real numbers contains an infinite monotone subsequence.

SOLUTIONS

The Paint Paradox

E 1151 [1955, 121]. *Proposed by C. S. Ogilvy, Hamilton College*

Resolve the following paradox. The area between the curve $y = 1/x$ and the x-axis, to the right of the line $x = 1$, is infinite. Yet the volume generated by rotating this area about the x-axis is π. Thus it would require an infinite quantity of paint to cover the area; yet the volume, which completely contains and sur-

rounds the area, could be filled with π cubic units of paint!

Editorial Note. The general conclusions arrived at seem to be the following:

(1) If paint is molecular, then the paradox is explained by the fact that one can neither paint the area nor fill the volume after y becomes less than the molecular dimensions.

(2) If paint is continuous, then the paradox is explained by the fact that one needs but a finite amount of paint either to paint the surface or to fill the volume.

(3) If paint is mathematical, so that the paint on a painted surface has no thickness, then the paradox is explained by the fact that one cannot compare an area with a volume.

Contributions by R. V. Andree's Calculus Class, Julian Braun, M. L. Clinnick, Richard Courter, Hüseyin Demir, Fred Discepoli, J. M. Elkin, B. A. Fusaro, N. G. Gunderson, R. R. Gutzman, Vern Hoggatt, A. R. Hyde, E. S. Keeping, M. S. Klamkin, D. C. B. Marsh, J. V. Pennington, C. F. Pinzka, Anatol Rapoport, L. A. Ringenberg, Max Rosenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., Nina Schub, W. L. Shepherd, Lawrence Shepp, D. D. Strebe, D. R. Sudborough, J. A. Tierney, W. R. Van Voorhis, R. M. Warten, A. F. Wilson, R. J. Wisner, and the proposer.

Inconsistency of a Certain Postulate Set

E 1152 [1955, 122]. *Proposed by J. J. Bowers, Student, Wesleyan University*

If, in the postulates of a field, the distributivity of multiplication over addition is replaced by distributivity of addition over multiplication, is the resulting set of postulates consistent?

I. *Solution by C. J. Vanderlin, University of Wisconsin.* We shall denote by a *Bowers field*, or a *B-field*, a system $(F, +, \cdot)$ consisting of a non-empty set F together with two binary operations, $+$ and $\cdot$, such that:

(i) $(F, +)$ is an abelian group, with identity 0.

(ii) $(F^*, \cdot)$ is a group, with identity 1, where F^* is the set $F-0$.

(iii) $ab=ba$ for all $a, b \epsilon F$.

(iv) $a+bc=(a+b)(a+c)$ for all $a, b, c \epsilon F$.

As is customary, we shall denote the inverse of an element $a \epsilon F$ with respect to $+$ by $-a$, and the inverse of an element $b \epsilon F^*$ with respect to $\cdot$ by b^{-1}.

Lemma 1: Let $(F, +, \cdot)$ be a B-field. If $a \epsilon F$ and $a+1 \neq 0$, then $a=0$.

We have, by (iv),

$$a + 1 = a + 1\cdot 1 = (a + 1)(a + 1).$$

Since $a+1 \epsilon F$ and $a+1 \neq 0$, then $a+1 \epsilon F^*$ and $(a+1)^{-1}$ exists. Therefore

$$(a + 1)^{-1}(a + 1) = (a + 1)^{-1}(a + 1)(a + 1),$$

or

$$1 = 1\cdot(a + 1) = a + 1,$$

whence

$$1 - 1 = (a + 1) - 1 = a + (1 - 1),$$

or

$$0 = a.$$

Lemma 2: If $(F, +, \cdot)$ is a B-field, then the group $(F, +)$ consists of the elements 0 and 1 with $0+0=1+1=0$, $0+1=1+0=1$.

If $a\epsilon F^*$, then $a\neq 0$; so, by Lemma 1, $a+1=0$. But $1\epsilon F^*$; so $1+1=0$. Therefore $a=1$, and 1 is the only element in F^*, and F consists of just the elements 0 and 1. Finally, $0+0=0$, $0+1=1+0=1$, since 0 is the identity of $(F, +)$.

Theorem: There exist no B-fields.

Let $(F, +, \cdot)$ be a B-field. Then

$$\begin{aligned} 1 + 1\cdot 0 &= (1 + 1)(1 + 0) \quad \text{(by (iv))} \\ &= 0\cdot 1 \quad \text{(by Lemma 2)} \\ &= 1\cdot 0, \quad \text{(by (iii))} \end{aligned}$$

whence

$$1 = 0,$$

which is absurd. Therefore, there can exist no B-field.

It follows that the above postulate set is inconsistent.

Remark: If one allows the trivial system $(F, +, \cdot)$ with $F=\{0\}$, then this system satisfies the postulates for a B-field if (ii) is thought of as being vacuously satisfied. However, as is customary in field theory, we have assumed that F^* is a bona-fide group , that is, that it consists of a non-empty set.

II. *Addendum by C. N. Campopiano, Sperry Gyroscope Co., Great Neck, L. I.* Weakening the above postulate set for a B-field by deleting the commutivity postulate (iii), we find that the system $(F, +, \cdot)$, where $F=\{0, 1\}$ and $0+0=1+1=0$, $0+1=1+0=1$, $0\cdot 0=0\cdot 1=0$, $1\cdot 1=1\cdot 0=1$, satisfies the weakened set. Therefore, although the original set is inconsistent, the weakened set is not.

Also solved by W. J. Blundon, G. U. Brauer, Richard Courter, A. S. Gregory, J. D. Haggard, Douglas Holdridge, D. C. B. Marsh, C. S. Ogilvy, L. A. Ringenberg, Azriel Rosenfeld, H. M. Starr, D. R. Sudborough, and R. J. Wisner.

Editorial Note: It was impossible to check most of these solutions because the solver failed to indicate just what postulate set for a field was being employed. For example, many solvers used the fact that for all a, $a\cdot 0=0$. In the postulate

set for a field used above, this is a theorem whose proof depends upon the discarded distributivity law. As a matter of fact, the problem itself is at fault from this point of view; the proposer should have stated the postulate set for a field that he wished altered.

Trisection of Angles Arbitrarily Close to a Given Angle

E 1153 [1955, 122]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For any angle θ, show that arbitrarily small constructible angles ϕ exist such that $(\theta-\phi)$ can be trisected.

Solution by C. S. Ogilvy, Hamilton College. Let $\phi_n=\theta/4^n$, $n=1, 2, \cdots$. For all n, ϕ_n is certainly constructible. Obviously n can be selected so that $\phi_n<\epsilon$ for any ϵ. But

$$(\theta - \phi_n)/3 = \phi_1 + \phi_2 + \cdots + \phi_n.$$

Also solved by N. G. Gunderson, A. R. Hyde, J. V. Pennington, C. M. Sandwick, Sr., M. R. Spiegel, and the proposer.

A Property of the Regular Heptagon

E 1154 [1955, 122]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The distance from the midpoint of side AB of a regular convex heptagon $ABCDEFG$, inscribed in a circle, to the midpoint of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle.

I. *Solution by Leon Bankoff, Los Angeles, Calif.* Let d be the required distance, R the circumradius, and let $\theta=\pi/7$. By the cosine law,

$$d^2 = R^2(1/4 + \cos^2\theta - \cos\theta\cos 2\theta).$$

Since $\sin 3\theta=\sin 4\theta$, it follows that

$$3\sin\theta - 4\sin^3\theta = 4\sin\theta\cos\theta\cos 2\theta,$$

which reduces to

$$\cos^2\theta = \cos\theta\cos 2\theta = 1/4.$$

Hence $d=R\sqrt{2}/2$.

II. *Solution by Hüseyin Demir, Zonguldak, Turkey.* Let the vertices F, G, $A, \cdots, E$ be the affixes of the 7th roots $1, e, e^2, \cdots, e^6$ of unity. Then the midpoints U, V of AB and the concerned radius correspond to $u=(e^2+e^3)/2$ and $v=-\frac{1}{2}$, whence

$$\begin{aligned} UV^2 &= (u-v)(\bar{u}-\bar{v}) = (1+e^2+e^3)(1+e^5+e^4)/4 \\ &= (2+1+e+\cdots+e^6)/4 = 1/2, \end{aligned}$$

and the property is established.

Extending our method to diagonals, we may state the following: *The midpoints of the sides of the hexagon ABGDCEA are equidistant from the point V, the common distance being half the side of the inscribed square.*

Also solved by W. B. Carver, A. L. Epstein, Vern Hoggatt, A. R. Hyde, Edgar Karst, Josef Langr, D. C. B. Marsh, Walter Penney, J. V. Pennington, L. A. Ringenberg, C. M. Sandwick, Sr., E. P. Starke, Chih-yi Wang, and the proposer.

The Proposer mentioned the following additional properties of the regular heptagon $ABCDEFG$. Let O be the center of the heptagon, W the midpoint of OF, M the point diametrically opposite F, U the midpoint of AB, V the midpoint of OM, and J the point on UB produced such that $UJ=UM$. Then:

(1) UW is equal to the diagonal of the square constructed on an apothem of the heptagon as a side.

(2) OJ is equal to the diagonal of the square constructed on half the side of the inscribed equilateral triangle.

(3) UV is tangent to the circle through U, V, W.

One cannot help but wonder if these properties, and the property of the problem, are just remarkable geometric accidents, or are they special cases of more general theorems involving, perhaps, other regular polygons. Karst did some work on the regular inscribed nonagon $ABCDEFGHI$. Here, if U is the midpoint of side AB and V is the midpoint of the radius perpendicular to and cutting side BC, it can be shown that $\measuredangle OUV=30°$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4653. *Proposed by Albert Wilansky, Lehigh University*

Consider the three statements:

(a) $\sum U_n(x)$ converges uniformly, (b) $\sum |U_n(x)|$ converges,

(c) $\sum |U_n(x)|$ converges uniformly.

Consideration of $U_n(x)=(-x)^n/n$ shows that if we require (a) and (b) to

hold on an open interval (in this case $0<x<1$) we cannot deduce (c).

Does (c) follow if (a) and (b) hold on a closed interval?

4654. *Proposed by Harley Flanders, University of California at Berkeley*

Let $X=\{1, 2, 3, \cdots\}$ be the set of whole numbers, S the class of all subsets of X. Prove that there is a subclass T of S such that (a) if $Y_1, \cdots, Y_n$ are arbitrarily chosen distinct members of T, then there is at least one integer x which belongs to exactly an odd number of the Y_j, and (b) if Y is a non-empty member of S, then there are distinct sets $Y_1, \cdots, Y_n$ in T such that Y is the set of all integers x which belong to exactly an odd number of the Y_j.

4655. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Solve the difference equation

$$F(n+1) = F(n) - 2(n+1)\text{sg}\{F(n) + 3n - n^2\}, \qquad F(0) = 0,$$

where sg $(x)=1, 0, -1$ according as $x>0$, $x=0$, $x<0$.

4656. *Proposed by H. S. Shapiro, New York University*

(1) If α is a root of unity and its real part is an algebraic integer, then $\alpha^4=1$.

(2) Show that there exists an algebraic integer of absolute value 1 which is not a root of unity.

4657. *Proposed by Peter Swerling, Los Angeles, California*

Consider sets of real numbers which are subsets of the closed unit interval, which is denoted by I.

(1) Show that there exists no (Lebesgue) measurable set $S\subset I$ with the properties (i) $m(S)=M>0$ and (ii) there exists a number Δ, $0<\Delta<1$, such that for any interval $J\subset I$, $m(S\cap J)\leqq\Delta m(J)$. Here m denotes Lebesgue measure.

(1′) (Corollary) Given any measurable set $S\subset I$ such that $m(S)>0$ and given any number Δ, $0<\Delta<1$, there exists an interval $J\subset I$ such that $m(S\cap J)>\Delta m(J)$.

(1″) (Corollary) There is no measurable set $S\subset I$ with $m(S)=M>0$ such that S is "uniformly distributed" in I, in the sense that for every interval $J\subset I$, $m(S\cap J)=Mm(J)$.

(2) Show that statement (1′) cannot be improved in the following sense: one cannot say that for every measurable set $S\subset I$ with $m(S)>0$, there must exist a non-degenerate interval $J\subset I$ such that $m(S\cap J)=m(J)$.

(3) Given $S\subset I$; $m(S)>0$; and given Δ, $0<\Delta<1$, prove that S can be represented as the union of two disjoint sets: $S=S_1\cup S_2$ where (i) $m(S_1)=0$ and (ii) S_2 can be covered by a denumerable collection of disjoint non-degenerate intervals $\{I_n\}:S_2\subset\bigcup_n I_n$ such that for each I_n, $m(S\cap I_n)>\Delta m(I_n)$.

SOLUTIONS

Centroids of Parallel Convex Sets

4586 [1954, 264]. *Proposed by Robert Steinberg and F. A. Valentine, University of California at Los Angeles*

Consider a bounded closed convex set S_0 in the Euclidean plane. Let S_r be the parallel convex set to S_0, that is, the set of all points whose minimum distance from S_0 is at most r. Designate the centroid of S_r by g_r.

Show that the set of points $g_r(0 \leqq r < \infty)$ is a point or an arc of an hyperbola (possibly degenerate) lying in S_0. Does g_∞ have a simple geometric relationship with S_0?

Solution by the Proposers. Select a rectangular coordinate system (x_1, x_2), let $\overline{S}_0$ be the boundary of S_0, and let ϕ be the angle which the outward normal $N(\phi)$ to $\overline{S}_0$ makes with the x_1-axis. A point p in $S_r - S_0$ corresponds to a pair (r_1, ϕ), where r_1 is the distance of p from $N(\phi) \cdot \overline{S}_0$. Then the area of the ring-like set $S_r - S_0$, denoted by A_r is

$$A_r = \int_0^{2\pi} \int_{\rho_0}^{\rho_0 + r} \rho d\rho d\phi = \int_0^{2\pi} \int_0^r (r_1 + \rho_0) dr_1 d\phi = \pi r^2 + L_0 r,$$

where $\rho_0 = \rho_0(\phi)$ is the radius of curvature of $\overline{S}_0$ at $N(\phi) \cdot \overline{S}_0$, L_0 is the length of $\overline{S}_0$, and $\rho_0 d\phi = ds$ on $\overline{S}_0$.

In a corresponding manner, the moment of $S_r - S_0$ about the x_2-axis can be shown to be

$$\int_0^{2\pi} \int_{\rho_0}^{\rho_0+r} (x_1 + r_1 \cos \phi) \rho d\rho d\phi = \int_0^{2\pi} \int_0^r (x_1 + r_1 \cos \phi)(\rho_0 + r_1) dr_1 d\phi$$
$$= r \int_0^{L_0} x_1 ds + \frac{r^2}{2} \int_0^{2\pi} x_1 d\phi,$$

where (x_1, x_2) is the point $N(\phi) \cdot \overline{S}_0$. A corresponding result holds for the x_1-axis. Hence, if we let A_0 designate the area of S_0, then the x_i-coordinate of g_r is

$$\bar{x}_i = \frac{\iint_{S_0} x_i dA + r \int_0^{L_0} x_i ds + \frac{r^2}{2} \int_0^{2\pi} x_i d\phi}{A_0 + rL_0 + \pi r^2}, \qquad (i = 1, 2). \tag{1}$$

If $\overline{S}_0$ is a circle, clearly g_r is always its center. Otherwise, the isoperimetric inequality $(L_0^2 - 4\pi A_0 > 0)$ implies that equations (1), with r as parameter, represent an hyperbola, a straight line or a point. Moreover, equations (1) imply

$$\min_{x_i \in S_0} x_i \leqq \bar{x}_i \leqq \max_{x_i \in S_0} x_i, \qquad (i = 1, 2).$$

Hence $g_r \in S_0$.

If S_0 has interior points, by choosing (0, 0) on $\bar{S}_0$, and by choosing x_1 along a normal to $\bar{S}_0$, it is easy to see that g_r is in the interior of S_0. Equations (1) also yield the fact

$$\lim_{r\to\infty} \bar{x}_i = \frac{1}{2\pi}\int_0^{2\pi} x_i d\phi.$$

This describes $\lim g_r$ in terms of S_0. It is interesting to observe that the set of all g_r is a point if and only if, relative to g_r as origin, we have

$$(2) \qquad \iint_{S_0} x_i dA = 0, \qquad \int_0^{L_0} x_i ds = 0, \qquad \int_0^{2\pi} x_i d\phi = 0.$$

The question naturally arises: What are the sets S_0 satisfying equations (2)?

An Impossible Property of Solutions of a Differential Equation

4591 [1954, 350]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(z)$ be analytic. For $a=1$, the equation $f'(ax)+f(x)=0$ $(-\infty<x<\infty)$ has the property that its solution, $f(x)=e^{-x}$, approaches zero as $x\to\infty$. Is this true for any real $a>1$?

Solution by Paul Erdös, University of Notre Dame, and Melvin Henriksen, Purdue University. An elementary computation with power series shows that the only non-zero solutions of $f(x)+f'(ax)=0$ are necessarily analytic and are given by

$$f(x) = A\sum_{n=0}^{\infty}(-1)^n(n!)^{-1}a^{-n(n-1)/2}x^n, \qquad A \neq 0.$$

It is easily seen that if $a>1, f(x)$ is an entire function of order zero. By a theorem of Wiman (see Titchmarsh, *Theory of Functions*, p. 274), there exists a sequence x_n, approaching infinity, such that $f(x_n)\to\infty$. (Such a sequence could be explicitly constructed in this special case.)

Note: The trivial exception $f(x)=0$ is not explicitly excluded in the statement of the problem.

Also solved by I. N. Baker, D. S. Greenstein, Edgar Reich, O. E. Stanaitis, and the Proposer.

Dirichlet Series

4592 [1954, 350]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the sum

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}\log r}{r}.$$

I. *Solution by H. F. Sandham, Dublin Institute for Advanced Studies, Ireland.* Write

$$f(n) = \sum_{r=1}^{n} \frac{\log r}{r} - \frac{1}{2}\log^2 n,$$

then from Cauchy's integral test it follows that $f(n)$ tends to a limit as $n\to\infty$: hence

$$f(2n) - f(n) \to 0, \qquad n \to \infty.$$

Now, since $\log 2r = \log 2 + \log r$,

$$\sum_{r=1}^{n} \frac{\log r}{r} = 2\sum_{r=1}^{n} \frac{\log 2r}{2r} - \log 2 \sum_{r=1}^{n} \frac{1}{r}:$$

hence

$$f(2n) - f(n) = \sum_{r=1}^{2n} \frac{(-1)^{r+1}\log r}{r} + \log 2\left(\sum_{r=1}^{n}\frac{1}{r} - \log n\right) - \frac{1}{2}\log^2 2.$$

Thus, letting n tend to infinity, we have

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log r}{r} = \frac{1}{2}\log^2 2 - \gamma \log 2,$$

where γ denotes Euler's constant.

II. *Solution by M. R. Spiegel, Rensselaer Polytechnic Institute.* Choose $a>0$ and consider

$$\begin{aligned}
\int_0^\infty \frac{\log x}{e^{ax}+1}dx &= \sum_{r=1}^{\infty}(-1)^{r+1}\int_0^\infty e^{-rax}\log x\,dx \\
&= \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{ra}\int_0^\infty e^{-u}\log(u/ra)du \\
&= \left\{\int_0^\infty e^{-u}\log u\,du\right\}\left\{\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{ra}\right\} - \sum_{r=1}^{\infty}\frac{(-1)^{r+1}\log ra}{ra}.
\end{aligned}$$

Since

$$\int_0^\infty e^{-u}\log u\,du = \Gamma'(1) = -\gamma,$$

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{ra} = \frac{\log 2}{a},$$

$$\int_0^\infty \frac{\log x}{e^{ax}+1}dx = -\frac{1}{2a}\log 2\log 2a^2,$$

(see solution to problem 4394, [1951, 705]) we obtain

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} \log ra}{ra} = \frac{1}{2a} \log 2 \log 2a^2 - \frac{\gamma \log 2}{a} \cdot$$

The present problem is the special case $a=1$.

Also solved by P. T. Bateman, Ranko Bojanić, W. E. Briggs, L. Carlitz, S. Chowla, Oscar Goldman, S. W. Golomb, Norman Greenspan, J. W. Lawson, D. E. Russell, O. E. Stanaitis (two solutions), Arnold Walfisz, C. B. Walton, Chih-yi Wang, R. E. Wild, and the Proposer.

Editorial Note. In a note on *The Power Series Coefficients of* $\zeta(s)$, Briggs and Chowla obtain, among other results, formulas for the coefficients A_n in the expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} A_n(s-1)^n.$$

See this MONTHLY, May 1955, pp. 323–325.

Residue Systems

4593 [1954, 427]. *Proposed by S. W. Golomb, Harvard University*

Let $0, a_1, a_2, \cdots, a_{p-1}$ be any complete residue system modulo the odd prime p. Show that

$$(1) \qquad 0, a_1, 2a_2, \cdots, (p-1)a_{p-1}$$

is never a complete residue system modulo p.

Show further that if any non-zero residue r is specified, the a_i's can be so chosen that every residue except r occurs in (1). What relation will the residue s occurring twice bear to r?

Solution by D. H. and Emma Lehmer, Berkeley, California. It is clear that the zero in the problem may be dispensed with by replacing "complete system of residues" by "the set S of numbers less than p." Now if corresponding elements of two permutations of S are multiplied together and reduced modulo p the resulting set is not a permutation of S since, by Wilson's theorem, the product of the members of S is congruent (mod m) to -1.

For the second part of the problem we may take the set defined by

$$a_k \equiv \begin{cases} 1 & \text{if } k = p - r, \\ 1 + r\bar{k} & \text{otherwise,} \end{cases} \qquad k\bar{k} \equiv 1 \pmod{p}.$$

Since $r \not\equiv 0$, it is clear that this set of a's is a complete set of residues prime to p. Moreover

$$ka_k \equiv \begin{cases} p - r & \text{if } k = p - r \\ k + r & \text{otherwise} \end{cases} \pmod{p}$$

is also such a set except that r is missing and $s=p-r$ occurs twice, once for $k=p-r$ and again for $k=p-2r$.

Also solved by Kurt Bing, W. J. Blundon, L. Carlitz, S. H. Gould, D. S. Greenstein, Virginia S. Hanly, B. A. Hausmann, W. H. Jones, D. C. B. Marsh, R. R. Phelps, and the Proposer.

Zero of a Complex Function

4594 [1954, 427]. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z)=z$ whenever $|z|=1$, show that $f(z)$ has at least one zero in $|z|<1$.

I. *Solution by A. R. Hyde, West Hartford, Connecticut.* If $-1<x_1<1$, from the definition of $f(z)\equiv u+iv$, $u=x_1$ at a pair of points on the circle $|z|=1$, symmetric with respect to the axis of reals. Because of the continuity of $f(z)$ there is at least one continuous path which connects these points and lies within the circle and along which $u=x_1$. Correspondingly for $v=y_1$ (where the point pair on the circle are symmetric with respect to the axis of imaginaries). The paths $u=x_1$ and $v=y_1$ must intersect (within the circle) provided $|z_1|\equiv|x_1+iy_1|<1$, and at such point or points of intersection $f(z_1)=x_1+iy_1=z_1$. Thus at least once within $|z|=1$, $f(z)$ assumes every value assumed by z, including $f(z)=0$, which occurs at the intersection of $u=0$ (connecting $z=\pm i$) and $v=0$ (connecting $z=\pm 1$).

II. *Solution by S. K. B. Stein, University of California at Davis.* If $f(z)=0$ has no solution in $|z|<1$, then the continuous map $z\rightarrow f(z)/|f(z)|$ provides a retraction of the disk into its boundary. For any dimension this is known to be impossible. See Hurewicz and Wallman, *Dimension Theory*, Proposition B, p. 40.

III. *Solution by Albert Wilansky, Lehigh University.* Let

$$g(z) = \begin{cases} z & \text{for } |z| \geqq 1, \\ f(z) & \text{for } |z| < 1. \end{cases}$$

Then g is continuous for all z and $g(z)/z\rightarrow 1$ as $z\rightarrow\infty$. By the result of Problem 4475 [1953, 271], g has a zero. It has none on or outside the unit circle, hence has one inside.

Also solved by J. L. Brown, Jr., H. J. Cohen, A. J. Goldman, C. D. Gorman, R. H. Kasriel, G. J. Kleinhesselink, Norman Miller, L. A. Ringenberg, Azriel Rosenfeld, Michael Skalskyj, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent to E. P. Vance, Oberlin College, Oberlin, Ohio

An Introduction to Statistics. By C. E. Clark, New York, John Wiley and Sons, 1953. x+266, pages. $4.25.

This text is written for students with a minimal mathematical background, presupposing less than a mastery of high school algebra. Yet, unlike many such texts, the concern here is primarily with statistical inference and only those descriptive statistics most frequently used in statistical inference are introduced. Thus measures of central tendency and dispersion other than the mean and standard deviation are omitted entirely, as are index numbers, time series analysis, measures of skewness and kurtosis, methods of machine computation, *etc*. Whether or not one believes, as this reviewer does, that the inclusion of such topics in an introductory course tends only to obscure the more important ideas of estimation and tests of hypotheses, it is clear that the author's intention is to introduce statistical inference to the mathematically immature student, and it is with respect to this intention that the book must be evaluated.

The first four chapters are devoted to an introduction to the concept of statistical inference, permutations and combinations, probability, and frequency and probability distributions. The exposition in this section, and throughout the book, is clear and well organized and is supplemented by a carefully chosen set of exercises, half of which are solved for the reader in an appendix.

In the chapter on probability a distinction is made between three kinds of probability—empirical, statistical, and *a priori*—and an attempt is made to maintain this distinction throughout the text, but it seems to this reviewer that the use of a single definition would have been preferable. What the author defines as empirical probability could better be called relative frequency or sample proportion, the term used in later chapters in which the problem of the estimation of the unknown parameter of a binomial population is considered. The concept of *a priori* probability, as defined by the author, is applicable only to idealized games of chance which can be analyzed in terms of a finite set of equally likely outcomes. We are left, then, with statistical probability, defined as a subjective expectation of the frequency of occurrence of an event in future trials. This reviewer prefers a single definition similar to that used by Feller in his *Probability Theory and its Applications*, with the author's empirical probability treated as a sample estimate, his *a priori* probability treated as a special case of "fair" coins and dice, and his statistical probability treated as descriptive of the kinds of probability statements made in problems of estimation by confidence limits and tests of hypotheses.

The text follows a pattern of heuristic discussion, formal statement of a theorem, and formal proof whenever possible within the limits set by the low

level of mathematical training required of the reader. When a rigorous proof is not presented, a theorem is made plausible by unusually clear informal arguments. This mathematical format serves well to emphasize the important results obtained: a section is summarized in a brief and usually very careful statement in the form of a theorem. However, this technique of emphasis is occasionally misused. For example, the author chooses to define a frequency or a grouped probability distribution to be a normal distribution if the probabilities of a pair of tables hold. These tables give areas under the normal curve between ordinates symmetrically placed about the mean. (The normal curve is pictured only once, above a table of normal distribution areas at the end of the book.) A frequency distribution is defined to be "roughly normal" if the probabilities of these tables hold approximately. Then the following is stated as a formal theorem (4.25.3): "Almost all the numbers of a roughly normal frequency distribution differ from the mean by less than 3 times the standard deviation." It seems to this reviewer that to make this a formal theorem is to give a spurious validity to a statement which is in fact almost devoid of content.

Problems of statistical inference are discussed in four chapters titled The Reliability of Sample Means and Proportions, The Significance of the Difference between Two Sample Means or Percentages, The Analysis of Variance, and Inferences from Chi-Square. Primary emphasis is placed upon the estimation of population parameters by means of confidence limits. Since the concept of the null hypothesis is first mentioned in the chapter on the chi square test, tests of hypotheses concerning the mean are limited to statements of the risk (one minus the confidence) with which one can infer from a random sample that the population mean is less than (or greater than) a particular value. This treatment of tests of hypotheses makes difficult the introduction of the concept of the power of a test, and in fact the possibility of error of the second kind, acceptance of a false hypothesis, is never treated. Tests of hypotheses concerning a binomial population are also handled by the use of confidence limits based upon the normal approximation to the binomial. Here a serious error is introduced in the text by the estimate of the standard error using the sample proportion rather than the population proportion under the null hypothesis. However this problem is handled correctly and clearly in several of the exercises.

The analysis of variance is treated briefly in a short chapter which describes only the case of a single variable of classification. The exposition is clear and the usual notational complications are kept to a pleasant minimum, but it seems unfortunate even in so brief a treatment that no justification is given for using as a statistic the ratio of the between means and within samples estimates of variance and that no mention is made of the conditions under which the analysis of variance is valid.

After a short chapter on the chi-square test, the author turns in the last chapter from statistical inference back to the description and analysis of empirical data by means of correlation and linear regression. That these should be handled only as descriptive statistics in a text largely devoted to statistical inference

seems regrettable. In fact, problems of correct inference certainly are raised by statements such as that on page 204: "A value of $r(x, y)$ between .6 and .7 or $-.6$ and $-.7$ indicates a significant connection." Either this only defines significant connection or else an inference to a population parameter is being made, with no mention of sample size, upon which the confidence with which an inference can be made depends.

J. A. DUDMAN
Reed College

Partial Differential Equations in Engineering Problems. By K. S. Miller. New York, Prentice Hall, Inc., 1953. viii+254 pages. $4.75.

According to the author's introduction, this book is directed to upper classmen or beginning graduate engineering students. It deals with the elementary aspects of the theory of partial differential equations of mathematical physics without pretense of rigor, and presupposes an elementary knowledge of ordinary differential equations.

Chapter I contains the derivation of some of the equations in "typical engineering fashion." It is concluded with a three page list of equations. In Chapter II Fourier series are discussed. Except for partial statement (without proof) of two finer points (Section 26 on Points of Discontinuity, and Section 40 on Mathematical Conditions on Fourier Series Expansions) this discussion consists essentially of derivation of formulae for Fourier coefficients, and of computation of various examples, spread over 40 pages. A natural continuation of the topic is found in Chapter V on Legendre, Bessel, and Mathiew functions. The defining equations for these functions are obtained in the process of applying the method of separation of variables to equations involving the Laplacian in spherical, cylindrical, and elliptical coordinates. Orthogonality relations are obtained, and the theory of representation of functions as linear superpositions over a complete orthonormal set is explained. Chapter V ends with a brief survey of the Sturm-Liouville Theory. More material of preparatory character is found in Chapter IV on the Fourier Integral.

The discussion of partial differential equations in this book is of rather limited scope. Chapter III on Separation of Variables contains the discussion of four boundary value or mixed boundary value-initial value problems for second order linear equations with constant coefficients in two variables leading to Fourier series expansions; an example of a fourth order equation in two variables (transverse vibrations of a beam); and finally, a discussion of the rectangular vibrating membrane without consideration of the possibility of degeneracy of eigenvalues. Some nonhomogeneous equations are also considered and the method of reduction of a problem with nonhomogeneous equation to a problem with homogeneous equation but with nonhomogeneous boundary conditions is explained. In Chapter IV there are two sections in which Fourier transforms are used to solve the Cauchy problem for the heat and the wave equation. In Chapter V there are three sections devoted to solutions by the method of separation of variables

in which generalized Fourier series are used. Lastly, in the concluding Chapter VI the reduction to normal form of second order equations in two variables with constant coefficients is carried out.

The book is written in a clear, unhurried style, and there is no doubt that the author wanted the reader to understand everything without effort. Yet one cannot escape the feeling that this book constitutes a literal printing of unpolished lecture notes. Uniformity of presentation is lacking. For example, in setting up problems, in some cases units are used (as if they were of importance) while in other cases they are not used, and certain definitions are entirely forgotten (compare Sections 2, 4, 5, and 7). Similar comments may be made concerning what is considered as the prerequisite for the reader. For example, it is assumed that the reader may not know the mean value theorem of the differential calculus, and half of page 2 is given to the proof assuming the knowledge of Taylor series. On page 187 it is shown that $\lim_{x\to\infty} x^a e^{-x}=0$, while on page 190 the author assumes that the method of Frobenius is known. Matter having no bearing on the subject is sometimes considered. For example, there is in Chapter V a four page section on the gamma function, but the function itself is used only in connection with expansions for Bessel functions (pages 202 to 205) where it appears with integral arguments. Likewise, in connection with the study of Bessel functions, functions of negative index are considered.

The author made an attempt to be specific and precise, but often misses the point. In setting up problems, he carefully lists all the assumptions (although no distinction is made between assumptions of physical nature and mathematical simplifications). Then on page 4 we find:

"Assumption 2. The deflections u are small compared with the length L of the string.

Assumption 3. The slope at any point of the deflected string is small compared to unit."

as if the former were not implied by the latter. Incidentally, this particular list of assumptions is followed by a comment that they are reasonable and consistent and then one reads a few lines below: "An example of a pair of inconsistent assumptions would be to assume that the displacements u are large compared with the length of the string, but that the horizontal displacements are small."

One usually expects a textbook on a mathematical subject to be mathematically palatable. Yet we find in this one a whole spectrum of sins from loose language to outright errors. In Chapter I the use of such undefined phrases as "elementary," "infinitesimal," "in the first approximation" is commonplace. On page 29 a too crude differentiation of integrals is punished by obtaining the Laplace equation in place of the Poisson equation. From Chapter II the student may get the impression that only periodic functions can be expanded in Fourier series, while in Chapter IV he is told that Fourier transforms are considered to take care of the aperiodic case. In Section 49 the notion "nonhomogeneous boundary conditions" does not agree with the accepted meaning of this phrase.

In Chapter V, the study of the Legendre equation is started on the wrong foot (by deriving the indicial equation in a regular case as the Legendre equation is at $x=0$) and is ended likewise.

Our main criticism concerns, however, the presentation of material. One gets the feeling that by remaining on a too elementary level, the issues in question are only confused. Would it not be better to start the Chapter on Fourier series with a discussion of trigonometric approximations (as in Section 27), and after a few applications to Fourier series, to proceed to generalized Fourier series? This approach, while clarifying the purpose of the subject, would considerably shorten the presentation of preparatory material, and would allow the author to expand on the main subject. As it is, the selection of material is unbalanced and is certainly inadequate for a student who may not have a chance to return to the subject.

I. I. KOLODNER
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New York University

Mathematical Thinking in the Social Sciences. Edited by Paul F. Lazarsfeld. Glencoe, Illinois, The Free Press, 1954. 444 pages. $10.00.

Decision Processes. Edited by R. M. Thrall, C. H. Coombs and R. L. Davis. New York, John Wiley and Sons, Inc., 1954. viii+332 pages. $5.00.

Problems in economics, psychology, or sociology do not yet generally leap to mind when applications of mathematics are under discussion. Nevertheless, there is a growing awareness, among both mathematicians and social scientists, that mathematical thinking and methods are important in these fields. The volumes under review should make significant contributions to this growth. Each is a collection of articles concerned with the development of mathematical theories and techniques, with applications of this mathematics to social science problems, or with methodological questions of how mathematical models may best be used by social scientists.

Mathematical Thinking in the Social Sciences contains 8 long papers in addition to an introductory essay by the editor. T. W. Anderson defines an attitude as a choice from among a set of responses to a question and then describes a Markov chain probability model for analyzing changes in people's attitudes over time. This analysis is then applied to data obtained in a panel survey of voters. Nicholas Rashevsky, with his work in mathematical biology as background, derives differential equations for studying the behavior of large social groups, especially imitative and mob behavior. He also develops a theory of social status distributions in human populations. The third paper, by James C. Coleman, is an analysis of four of Rashevsky's social behavior models. Eliminating mathematical details, he compares these models in order to explain the interplay between their formal mathematical and their empirical sociological content.

The nature and role of the probability concept in social science are reviewed by Jacob Marschak. He discusses subjective probabilities, the statistical character of descriptive social science, and the use of probability models in forming policy decisions. The problems of aggregation and identification in economics are outlined and relations of these with the sociologist's latent structure and the psychologist's factor analysis are indicated.

In the first of two articles, Louis Guttman interprets the characteristic vectors of a certain matrix as so-called principal components of scalable social attitudes. This theory of scale analysis is a remarkable example of productive interplay between mathematics and social science. The author refers elsewhere for the mathematical details and limits himself to the psychological theory and its relation to the mathematical formalism. In his second article, Guttman uses quantitative data from mental testing as illustrative material and develops a new general theory for linking test scores and the factors studied by these tests. This radex theory extends the classical factor analysis approaches of Spearman and Thurstone, using multivariate statistical analysis as its main mathematical tool.

The central problem considered in the paper by Paul F. Lazarsfeld is how to analyze a set of qualitative responses to test questions or observed items of behavior (the so-called manifest data) in order to yield measurements of underlying characteristics (the so-called latent structure of the subject group). In this presentation of latent structure analysis we find discussions of the notion of probability as used in the theory, of response patterns, and of the "accounting equations", by which inferences are made from the manifest data to the latent structure.

The final contribution, by Herbert A. Simon, expresses the need for translating into the language of mathematics some of the central laws in social-psychological theory and for considering many complementary models for phenomena to be studied. Two main examples of model building are given, one a study of motivation and learning processes and the other an examination of equilibrium states in a system of social interaction.

Decision Processes is a volume on the proceedings of a 1952 summer seminar which brought together a group of mathematicians, psychologists, economists, and philosophers. After two introductory articles, one a discussion of the papers to follow and their interrelationships, the other an exposition of the nature of mathematical models with special emphasis on models in measurement theory, the remaining 17 papers are divided into four subject matter groups I. Individual and Social Choice, II. Learning Theory, III. Theory and Applications of Utility, IV. Experimental Studies. In general, all of these papers, ranging in character from pure mathematics to experiments in group dynamics, deal with one aspect or another of research into procedures and rationale for making decisions or into how decisions are actually made. All told, one obtains that variety, in both subject matter and approach, necessary to present a fair sample of the recent work in the field of decision problems.

Questions may be raised as to the advisability of presenting, as both volumes do, so uneven a product in terms of prerequisites needed for successful reading. We believe this method of presentation is advisable. It enables the editors not only to point out the rich possibilities of mathematical social science, but at the same time to emphasize the fact that mathematician and social scientist need to work together to continue to explore these possibilities.

SAMUEL GOLDBERG
Oberlin College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

OFFICERS OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 20–24, 1955, at Pennsylvania State University. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor Haim Reingold, Illinois Institute of Technology; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Professor C. O. Oakley, Haverford College. Dr. R. S. Burington, Bureau of Ordnance, Navy Department, and Dr. C. V. Newsom, Executive Vice-Chancellor, New York University, will continue to serve as Directors. Dr. Burington was elected Council Member of ASEE. The next annual meeting of the ASEE Mathematics Division will be held in June, 1956, at Iowa State College. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

PERSONAL ITEMS

Professor C. A. Hutchinson of the University of Colorado represented the Association at the dedication of the United States Air Force Academy in Denver, Colorado, on July 11, 1955.

Assistant Professor B. F. Bryant of Vanderbilt University is on leave of absence for 1955–1956 on a Ford Foundation Faculty Fellowship and is at Princeton University.

Georgetown University announces: Dr. Anne Scheerer of Washington University has been appointed to an assistant professorship; Mr. L. N. Zaccaro of Temple University has been appointed to an instructorship.

Kansas State College reports the following: Dean R. W. Babcock of the School of Arts and Sciences retired from the deanship July 1, 1955, after twenty-five years of such service but will continue as Professor of Mathematics; Professor R. G. Sanger has been appointed Acting Dean; Mr. J. J. Harris and Miss Dorothy Powell have been appointed to instructorships.

Los Angeles City College announces that the fifth annual William B. Orange Mathematics Prize Competition for Los Angeles high school students was held May 13, 1955, with 171 students from 38 schools participating.

Assistant Professor N. E. Albrecht of Hamline University has a position as a mathematician with Remington-Rand Corporation, St. Paul, Minnesota.

Associate Professor C. E. Amos of the University of Toledo has been promoted to a professorship.

Dr. E. L. Arnoff, recently a research associate at Case Institute of Technology, has been appointed to an assistant professorship at the Institute.

Mr. F. S. Badger, dean of students at Alliance College, has been appointed to an associate professorship at Allegheny College.

Dr. W. E. Barnes, previously head of Ballistics and Statistical Theory Branch, U. S. Naval Proving Ground, Dahlgren, Virginia, has a position as a mathematical consultant at General Electric Company, Schenectady, New York.

Associate Professor Max Beberman of Florida State University has been appointed Associate Professor of Education at the University of Illinois.

Dr. Barbara J. Beechler of the State University of Iowa has been appointed to an assistant professorship at Wilson College, Chambersburg, Pennsylvania.

Associate Professor J. H. Blau of Antioch College is Acting Chairman of the Department of Mathematics until September 1956.

Dr. S. R. Bodner, formerly a research assistant at the Polytechnic Institute of Brooklyn, has a position as a senior mathematician at Republic Aviation Corporation, Farmingdale, New York.

Mr. H. A. Bott, previously a student at Illinois Institute of Technology, is employed as a research engineer at Minneapolis-Honeywell, Chicago, Illinois.

Dr. J. W. Brace of the University of Maryland has been promoted to an assistant professorship.

Assistant Professor F. R. Brown of Illinois State Normal University has been promoted to an associate professorship.

Professor W. F. Cheney, Jr., of the University of Connecticut has retired from his position at the University and is Head of the Department of Mathematics of Hillyer College.

Associate Professor R. N. Cobb of Worcester Polytechnic Institute has been promoted to a professorship.

Mr. S. H. Cohn, formerly an instructor at Fournier Institute of Technology, is now in the Aerodynamics Computing Centre, Avro Aircraft Limited, Malton, Ontario, Canada.

Associate Professor V. F. Cowling of the University of Kentucky has been promoted to the position of Professor of Mathematics and Astronomy.

Mr. D. W. Crowe, previously a research assistant at the University of Michigan, has been appointed to an instructorship at the University of British Columbia.

Mr. R. S. Dick, recently a student at Queens College, New York, is a University Fellow in Mathematical Statistics at Columbia University.

Assistant Professor M. H. M. Esser of Georgia Institute of Technology has been appointed to an instructorship at the University of Maryland.

Mr. A. R. Friedenheit, formerly a mathematician at Douglas Aircraft Corporation, Santa Monica, California, is employed now as a mathematician-programmer at the Burroughs Corporation, Paoli, Pennsylvania.

Dr. R. F. Gabriel of Rutgers University has been promoted to an assistant professorship.

Assistant Professor David Gans of New York University has been promoted to an associate professorship.

Dr. E. F. Gillette of Harpur College has been appointed to an assistant professorship at Union College.

Assistant Professor Samuel Goldberg of Oberlin College has been promoted to an associate professorship.

Dr. J. K. Goldhaber of Cornell University has been appointed to an assistant professorship at Washington University.

Associate Professor A. W. Goodman of the University of Kentucky has been promoted to a professorship.

Dr. R. T. Gregory, formerly an assistant at the University of Illinois, has been appointed to an assistant professorship at the University of California, Santa Barbara College.

Assistant Professor Dilla Hall of Southern Illinois University has been promoted to an associate professorship.

Associate Professor J. R. Hammond of the United States Naval Academy has been promoted to a professorship.

Mr. C. L. Hassell, Jr., previously a supervisor of the Mathematical Operations Section, Sandia Corporation, Albuquerque, New Mexico, has a position as a research mathematician at the Continental Oil Company, Ponca City, Oklahoma.

Assistant Professor F. V. Higgins of Fenn College has been promoted to an associate professorship.

Mr. H. L. Hunzeker, instructor at DePauw University, has been promoted to an assistant professorship.

Dr. D. A. Kearns of the University of Maine has been promoted to an assistant professorship.

Mr. J. D. E. Konhauser, recently an instructor at Pennsylvania State University, has a position as a mathematician at Haller, Raymond and Brown, Inc., State College, Pennsylvania.

Mr. E. L. Kretschmar, Jr., formerly a mathematics teacher at Pasco High

School, Dade City, Florida, has been appointed Assistant Principal of Zephyrhills Public Schools, Florida.

Dr. E. B. Leach of Massachusetts Institute of Technology has been appointed to an assistant professorship at Case Institute of Technology.

Professor Jean Leray of the Institute for Advanced Study and College de France lectured in the Department of Mathematics of the Institute of Technology, University of Minnesota, on topics in partial differential equations, September 9–25, 1955.

Mr. R. L. Liboff, formerly a physicist at the Army Chemical Center, Maryland, is now a junior research physicist at Stevens Institute of Technology.

Mr. K. L. Loewen, previously an instructor at Freeman Junior College, South Dakota, has been appointed to an assistantship at Pennsylvania State University.

Mr. J. N. Mangnall, associate mathematician at Cornell Aeronautical Laboratory, Buffalo, New York, has been appointed to an instructorship at the University of Buffalo.

Mr. G. J. Marks, recently a test engineer with Curtiss-Wright Corporation, Woodridge, New Jersey, is employed now as a technical engineer in the Jet Engine Department of General Electric Company, Evendale, Ohio.

Mr. R. L. May, previously a technical assistant at Socony-Vacuum Oil Company, New York City, is now a naval architect for California Texas Oil Company, New York City.

Professor C. T. McCormick of Illinois State Normal University has been appointed Head of the Department of Mathematics.

Assistant Professor S. W. McInnis of the University of Florida has been promoted to an associate professorship.

Assistant Professor E. K. McLachlan of Baylor University has been promoted to an associate professorship.

Mr. V. A. Miculka, chairman of the Department of Mathematics of Frank Phillips College, has been appointed to an instructorship at Texas Western College.

Dr. Knox Millsaps, chief of the Applied Mathematics Research Branch of the Wright Air Development Center, Wright-Patterson Air Force Base, Dayton, Ohio, is Visiting Professor at Massachusetts Institute of Technology for the academic year 1955–1956.

Dr. Marian A. Moore, chairman of the Department of Mathematics of Glenbrook High School, Northbrook, Illinois, has been appointed to an assistant professorship at Southern Illinois University.

Mr. W. E. Moore, previously chief of the Data Reduction Section of Redstone Arsenal, has a position as a senior computer engineer in the Engineering Division of Republic Aviation Corporation, Farmingdale, New York.

Dr. Z. I. Mossesson, chief actuarial assistant at Prudential Insurance Company of America, Newark, New Jersey, has been promoted to the position of associate actuarial director.

Mr. Herbert Nadler, recently in military service, has been appointed to the position of actuary with the Fire Department Pension Fund, New York City.

Mr. Hiram Paley, formerly a student at the University of Rochester, is now a National Science Foundation Fellow at the University of Wisconsin.

Dr. B. J. Pearson, previously a teaching fellow at the University of Texas, has been appointed to an instructorship at Illinois Institute of Technology.

Dr. L. L. Philipson of the Lockheed Aircraft Corporation has accepted a position as research engineer with the Hughes Research and Development Laboratories, Culver City, California.

Miss Lois J. Roper, formerly instructor at Trenton Junior College and Senior High School, is teaching at St. Joseph Junior College, Missouri.

Associate Professor E. H. Rothe of the University of Michigan has been promoted to a professorship.

Mr. M. J. Sendrow of the Department of Defense, Washington, D. C., has a position as an engineer at the R. C. A. Engineering Products Division, Camden, New Jersey.

Assistant Professor R. L. Shively of Western Reserve University has been appointed Oppenheim Professor and Head of the Department of Mathematics of Manchester College.

Assistant Professor E. C. Smith, Jr., of the University of Utah has a position as Applied Science Representative with the I. B. M. Corporation, San Francisco, California.

Dr. O. K. Smith of Princeton University has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Dr. J. K. Sterrett has accepted a position with the Office of Naval Research, Operations Research Division, Navy Department, Washington, D. C.

Professor W. R. Van Voorhis of Fenn College has been appointed Chairman of the Department of Mathematics.

Mr. W. D. Ward, previously a student at the University of Michigan, is now a student actuary for Aetna Life Insurance Company, Hartford, Connecticut.

Mr. C. B. H. Watson, formerly a student at the University of Toronto, is now an actuarial student at Canada Life Assurance Company, Toronto, Ontario, Canada.

Associate Professor H. P. Wirth of the City College of the City of New York has been promoted to a professorship.

Assistant Professor Rhoda Manning Wood of Oregon State College has been promoted to an associate professorship.

Professor Emeritus N. R. Bryan of the University of Maine died on May 22, 1955. He was a member of the Association for thirty-five years.

Professor Emeritus Cora Strong of Woman's College, University of North Carolina, died on June 3, 1955. She was a member of the Association for thirty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 119 persons have been elected to membership by the Board of Governors on applications duly certified.

SMBAT ABIAN, M.S. (Chicago) Grad. Student, University of Cincinnati.
LUCILLE J. ALBERS, B.S. (Dayton) Data Processing Mathematician, Land-Air Inc., Wright Patterson Air Force Base, Ohio.
M. C. ALTSCHULER, Student, Yale University.
J. J. ANDREWS, B.A. (Hofstra) Teaching Asst., University of Georgia.
J. F. ANDRUS, B.S. (Charleston) Grad. Student, Emory University.
C. E. ANTLE, B.S. (Eastern Kentucky S.C.) Grad. Asst., University of Kentucky.
ZAMIR BAVEL, B.A. (Southern Illinois) Grad. Asst., Southern Illinois University.
W. H. BENSON, B.S. (U.S. Naval Acad.) Captain, United States Navy.
A. F. BOND, Jr., Student, West Virginia University.
R. V. BORCHERS, B.S. (Iowa S.C.) Cartographer, U.S. Forest Service, Alexandria, Va.
I. W. BOXER, M.A. (Calif. State Poly. C.) Instr., California State Polytechnic College.
C. B. BROWN, Student, Oklahoma Agricultural and Mechanical College.
DONALD BROWN, Student, Hofstra College.
R. T. BUMBY, Student, Massachusetts Institute of Technology.
F. A. CENEY, JR., Student, Southern Illinois University.
N. H. CLARK, Student, University of California at Los Angeles.
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K. L. CONRAD, M.S. (Iowa S.C.) Jr. Dev. Engr., Goodyear Aircraft Corp., Akron, O.
ELAINE COOK, B.A. (Indiana) Teaching Asst., University of Wisconsin.
W. C. COOK, Student, State College of Washington.
A. H. CORNELIUSSEN, JR., Student, Massachusetts Institute of Technology.
C. G. CULLEN, B.A. (N.Y.S.C. for Teachers, Albany) Grad. Asst., University of New Hampshire.
C. R. DEETER, B.S. (Fort Hays Kansas S.C.) Grad. Student, Fort Hays Kansas State College.
R. T. DONNELL, M.A. (Vanderbilt) Department Head, Union University.
M. D. DONSKER, Ph.D. (Minnesota) Asso. Professor, University of Minnesota.
E. S. EBY, Student, University of Illinois.
CATHERINE G. EHREN, Student, Cardinal Stritch College.
R. M. ELASHOFF, A.M. (Boston U.) Grad. Student, Boston University.
E. A. ENRIONE, Student, University of Miami.
HERBERT ERICSON, JR., Lieutenant, United States Air Force.
C. G. FAIN, B.A. (Tulsa) Electronic Calculator Programmer, Eglin Air Force Base, Fla.
R. B. FOSTER, JR., Student, University of the South.
NORMA M. FREY, B.S. (Nevada) Grad. Student, University of Colorado.
W. P. GANLEY, Student, University of Buffalo.
A. M. GARNER, JR., B.A. (A.&.M.C. of Texas) 2/Lt., United States Army.
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ROBERT GOLD, M.S. (M.I.T.) Everett, Massachusetts.
H. G. GRAEBNER, M.S. (Ohio S.U.) Lofting Mathematician, North American Aviation, Columbus, O.
K. W. GRISWOLD, Student, College of Puget Sound.
WALTER GUBER, New York, New York.
G. R. HAGAN, Student, St. John's College.
PETER HAGIS, JR., M.A. (Temple) Instr., Temple University.
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D. J. HENDERSON, Student, University of British Columbia.

K. E. HOFER, JR., Student, Illinois Institute of Technology.

R. W. HOFFARTH, Student, Southern Illinois University.

H. J. HOLZMAN, Student, University of California at Los Angeles.

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L. G. HOYE, B.S. (Alberta) Edmonton, Alberta, Canada.

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G. E. MARTIN, M.A. (N.Y.S.C. for Teachers, Albany) Batavia, New York.

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HALCYON E. MCNEIL, B.S. (Kansas S.C.) Carter Oil Company, Tulsa, Okla.

R. J. MCQUILLIN, Student, College of Puget Sound.

D. A. MERLINE, Student, University of South Carolina.

A. D. MILLER, Ph.D. (Iowa S.C.) Asso. Professor, Wisconsin State College.

VIVIAN MORGAN, Student, State College of Washington.

MISS MASAKO OBA, Student, University of California at Los Angeles.

M. A. OMAR, Student, Colorado School of Mines.

M. L. PATRICK, Student, Eastern Kentucky State College.

PAUL PAYETTE, Student, Ecole Polytechnique de Montréal, Université de Montréal.

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S. L. PROSSER, M.A. (Stanford) Grad. Asst., University of Alabama.

A. W. RANSOM, Student, University of Rochester.

W. K. RAWDON, Student, Yale University.

W. A. REES, M.A. (Texas) Asst. Professor, University of Houston.

BARNETT RICH, Ph.D. (Columbia) Chairman, Brooklyn Technical High School, Brooklyn, N. Y.

J. I. RICHARDS, Student, University of Minnesota.

GEORGE RICHMAN, Student, University of California at Los Angeles.

R. A. RIEGER, Asst. Personnel Manager, Comar Electric Co., Chicago, Ill.

R. B. RODMAN, Student, Reed College.

K. W. ROESSING, Student, West Virginia University.

I. S. RUBIN, M.S(N.Y.U.) Mathematician-Programmer, Remington Rand, New York, N. Y.

DAVID SACHS, Student, Illinois Institute of Technology.

R. L. SANDERS, B.S. in Min.E. (Missouri School of Mines) Geophysicist, Stanolind Oil and Gas Co., Tulsa, Okla.

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INEZ I. SAUSEN, A.B. (Marymount) Instr., Garden City Junior College, Kansas.

R. J. SCHEFFEL, Student, Hofstra College.

E. C. SCHLUTER, Student, Municipal University of Omaha.

C. E. SEABOLD, B.A. (Heidelberg) Mathematician, Wright-Patterson Air Force Base, Ohio.

RAYMOND SEICK, Student, College of Puget Sound.

BEN SILVER, S.B. (M.I.T.) Providence, R. I.

A. H. SIMON, Student, Cooper Union.

MRS. JANET M. SMITH, Student, Brown University.

O. P. STACKELBERG, Student, Massachusetts Institute of Technology.

J. R. STALLINGS, JR., Student, University of Arkansas.

J. C. STROUD, B.A. (Colorado C.) Mathematician, National Bureau of Standards, Boulder, Colo.

GEORGE SWARTFIGURE, Student, University of Buffalo.

D. F. TEMPLETON, Jr., Student, American University.

BARBARA J. THOMSON, Student, University of Oregon.

J. K. THURBER, Student, Brooklyn College.

C. W. TRAFTON, Undergraduate Asst., Iowa State College.

L. P. TUELL, Student, Northeastern State College.

R. J. TUKOVITS, Student, Hunter College of New York.

P. L. WARNSHUIS, JR., M.S. (Stanford) Mathematician, U. S. Naval Ordnance Test Station, Pasadena, Calif.

IMELDA WASINGER, Student, Marymount College.

J. W. WEIHE, Ph.D. (California, Berkeley) Staff Member, Sandia Corp., Albuquerque, N. M.

L. R. WELCH, B.S. (Illinois) Grad. Student, California Institute of Technology.

H. R. WESSON, B.S. (Birmingham-Southern) Grad. Asst., University of Alabama.

D. R. WILDER, B.A. (Oberlin) U. S. Naval Reserve.

BENJAMIN WILLIAMS, Student, Virginia State College.

R. G. WILLIAMS, M.S. (Illinois) Instr., Sullivan School, Washington, D. C.

L. L. WIMP, M.A. (Missouri) Lecturer, Southern Illinois University.

LEM WONG, Student, University of California at Los Angeles.

M. D. WOODARD, M.S. (Washington) Grad. Asst., University of Florida.

C. R. B. WRIGHT, Student, University of Nebraska.

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-first annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on April 24, 1955. Professor L. K. Jackson of the University of Nebraska presided.

There were 40 persons in attendance, including the following 26 members of the Association:

M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Camp, Morris Dansky, H. W. Doss, Jr., J. M. Earl, M. G. Gaba, Edwin Halfar, L. K. Jackson, M. L. Keedy, W. G. Leavitt, K. L. Loewen, E. J. Lowry, R. L. Moenter, T. A. Newton, Florence E. Pool, H. B. Ribeiro, H. L. Rice, Lulu L. Runge, E. C. Schluter, George Seifert, T. C. Smith, H. P. Thielman, J. F. Wampler.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor E. J. Lowry, Hastings College; Vice-Chairman, Professor L. K. Jackson, University of Nebraska; Secretary, Professor Edwin Halfar, University of Nebraska.

The following papers were presented:

1. *An infinite series analogue to the gamma function*, by Dr. T. A. Newton, University of Nebraska.

2. *An application of independence*, by Professor W. G. Leavitt, University of Nebraska.

The idea of independence of sets of functions may be used to give rather simple proofs of several theorems from elementary analysis. Suppose $\{f_i\}$ $(i=1, \cdots, n)$ is an independent set of analytic functions, which is also closed under differentiation (that is, $f_i' = \sum_{j=1}^{n} C_{ij}f_j$). Let $F(x) = \sum_1^n C_i f_i$ for an arbitrary set $\{C_i\}$ of constants. The following theorems may be proved:

THEOREM 1. *If a is a constant such that $e^{-ax} \neq \sum k_i f_i$ for any set $\{k_i\}$ of constants, then integration by parts furnishes a method of evaluating the integral $\int e^{ax}F(x)dx$.*

THEOREM 2. *If $\{a_i\}$ $(i=1, \cdots, k)$ is a set of constants none of which are characteristic values of the matrix $[C_{ij}]$ then the method of undetermined coefficients may be used to solve the differential equation $(D-a_1) \cdots (D-a_k)y = F(x)$.*

3. *An isoperimetric inequality*, by Professor M. A. Basoco, University of Nebraska.

Let $p(\theta)$ be the supporting function of a convex curve C in the plane. This function is characterized by the conditions that (1) it be periodic, period 2π, and (2) it be "sub-sine" in the sense of Pólya and J. W. Green. If $p(\theta)$ is sufficiently regular, the length L of C, the area F of the region bounded by C and the area E of the evolute of C may be expressed in terms of the Fourier coefficients of $p(\theta)$. Following methods initiated by A. Hurwitz, it is shown that the isoperimetric deficit $(L^2/4\pi) - F \leqq E/4$. A characterization of those convex curves for which equality holds is obtained.

4. *Generalizations of the concept of continuity*, by Professor H. P. Thielman, Iowa State College. (By invitation.)

Let f be a function on a neighborhood space X onto a subset of a neighborhood space Y. The neighborhoods in X are denoted by N_x, those in Y by M_y. The symbol $f^{-1}(M_y)$ indicates the subset of X which consists of those elements x of X for which $f(x)$ is an element of M_y. Let λ be a property of subsets S of X. The property λ is said to be an ascending set property if λ is such that if S has the property λ and T is any subset of X, then $S+T$ also has the property λ. In general, $f(a)$ is said to be λ-approached if for every $M_{f(a)}$ the set $F^{-1}(M_{f(a)})N_a$ has the property λ. The speaker showed that a number of well-known concepts such as continuity almost everywhere, partial continuity, neighborliness and many others are special types of λ-approaches. It was shown that a number of theorems on convergent sequences of continuous functions can be extended to sequences of λ-approach functions if λ is an ascending set property. Various examples of λ-approach functions were given. In particular, an example was given of a function which has the Darboux property (*i.e.*, the property of taking on all values between any two functional values) which is cliquish but is not neighborly. For the definition of these concepts the reader is referred to: Thielman, H. P., *Types of Functions*, this MONTHLY, vol. 60, 1953, pp. 156–161.

5. *A class of spherical hyperbolas*, by Professor O. C. Collins, University of Nebraska.

The locus of constant angle of the sphere has been explored for all combinations of the two parameters, subtended angle and subtending arc. This curve has been employed as a tool in the study of one of its own special cases, where it degenerated into a spherical hyperbola of a particular class, in which the minor axis subtends a right angle at all points on the curve. A formula was found for the angle subtended at any point on this curve by any chord perpendicular to the major axis. The angles subtended at the ends of any diameter were found to be either supplementary or equal according as that diameter does or does not intersect at the chord. The minimum angle was found to be subtended at either end of the minor axis.

6. *On the misuse of accuracy*, by Professor Robert M. Kozelka, University of Nebraska.

This inquiry into certain spurious, or merely misleading, uses of "significant" figures included examination of some possible causes and effects of such misuse and examples from advertising, teaching, and government documents.

7. *Engineers need competence in calculation*, by Professor T. C. Smith, University of Nebraska.

Design today implies dozens of variables that must be dealt with simultaneously. Engineers must set up and solve the corresponding simultaneous equations. Engineers should be taught efficient methods for solving such equations. To illustrate this point, the author explained a method of condensation for solving n simultaneous linear equations with n unknowns. This method has been used by his descriptive geometry class to verify the solution of certain space problems.

Furthermore, one need not hesitate to work with numbers having three or more digits if a calculating machine is used. The author predicted that in the not too distant future a fairly elaborate calculating machine will be a necessary tool for mathematicians and engineers.

8. *Academic background of college students at the University of Nebraska*, by Professor H. M. Cox, Director, Bureau of Instructional Research, University of Nebraska.

The patterns of courses taken in high school by entering freshmen at the University of Nebraska are significantly different in certain areas by sex and by choice of curriculum. They doubtless reflect, in part, the pattern of requirements for admission to the college or curriculum chosen by the students. The differing patterns of courses taken by student groups explain, in part, the differences in distributions of scores made by the students on placement examinations in mathematics, English, *etc.*

9. *Late developments in arithmogeometry*, by Mr. H. W. Becker, Omaha, Nebraska.

(A) It was proved that if one side of a Pythagorean triangle is an mth power not divisible by m, then neither of the other two sides can be an mth or any other power.

(B) 200 errata in Dickson's *History*, v. II, 1st edition, were classified in six categories. These include some noted by Alliston, Bell, Carmichael, Lehmer, Ward, and Dickson himself later.

(C) By five different methods of proof, Pythagorean kites, tetrahedra, cuboids, *etc.*, can have no general solution in polynomials, hence neither can the underlying equation "general quartic $=\square$." This liberates Diophantine Analysis from its most modern supersitition.

EDWIN HALFAR, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The fifteenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, New Mexico, on April 8, 1955. Professor D. L. Webb, Chairman of the Section, presided.

There were 53 persons in attendance, including the following 34 members of the Association:

C. E. Aull, C. E. Buell, J. H. Butchart, Louis Child, R. B. Crouch, M. E. Drummond, Jr., G. E. Forsythe, J. H. Fountain, F. C. Gentry, R. F. Graesser, Elzie A. Greene, J. D. Hankins, P. W. Healy, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, M. H. Hoehn, Carl Holtom, T. L. Jordan, Jr., Max Kramer, R. B. Lyon, W. W. Mitchell, Jr., D. R. Morrison, E. A. Propes, E. J.

Purcell, B. D. Roberts, C. B. Rogers, Rafael Sanchez-Diaz, E. A. Voorhees, Earl Walden, D. L. Webb, R. L. Westhafer, H. H. Wicke, G. M. Wing.

The following officers were elected for the year 1955: Chairman, Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanical Arts; Vice-Chairman, Professor R. B. Lyon, Arizona State College at Tempe; Professor W. W. Mitchell, Jr., Phoenix College, continues for the second year of a four year term as Secretary-Treasurer. The traveling lectureships were abolished.

An invited address "Digital Computing at U.C.L.A." was given following the banquet by Dr. G. E. Forsythe of the University of California at Los Angeles. Dr. Forsythe also read a paper on "Computing Constrained Maxima with Lagrange Multipliers" during the morning session.

The following papers were presented:

1. *A new treatment for the parabola of surety*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

Professor Butchart showed that the envelope of the paths of projectiles issuing from a fixed point with a fixed initial velocity, which is a parabola with its focus at the point of departure, can be developed using only elementary physics, polar coordinates, and pure geometry. He noted the ease with which incidental properties can be proved such as the following: (a) the locus of the foci of the trajectories is a circle; (b) the directrices of the trajectories coincide with the tangent at the vertex of the envelope; (c) if angles of departure differ by 90°, the points of contact with the envelope are at ends of a focal chord.

2. *Cross ratio in n dimensions*, by Professor F. C. Gentry, University of New Mexico.

The cross ratio of two points a and b and two hyperplanes $(mx)=0$ and $(nx)=0$ in projective space of n dimensions was defined as $[(ma)(nb)]/[(mb)(na)]$. From this it was proved that the cross ratio of 4 hyperplanes is constant if they belong to a pencil and that the cross ratio of 4 points is constant if they are collinear. The definition was shown to be consistent with the usual definition for one dimensional space.

3. *Bi-geometric progressions*, by Dr. D. R. Morrison, Sandia Corporation, Albuquerque, New Mexico.

A progression is bi-geometric if there exist constants p and q, such that for every three consecutive terms, x, x', x'' of the progression, $x''=px'+qx$. Rules for geometric progressions presented in standard algebra texts are extended to bi-geometric progressions, using only elementary algebra and analytic geometry. Formulas are developed for the sum and the nth term and for interpolation. Applications are made to oscillating circuits, vibrating strings, pendula, and hanging cables, which would otherwise require differential equations, and to discrete problems to which differential equations are not applicable, such as compound interest with modified interest rates and annual population growth in symbiotic situations.

4. *Sextactic points of a plane curve*, by Professor Louis Child, New Mexico College of Agriculture and Mechanical Arts.

At the point P of a base curve Γ, at which the radius of curvature ρ and its first four derivatives $\rho_1, \rho_2, \rho_3, \rho_4$ with respect to the arc-length s exist and are finite, and $\rho\neq 0$, there exists a one-parameter family D_7 of seven-pointic osculating nodal cubics. Every member of D_7 is nondegenerate if $q\equiv 36\rho_1+9\rho^2\rho_3-9\rho\rho_1\rho_2+4\rho_1^3\neq 0$. If $q=0$, then every member of D_7 is composite, consisting of the osculating conic and a line on P. The osculating conic is then sextactic; it is not sextactic when $q\neq 0$.

5. *On computing constrained maxima with Lagrange multipliers*, by Dr. G. E. Forsythe, University of California, Los Angeles.

Let R be the n-dimensional space of points x. Let G be the intersection of m smooth surfaces $g_r(x)=0$ $(r=1, \cdots, m)$. A computer wishes to locate a certain point $x_{\max}$ which locally maximizes $f(x)$ with respect to G. A classical method of Lagrange multipliers suggests defining $F(x, \lambda)=f(x)+\sum_{r=1}^{m} \lambda_r g_r(x)$, where $\lambda=(\lambda_1, \cdots, \lambda_m)$ are constants. One then seeks to adjust λ until the appropriate stationary value y of $F(x, \lambda)$ with respect to R lies in G. Then ordinarily $y=x_{\max}$.

It is shown by an example $(n=2, m=1)$ and a theorem that, *although $x_{\max}$ is a local maximum of $f(x)$ with respect to G, it may be only a saddle point of $F(x, \lambda)$ with respect to R.* When so, $x_{\max}$ could not be found by ordinary ascent methods applied to $F(x, \lambda)$ in R. A possible cure is to apply an ascent method to $f(x)$ within G.

6. *The splitting of monomial groups*, by Professor R. B. Crouch, New Mexico College of Agriculture and Mechanical Arts.

This paper will be published in the *Transactions of the American Mathematical Society.*

7. *A development of cardinals in "The Consistency of the Continuum Hypothesis,"* by Professor H. D. Sprinkle, University of Arizona, introduced by the Secretary.

In *The Consistency of the Continuum Hypothesis* the axiom of choice is used to construct the theory of powers within that of cardinals. The main purpose here is to develop cardinals without such an axiom. This work could be formalized with the use of the predicate calculus, but, as is done in the book, the proofs are presented rather informally.

8. *The mathematical history of 781 integral calculus students*, by Professor D. L. Webb, University of Arizona.

A summary of the courses and grades of 781 University of Arizona students who have completed integral calculus in the years from 1948 to 1954.

9. *Arizona mathematics contest*, by Professor W. W. Mitchell, Jr., Phoenix College.

The development, execution, and preliminary results of the first annual Arizona Mathematics Contest sponsored by the Arizona Association of Teachers of Mathematics were explained. Copies of the examination and contest rules were distributed and a short discussion period followed.

10. *The use of visual aids in mathematics courses*, by Professor P. W. Healy, University of New Mexico.

Professor Healy presented his views concerning the use of visual aids in mathematics courses. He pointed out that increased use of improved visual aids could be a partial answer to the dilemma facing us in the immediate future when there will be more students and fewer qualified teachers. His presentation was followed by some sound films designed to teach mathematical topics.

W. W. MITCHELL, JR., *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Abilene, Texas, on April 22–23, 1955, sponsored jointly by Abilene Christian College, Hardin-Simmons University, and McMurry College. Sessions were held Friday afternoon and Saturday morning, Professor E. A. Hazelwood and Professor M. E. Mullings presiding.

By invitation, Professor W. L. Duren, Jr., of Tulane University delivered an address at the afternoon session. The title of his address was "A Course on Sets." At the banquet Professor Dorothy McCoy of Wayland Baptist College gave an illustrated lecture on her teaching experiences in Iraq.

There were 74 persons in attendance, including the following 50 members of the Association:

T. A. Abouhalkah, C. P. Benner, Ina M. Bramblett, H. E. Bray, J. E. Burnam, L. A. Colquitt, Don Cude, H. B. Curtis, Jr., W. L. Duren, Jr., B. T. Goldbeck, Jr., R. G. Green, R. E. Greenwood, Jr., W. T. Guy, Jr., Lena L. Hays, E. A. Hazelwood, E. R. Heineman, R. F. House, P.W.M. John, Guy Johnson, Jr., E. C. Klipple, J. W. Lindsay, C. W. Long, H. A. Luther, Dorothy McCoy, R. V. McGee, A. B. McLemore, R. M. McLeod, V. A. Miculka, P. D. Minton, V. H. Morrill, M. E. Mullings, C. J. Pipes, C. L. Riggs, Virginia B. Roberts, C. R. Sherer, D. P. Shore, W. C. Sikes, S. A. Sims, Sister Antonietta, D. W. Starr, J. R. Swaffield, Jennie L. Tate, W. W. Taylor, J. I. Tracey, F. E. Ulrich, R. S. Underwood, Patricia A. Ward, Mabel Williams, C. B. Wright, Martin Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor M. E. Mullings, Abilene Christian College; Vice-Chairman, Professor Don Cude, Southwest Texas State Teachers College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The program consisted of the following papers:

1. *A Mathieu integral transform*, by Professor W. T. Guy, Jr., The University of Texas.

A Mathieu integral transform was defined and three useful operational properties were given. Then, these properties were used to solve a differential equation and derive a relation between two special functions.

2. *A Taylor's series*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

Let λ be a complex number and let $f(z-z_0)$, where $f(0)=1$, be a Taylor's series in powers of $z-z_0$. It can be shown that there exists a Taylor's series in $z-z_0$ which can consistently be interpreted as $[f(z-z_0)]^\lambda$. Moreover, the coefficients of this latter series can be exhibited. In this discussion, however, there is presented only an example of non-obvious character. The proof used is independent of the general results mentioned above.

3. *On bounded families of analytic functions*, by Mr. Guy Johnson, Jr., The Rice Institute.

Let F denote a uniformly bounded family of functions analytic in a domain D. The function $G(z)=\sup_f |f(z)|$, called the modulus of F, is continuous and satisfies a maximum principle. If $G_1(z)$ is defined in the annulus $R_1<r<R_2$ by $G_1(z)=M(r)$ for $|z|=r$, and $M(r)$ is a convex function of $\log r$ then there exists a family F_1 for which $G_1(z)$ is the modulus.

4. *A course on sets*, by Professor W. L. Duren, Jr. Tulane University.

The Committee on the Undergraduate Program of MAA has been consulting with mathematicians, engineers, and scientists on the content of the early college courses in mathematics. Present first year courses fall into about six diverse types. This paper presents an appeal for simplifying these offerings into one course for normally prepared students. Such a course already exists in the form of graphs and elementary calculus, and numerous text books are available for it. This

elementary calculus should be devoted to the study of polynomial, rational, exponential, and logarithmic functions, postponing analytic trigonometry.

Early calculus is feasible if certain traditional but unnecessary prerequisites are eliminated. A major source of these unessential subjects is in the analytic geometry of the euclidean plane. A more general "graphs" geometry in the xy-plane is better. The graphs geometry admits straight lines, slopes, and area under a curve but not angle, circle, and arc length.

5. *Combinatorial relations and chromatic graphs*, by Professor R. E. Greenwood, Jr., The University of Texas.

The results presented in this talk have been published in the following paper: Greenwood, R. E. and Gleason, A. M., *Combinatorial relations and chromatic graphs*, (Canadian Journal of Mathematics, vol. 7, 1955, pp. 1–7).

6. *Birth and death processes*, by Mr. P. W. M. John, University of Oklahoma.

The probabilities of population sizes in a generalized birth and death process are given by the infinite set of differential equations.

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t).$$

Complete solutions are known for those processes in which λ_n and μ_n are either constant or may be written as $\lambda(t)n$ and $\mu(t)n$. The state of knowledge of estimators for the parameters λ and μ for stationary processes of the latter type is discussed.

7. *An extension of gap theorems to expansions in Gontcharoff polynomials*, by Mr. R. M. McLeod, The Rice Institute.

The classical gap theorems on Taylor's series are extended to series $\sum_{n=0}^{\infty} c_nP_n(z)$ where $\{P_n(z)\}$ is the sequence of Gontcharoff polynomials corresponding to a sequence $\{z_n\}$ satisfying

$$\limsup_{n\to\infty} |z_n|^{1/n} = a, \qquad a < 1.$$

The method is to show that the region of regularity of the difference

$$\sum_{n=0}^{\infty} c_n(P_n(z) - z^n)$$

includes the closed disk $|z| \leqq R$ in its interior, R being the radius of convergence of $\sum c_nz^n$.

8. *Related algebraic and geometric curiosities*, by Professor R. S. Underwood, Texas Technological College.

One way to get often predictable loci on a plane for equations in n variables is to use an inflexible coordinate system. This method turns up an algebraic curiosity—the analogue of the "point-ellipsoid,"—whose single solution is known here but hard to find otherwise. A second method solves two quadratics simultaneously by adjusting the plotting rule so that the locus of one equation is a line or curve. The other locus turns out to be the "geometric curiosity"—a closed area bounded by an ellipse, two parabolas, and a straight line.

9. *How to solve it*, by Professor J. I. Tracey, Texas Christian University.

Among the problems included in this paper are some for which the solution can be obtained readily by the use of the harmonic mean. Others which are suggested include the famous Morley Triangle and some of the historical problems from Archimedes' *Method.*

10. *A theorem on the roots of the derivative of a polynomial*, by Professor H. E. Bray, The Rice Institute.

A theorem of A. Cohn states that if $P(z)$ is a polynomial of degree $2k$, with k pairs of roots, each pair having the same argument, ϕ_j, and moduli $(r_j<R_j)$ such that $r_jR_j=1$, $[j=1, 2, \cdots k]$, then the derivative $P'(z)$ has exactly k roots outside the unit circle. The theorem is generalized; the conclusion holds if the moduli r_j, R_j, are such that:

$$r_j < 1 < R_j; \qquad a_j \geqq 1; \qquad h_j < 9a_j/(a_j^2 + 8) \quad \text{if} \quad 1 \leqq a_j \leqq 2,$$

where a_j is the arithmetic mean of r_j, R_j, and h_j is their harmonic mean, $j=1, 2, \cdots k$.

11. *Engineers and mathematicians*, by Mr. T. A. Abouhalkah, Railroad Commission of Texas.

Expert teachers who will inspire and encourage prospective engineers and mathematicians at high school and college levels are badly needed for the sake of sciences and of civilization. Successful mathematicians have shown that the laws of nature are designed mathematically, that the relation of the universe with what is inside of its horizons is based on sound mathematics, and that the concepts of physical and social sciences have been influenced by the comprehension and understanding of the "Queen of the Sciences."

Successful engineers are not interested only in methodology and technique, but consider mathematics as an "intellectual occupation." Hence, they are not deprived of the charm of the subject and they do not resort blindly to the using of formulae in solving their engineering probems.

12. *Current legislation affecting higher education*, by Professor E. A. Hazelwood, Texas Technological College.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-ninth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 7, 1955, at Duquesne University, Pittsburgh, Pennsylvania. Professor F. H. Steen, Chairman of the Section, presided at the morning and afternoon sessions.

There were 63 in attendance, including the following 42 members of the Association:

A. G. Anderson, Thomas Bauserman, J. O. Blumberg, W. G. Brady, A. M. Bryson, F. C. Calabrese, J. G. Christiano, A. B. Cunningham, D. A. De Felice, Esther S. Dunkelberger, R. D. Edwards, Ruth O'Donnell Goodman, D. S. Hoffman, B. P. Hoover, M. A. Hyman, F. E. Justis, J. C. Knipp, E. C. Kovacs, George Laush, Gloria A. Martin, R. G. McDermot, David Moskovitz, L. T. Moston, B. H. Mount, Jr., E. F. Myers, L. A. Ondis, II, Morris Ostrofsky, M. C. Palmer, P. A. Penzo, I. D. Peters, F. H. Steen, Andrew Sterrett, Jr., Alexander Strasser, E. A. Sturley, W. C. Styslinger, Jr., J. S. Taylor, Jean E. Teats, V. E. Thomas, L. O. Thompson, E. A. Whitman, P. M. Whitman, B. H. Youell, Jr.

At the business meeting, the following officers were elected for a term of two years: Chairman, Dean L. T. Moston, Waynesburg College; Secretary, Professor I. D. Peters, West Virginia University; Executive Council: Professor F. H. Steen, Allegheny College; Professor R. D. Edwards, University of Pittsburgh.

The following papers were presented:

1. *A plastic theory for pre-sintered powdered metals*, by Mr. E. M. Shoemaker, Carnegie Institute of Technology, introduced by the Chairman.

A mathematical model is constructed which is intended to characterize the mechanical behavior of a pre-sintered powdered metal. Suitable plastic potential surfaces are discussed and the corresponding stress-strain laws derived. Uniqueness theorems and minimum principles are established.

2. *An experiment in probability*, by Professor Emeritus E. A. Whitman, Carnegie Institute of Technology.

In this paper the author shows one arrangement of successes where the probability of success is known and another arrangement of successes where the probability of success is determined experimentally. In both cases the interest is in the distribution of successes that are relatively infrequent.

3. *Symposium on automatic digital computers.*

High speed electronic computers are being more and more widely used to solve both scientific and industrial problems. The following three papers outlined some of the methods currently used in preparing problems for high speed computation:

(1) *Basic programming*, by Dr. Ruth Goodman,
(2) *Automatic coding*, by Dr. D. H. Shaffer,
(3) *Numerical integration*, by Dr. Morris Ostrofsy,

staff members of the Research Laboratory, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

L. T. MOSTON, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-second annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on May 7, 1955. Two sessions were held at which Professor H. W. Alexander of Earlham College, Chairman of the Section, presided.

There were 60 in attendance, including the following 52 members of the Association:

H. W. Alexander, W. C. Arnold, Juna L. Beal, L. G. Black, A. P. Boblett, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, W. W. Chambers, K. W. Crain, H. E. Crull, M. W. DeJonge, W. E. Edington, P. D. Edwards, C. B. Gass, E. L. Godfrey, S. H. Gould, G. H. Graves, Ralph Hafner, H. H. Hartzler, Ralph Hull, H. L. Hunzeker, M. W. Keller, E. L. Klinger, Florence Long, Gladys B. McColgin, D. M. Mesner, G. T. Miller, Vera T. Morris, J. E. Mueller, R. H. Oehmke, Theresa M. C. Oehmke, Gloria Olive, C. C. Oursler, P. W. Overman, T. P. Palmer, J. C. Polley, D. H. Porter, J. N. Rogers, R. M. Ross, A. R. Schmidt, K. J. Sidebottom, Sister Gertrude Marie, C. P. Sousley, Anna K. Suter, R. O. Virts, M. S. Webster, K. P. Williams, Herbert Wolf, Elizabeth S. Wolf, H. E. Wolfe, G. N. Wollan.

The following officers were elected: Chairman, Mr. R. O. Virts, Central High School, Fort Wayne, Indiana; Vice-Chairman, Professor C. F. Brumfiel, Ball State Teachers College; Secretary-Treasurer, Professor J. C. Polley, Wabash College.

Both sessions were held in the Holcomb Observatory, recently constructed and in use this year for the first time. Professor G. C. McVittee, Director of the Observatory at the University of Illinois, was guest speaker for the hour lecture. The title of the lecture was: *Why should an astronomer study relativity*?

Professor H. E. Crull of Butler University gave a planetarium demonstration following the lecture.

Professor P. D. Edwards, chairman of the Committee on Awards, reported that four Association medals had been awarded for high mathematical achievement in the Indiana Science Talent Search.

The following papers were presented:

1. *Some elementary properties of bonding mappings*, by Professor R. H. Oehmke, Butler University.

In any non-associative algebra A of characteristic not 2, with a subspace S closed under the operation $(x, y)=\frac{1}{2}(xy+yx)$, $U(S)$ denotes the subspace generated by all elements $xy-yx$ for x and y in S. If T is any linear mapping from $U(S)$ into S, a multiplication $x \circ y$ can be defined in S as $x \circ y=\frac{1}{2}(xy+yx)+(xy-yx)T$. Thus a new algebra $B(A, T)$ is defined which is in the same vector space as S and is closed under the product $x \circ y$. This algebra is said to be *bonded* to A by the *bonding mapping* T. The behavior of associative, Jordan, Lie, and power-associative algebras under a bonding mapping was examined. Such tools as ideals, idempotents, derivations, *etc.* used for the study of the structure of algebras were also examined.

2. *The mathematical theory of the Hatchet Planimeter*, by Professor P. D. Edwards, Ball State Teachers College.

The March 1954 issue of *The Professional Geographer* contained a short description of the "Hatchet Planimeter" by R. L. Williams. The mathematical theory was not given. The instrument has been used to a limited extent by cartographers and others since its description by Prytz in 1889. In this paper the mathematical theory is presented and comments made on the degree of accuracy to be expected.

3. *A summary of integral methods*, by Professor T. P. Palmer, Rose Polytechnic Institute.

Integration methods can be summarized under six topics: (1) the integral of $u^n du$, with $n=-1$ as a special case; (2) a collection of eight exact differentials (exponential and trigonometric; eleven, if including hyperbolic functions); (3) integration by parts; (4) substitution (chiefly trigonometric); (5) partial fractions; and (6) trigonometric identities. The last four topics are not really calculus, but provide ways of rearranging so that the first two topics apply. The only differential included which is not familiar from differential calculus is $d \ln (\sec x+\tan x)=\sec x dx$. By these methods, any form whose integral can be expressed in elementary functions can be integrated easily without reference to tables.

4. *Some embedding theorems for incidence matrices*, by Professor D. M. Mesner, Purdue University.

To a given incidence matrix A, matrices B, C, and D are to be adjoined so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an $m\times n$ incidence matrix with equal row totals T and equal column totals U. The required numbers of 0's and 1's in each row of B and column of C, and in the entire block D, are easily computed. It is obviously necessary for the existence of B, C, and D that these numbers be non-negative, and that $mT=nU$. These are shown to be sufficient conditions as well. This generalizes a theorem of Ryser (*Proc. Amer. Math. Soc.*, vol. 2, 1951, 550–552).

5. *Some applications of evenly convex sets*, by Dr. J. R. Blum, Indiana University, introduced by Professor K. P. Williams.

A convex set in finite-dimensional Euclidean space is called evenly convex (W. Fenchel, *A remark on convex sets and polarity*, Comm. du sem. math. de L'un. de Lund, tome supp., 1952) if it is the intersection of a family of open half-spaces. Criteria are given for a convex set to be evenly convex, and for a vector to belong to an evenly convex set. These are applied to obtain an existence criterion for unbiased tests of finite statistical hypotheses.

6. *Mohr space representation of algebraic equations*, by Professor M. O. Peach, University of Notre Dame.

The Mohr circle construction used by engineers to represent the stress tensor is extended to space of higher dimensions, hence to square symmetric matrices of arbitrary rank. It is then generalized to represent non-symmetric matrices. A step by step method for diagonalizing such matrices is interpreted geometrically, both for the case of real and the case of complex characteristic roots. The well known procedure for writing the matrix for which a given algebraic equation is the characteristic equation provides the connecting link whereby any algebraic equation can be given a unique geometrical representation in Mohr space.

7. *The structure of commutative semigroups*, by Professor R. E. MacKenzie, Indiana University, introduced by Professor H. E. Wolfe.

By suitably formulating the basic structure theorems of commutative rings it is possible to carry through their demonstration without the use of the operation of addition. These theorems then become statements about the structure of commutative semigroups. The formulation is such that the theorems on rings may then be obtained by assuming that the semigroup is a ring.

J. C. Polley, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-fourth annual meeting of the Illinois Section of the Mathematical Association of America was held at Monmouth College, Monmouth, Illinois, on May 13 and 14, 1955. Professor Rothwell Stephens, Chairman of the Section, presided at all sessions.

There were 47 in attendance, including the following 37 members:

Beulah M. Armstrong, H. G. Ayre, J. W. Beach, H. R. Beveridge, D. R. Bey, A. H. Black, A. O. Boatman, H. A. Bott, Joseph R. Brown, L. J. Burton, Paul Cramer, Allen Fenstermacher, S. R. Filippone, A. E. Gault, A. E. Hallerberg, M. C. Hartley, F. E. Hohn, M. R. Kenner, E. C. Kiefer, Rose Lariviere, A. O. Lindstrum, Jr., Saunders MacLane, W. G. Madow, W. C. McDaniel, A. W. McGaughey, E. B. Miller, M. G. Moore, C. E. Moulton, T. E. Rine, L. A. Ringenberg, W. C. Ross, Jr., M. Anice Seybold, W. H. Spragens, Jr., Rothwell Stephens, Gabriel Tsiang, L. L. Wimp, Alice K. Wright.

At the business meeting on Friday afternoon the following officers were elected for the coming year: Chairman, Professor H. R. Beveridge, Monmouth College; Vice-Chairman, Professor L. A. Ringenberg, Eastern Illinois State College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. Professor Joseph Stipanowich reported on the work of the "Committee on Contests and Awards" stating that the number of high schools participating increased over that of the preceding year by almost 80%. Professor A. O. Lindstrum, Jr., reported on the work done by the "Committee on the Strengthening of the Teaching of Mathematics" and proposed several resolutions which were adopted.

The following program was presented:

1. *Recent trends in operational calculus*, by Professor E. J. Scott, University of Illinois.

Heaviside's operational calculus and its development in the form of the Laplace transform were briefly discussed and the difference between them indicated. It was shown how certain shortcomings of the Laplace transform theory have been recently overcome by J. G. Mikusinski by returning to a direct interpretation of Heaviside's symbols and giving them a mathematical sense by generalizing the concept of number suitably.

2. *A logical basis for engineering calculus*, by Professor M. E. Munroe, University of Illinois, introduced by the Secretary.

A rigorous justification of the formal manipulation commonly referred to as "engineering calculus" can be based on the following consideration. Definitions of variables and differentials should be independent of the equations in which they appear. Let a variable x be a mapping of something (*e.g.*, times, distances) into numbers, and let its differential dx be a similar mapping. Equations relating variables (including differentials) are conditional equations, and justification of a formal manipulation amounts to checking for consistency the conditional equations that appear in the argument.

3. *A note on the indeterminate forms 0^0 and ∞^0*, by Professor J. W. Beach, Northern Illinos State Teachers College.

This paper presented four theorems concerning the formation of a function which at a point, a, is one of the indeterminate forms 0^0 or ∞^0, but the limit of the function as the variable approaches a is some value other than 0, 1, or ∞.

4. *Fibonacci numbers and Lucas series*, by Professor L. J. Burton, Lake Forest College.

$\tau=\frac{1}{2}(1+\sqrt{5})$ is defined by means of the golden section, and is related to the construction of a regular decagon, the continued fraction $1+1/1 \cdots$, and the safe combinations in Wythoff's game. The Fibonacci numbers 1, 1, 2, 3, 5, 8, $\cdots$, are introduced by means of the convergents to τ and the Lucas series 1, 3, 4, 7, 11, $\cdots$, is also defined in terms of τ, and certain simple properties of the series are explained. Finally a proof is outlined by means of congruences in $k(\sqrt{3})$ that if p is an odd prime, then $M_p=2^p-1$ is prime if and only if M_p divides S_{p-1}, where $S_1=4$, $S_n=S_{n-1}^2-2$. With this theorem the SWAC computer in Los Angeles showed in about an hour that $2^{2281}-1$ is a Mersenne prime, the largest specific prime yet known.

5. *What constitutes an effective program of student teaching in secondary mathematics?* by Professor T. E. Rine, Illinois State Normal University.

Improvement of student teaching programs is considered important to mathematics teachers and other educators. This is shown by statements of teachers in service, by the findings of national surveys, and by the work of professional committees.

About 15 years ago the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics gave its support to student teaching in these words: "The most important element in professional training is student practice teaching, carried out under the most competent supervision that can be procured. The commission considers this work so important that it urges that even greater attention be paid to it in the future than in the past." (N.C.T.M., Fifteenth Yearbook, 1940, p. 191).

Since that time very little has been done to improve the program of student teaching in secondary mathematics on a nationwide basis. An effective program of student teaching in secondary

mathematics must take into consideration the needs of prospective teachers in regard to four major aspects, namely; preparation for admission to student teaching, supervision of student teaching, the participation period of student teaching, and evaluation of student teaching. Any educational institution that is concerned with providing outstanding teachers of secondary mathematics should make a careful study and analysis of its program of student teaching in secondary mathematics in each of these major aspects.

6. *The revolution in mathematics instruction*, by Professor Saunders MacLane, University of Chicago.

At the dinner Professor MacLane spoke about the revolution. This revolution springs from the need to introduce new mathematical ideas and methods into the curriculum. Basically, mathematical instruction remains always the same process of the discovery by youth of the beauty of mathematical ideas, but the chances of success are better when youth is exposed to the best mathematical ideas. The process of introducing newer ideas (*e.g.* sets) into undergraduate courses is well under way, and is illustrated by the work of the Illinois Section and the Association at large. The problem of introducing newer mathematical ideas into high schools has just come to the foreground; it offers challenging prospects and difficult questions.

7. *Linear algebra for students of applied mathematics*, by Professor F. E. Hohn, University of Illinois.

This paper outlines a course in linear transformations and matrices given at the University of Illinois for students interested in the applications of the subject. The course is used to introduce the student to a variety of abstract concepts as well as to techniques of computation. The author feels that on the grounds of usefulness and mathematical importance, it might well replace the traditional course in the theory of equations.

8. *A report on a cultural mathematics course*, by Professor W. C. McDaniel, Southern Illinois University.

A course has been given at Southern Illinois University for the past three years to students in the general program who have no particular vocational needs for mathematics. These students have a weak mathematical background on the average. The course stresses understanding rather than techniques. The instructors of the course believe that it is moderately successful and should be continued with further modifications.

9. *A proof of the fundamental theorem of algebra*, by Professor H. E. Vaughan, University of Illinois.

In this proof it is shown that the fundamental theorem of algebra is a simple consequence of the following easily proved generalization of the Brouwer Fixed Point Theorem: If $R>0$ and n is a positive integer, and if f is continuous and $|f(z)| \leqq R^n$ in $|z| \leqq R$, then there exists a number z_0 with $|z_0| \leqq R$ such that $z_0^n = f(z_0)$.

10. *Mathematics for social scientists and its effect on undergraduate teaching*, by Professor W. G. Madow, University of Illinois.

This paper continues and amplifies the discussion of *Mathematics for Social Scientists* (this MONTHLY, vol. 61, 1954, pp. 550–561) and the report of the Social Science Research Council Committee on the Mathematical Training of Social Scientists soon to be published in the Items of the SSRC (copies may be obtained from the present author) with particular emphasis on undergraduate mathematics courses. The main point is that the two years of undergraduate mathematics suitable for social scientists are better for majors in mathematics, statistics, the natural sciences and humanities than the present curriculum; and, with some supplementation, are believed

also to be better for students in engineering and the physical sciences. Experimentation with "non-contractual" groups such as the social scientists and mathematicians is strongly recommended to the end that the necessary curriculum and books may be developed along the lines indicated in the above reports and in this paper.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Teresa in Winona, Minnesota, on May 7, 1955. Sessions were held in the forenoon, at luncheon and in the afternoon. Sister M. Thomas à Kempis, Professor M. D. Donsker, and Sister M. Leontius, Chairman of the Section, presided at the respective sessions.

There were 64 persons in attendance, including the following 37 members of the Association:

N. E. Albrecht, H. M. Anderson, Souren Babikian, K. H. Bracewell, G. U. Brauer, E. J. Camp, C. S. Carlson, H. D. Colson, Brother Louis De La Salle, R. J. Dowling, Mary I. Elwell, J. E. Hafstrom, Fulton Koehler, D. R. Lewis, C. B. Lindquist, W. S. Loud, K. O. May, W. R. McEwen, Reverend Florian Muggli, E. D. Nering, C. R. Perisho, P. C. Rosenbloom, Sister M. Bibiana, Sister M. Helen, Sister M. Joanne, Sister M. Thomas à Kempis, Sister Mary Leontius, Sister Mary Seraphim, F. C. Smith, O. E. Stanaitis, F. J. Taylor, Takashi Terami, Frances E. Walsh, Chih-yi Wang, K. W. Wegner, R. P. Winter, F. L. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Professor E. J. Camp, Macalester College; Secretary, Professor F. C. Smith, College of St. Thomas; Executive Committee, Professor M. D. Donsker, University of Minnesota, Professor H. B. MacDougal, South Dakota State College, Sister Mary Leontius, College of St. Teresa.

By invitation of the Executive Committee, Sister M. Helen of Mount St. Scholastica College delivered an address at the morning session entitled "Some Philosophic Considerations of Mathematics." Abstract of this address follows:

In the first section of this paper the author proposes to answer the following questions: (1) what is mathematics in itself? (2) what comprises its matter and its form? (3) how is mathematics regarded by the layman, the scientist, the mathematician and the philosopher?

The last half of the paper treats of those specific features which marked the field of mathematics in the past and contrasts these with mathematics as we know it today. The concluding paragraphs point out what in mathematics is universal and abiding and what is particular and transient. Special mention is given the role of the imagination in the field of speculative mathematics.

The following short papers were presented:

1. *Seminar in mathematics*, by Professor Frances E. Walsh, College of St. Teresa.

This paper presents a description of a course entitled Seminar in Mathematics offered for seniors majoring in mathematics at the College of St. Teresa. The position of the course in the curriculum is developed, showing the contribution made to the fulfillment of the aim of the college, *i.e.*, the training of apostolic women. The objectives of such a course must conform with the objectives of the institution in which the course is being developed. The ways in which the study of mathematics lends itself to developing intellectual maturity are considered and emphasis is placed on the spirit in which the course is presented.

finite number of points which are explicitly located and classified according to the types of singularities $P(t)$ has on them. The construction can be adapted to give an elementary derivation of the Jordan normal form of a constant matrix and the theorems concerning it.

6. *An integral test for convergence of series*, by Professor O. E. Stanaitis, St. Olaf College.

It is shown that $\sum_{x=1}^{\infty} f(x)$ is convergent if the integral $\int_1^{\infty} f(x)dx$ is convergent and if $\int_1^{\infty} f'(x) \sin 2k\pi x dx = O(1/k^p)$ where p is a positive constant.

7. *A certain class of summation methods*, by Dr. G. U. Brauer, University of Minnesota.

A regular matrix A is said to be of type P if it satisfies the following condition: if $\{s_n\}$ and $\{t_n\}$ are two bounded sequences which are evaluated to σ and τ, respectively, by A, then the sequence $\{s_n t_n\}$ is evaluated to $\sigma\tau$ by A. Some properties of matrices of type P are obtained.

8. *A problem on power series in two variables*, by Professor P. C. Rosenbloom, University of Minnesota.

Let $f(x, y)$ be a power series with non-negative coefficients and suppose that the values $f(1, 1) = 1$ and $f(x_1, y_1) = c$ are prescribed, where (x_1, y_1) is a given point in the unit square. What is the greatest possible value for $f(x_2, y_2)$, where (x_2, y_2) is another given point in the unit square? The problem can also be proposed for polynomials of given degree. The extremal function in both cases is a polynomial with at most two non-vanishing coefficients. Some other properties of this extremal polynomial are known, but it has not been determined explicitly.

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The eleventh annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Buffalo, Buffalo, New York, on May 14, 1955. The Chairman of the Section, Professor J. R. F. Kent of Harpur College, presided at the morning session, and the Vice-Chairman, Professor C. E. Rhodes of Alfred University, presided at the afternoon session.

There were 98 persons in attendance, including the following 76 members of the Association:

M. R. Bates, R. L. Beinert, H. F. Bennett, Dorothy L. Bernstein, W. W. Bessell, Jr., A. H. Blessing, H. D. Block, L. L. Brassaw, Jr., A. L. Buchman, F. J. H. Burkett, W. B. Carver, H. S. M. Coxeter, E. J. Downie, A. H. Ettelson, A. G. Fadell, O. J. Farrell, C. W. Foard, A. H. Fox, C. L. Gape, H. M. Gehman, B. H. Gere, D. T. Gianturco, Lillian Gough, N. G. Gunderson, J. R. F. Kent, Violet H. Larney, D. B. Larson, Caroline A. Lester, R. W. MacDowell, R. T. J. Mahoney, J. N. Mangnall, June M. McArtney, C. A. McCarthy, Rudolf Meyer, Irene P. Monahan, Harriet F. Montague, Mabel D. Montgomery, L. J. Montzingo, Jr., D. S. Morse, C. W. Munshower, W. L. Murdock, W. V. Nevins III, C. S. Ogilvy, F. R. Olson, F. D. Parker, J. M. Perry, C. W. Pflaum, Theresa L. Podmele, L. R. Polan, L. D. Potts, J. F. Randolph, A. W. Ransom, C. E. Rhodes, J. B. Rosser, Mary E. Rudin, Joy B. Russek, P. J. Schillo, Edith R. Schneckenburger, L. F. Scholl, Dorothy B. Shaffer, B. B. Sharpe, Sister Florence Marie, Sister Marion, Sister Mary Michael, Ruth W. Stokes, Irwin Stoner, R. F. Tidd, Nura D. Turner, J. P. van Alstyne, J. R. Vanstone, Rev. G. W. Walker, J. F. Wardwell, F. C. Warner, W. G. Warnock, C. B. H. Watson, Frederick R. White.

At the business meeting the following officers were elected: Chairman, Professor C. E. Rhodes, Alfred University; Vice-Chairman, Professor A. J. Coleman, University of Toronto; Secretary, Professor N. G. Gunderson, University of Rochester. The Executive Committee presented a set of proposed By-Laws which it had prepared, and, after discussion and minor amendment, these By-Laws were adopted. The Chairman pointed out that it might be possible to find a new name for the Section more descriptive of its geographical boundaries, and asked that suggestions be sent to the Executive Committee.

The following papers were presented:

1. *The need of mathematics in business research*, by Dr. W. L. Murdock, General Electric Company.

The speaker described the growth of management in business as a profession, emphasizing the present and potential use of mathematical methods by management. He discussed the mathematical topics which should be included in the education of the management trainee. Linear systems are of particular importance.

2. *Teaching computing without a computer*, by Professor J. F. Randolph, University of Rochester.

After considering many of the traditional computing techniques, the class is introduced to automatic computing machinery through a study of a hypothetical computer, the Hypac. Programming is studied, with the class working out various subroutines as problems. Finally, several actual machines are discussed briefly.

3. *Teaching computing with a computer*, by Professor J. B. Rosser, Cornell University.

In teaching computing when a computing machine is available, work must be scheduled so that the class members will not find it necessary to have access to the machine simultaneously. This requires planning and assigning the homework considerably in advance. It also necessitates outside finals, which might well be given out in parts, each part being given out as soon as the class has covered the relevant material. Another point that needs constant attention is the tendency of people who have a computer available to stop doing any thinking and to rely on the computer for everything. It is necessary to keep bringing in problems for which an easy solution can be found, and then upbraiding the class for solving the problem in a routine fashion on the machine instead of finding the easy solution.

4. *Rigor in undergraduate mathematics*, by Professor F. D. Parker, Clarkson College of Technology.

All students of mathematics will benefit from a more rigorous treatment of undergraduate mathematics than is common today. The sciences and professions which use mathematics have all benefited from the rigor which characterizes mathematics. The speaker believes that the present trend is toward the utilitarian aspects of mathematics, and that mathematics, the students, and eventually the sciences and professions will suffer. He also believes that college students are more capable than is generally realized of understanding a more rigorous treatment. The result will be a better understanding of the aims and nature of mathematics and a better use of mathematics.

5. *Problems of a blind student of mathematics*, by Mr. Irving Bentsen, University of Rochester, introduced by Professor N. G. Gunderson.

Most problems encountered by the blind student of higher mathematics may be classified in terms of literacy in mathematics, time consumed in the mechanics of study, unavoidable reliance

on extensive memorization, and perception of geometrical figures. The standard mathematical Braille notation is too cumbersome and limited for efficient use. Recording texts and lectures affords the student an adequate though time consuming means to the literature. The need for instant memorization arises when the classroom teacher states only once an expression written on the blackboard (memory board). In perception of geometrical figures, a distinction is made between those students who have once seen and those who have not. For a student blind from birth problems may arise when a teacher explains a geometrical concept in visual terms. On the other hand, this student is not distracted by representation of figures in perspective, since perspective is an almost meaningless concept to him.

6. *Hyperbolic triangles*, by Professor H. S. M. Coxeter, University of Toronto.

Elegant proofs are described for three well known properties of triangles in the hyperbolic plane. The first is Gauss's proof that the area of a triangle is proportional to its angular defect. The second is a slightly improved version of Liebmann's proof that this area remains finite even when the sides are infinitely long. The third is a projective proof (applying Brianchon's theorem to the absolute conic) for the hyperbolic counterpart of the familiar theorem that the six bisectors of the three angles of a triangle are the six sides of a complete quadrangle.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-third annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Cardinal Stritch College, Milwaukee, Wisconsin, on May 7, 1955. Professor A. E. May, Chairman of the Section, presided.

There were 75 present, including the following 45 members of the Association:

R. H. Bardell, W. R. Brittenham, Leonard Bristow, R. C. Buck, G. L. Bullis, K. L. Clark, E. G. H. Comfort, Miriam E. Connellan, B. F. Dostal, H. P. Evans, J. V. Finch, C. E. Flanagan, Harold Glander, P. C. Hammer, C. B. Hanneken, E. G. Harrell, Fannie Hopkins, R. C. Huffer, Rev. M. L. Jautz, E. R. Johnston, Frances W. Jones, W. E. Lawrence, A. P. Loomer, Morris Marden, A. E. May, J. J. McLaughlin, Karl Menger, Genevieve T. Meyer, A. C. Moeller, O. E. Overn, H. P. Pettit, Berthold Schweizer, Sister M. Elizabeth, Sister Mary Corona, Sister Mary Felice, Sister Mary Petronia, Sister Ruthmary, Abraham Spitzbart, J. C. Spradling, Dorothy J. Stodola, E. W. Swokowski, J. V. Talacko, C. J. Vanderlin, Jr., R. D. Wagner, Louise A. Wolf.

The following officers were elected for the coming year: Chairman, Professor A. C. Moeller, Marquette University; Vice-Chairman, Professor Raphael Wagner, University of Wisconsin; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Program Committee Chairman, Professor Raphael Wagner.

It was decided at the business meeting that the Section would undertake a state-wide mathematics contest for high school students each year. The Executive Committee will appoint a Committee on Contests, which will be responsible for organizing and administering the first of these contests this year.

The following papers were presented:

1. *Antiderivatives, integrals, cumulations*, by Professor Karl Menger, Illinois Institute of Technology. (By invitation.)

(1) $\int \cos x dx = \sin x + c$ is correct when considered as an expression of the idea that any function having the cosine as its derivative differs from the sine by a constant function. As a formula connecting three terms (1) is false. Indeed, (1) connects neither three specific functions as does $\cos^2 x = 1 - \sin^2 x$ for any x, nor function variables, as does the inequality $-f^2(x) \leqq \sin^2 x + c^2$ for any x. Here f may be replaced by any function, *e.g.*, $f(x)$ by $\sin x+7$, and c by any constant function, say 6. However, in (1) if $\int \cos x dx$ is replaced by $\sin x+7$, then c must be replaced by 7 only.

Let $\mathbf{D}_{(a,b)}^{-1}f$ denote the *definite antiderivative* of f that assumes the value b at a; $\{\mathbf{D}^{-1}f\}$, the *antiderivative class* consisting of all antiderivatives of f; and $\mathbf{D}^{-1}f$, an *indefinite antiderivative, i.e.*, a function variable that may be replaced by any function in $\{\mathbf{D}^{-1}f\}$. Each of these three concepts lends itself to a correct formulation of the idea behind (1).

For details concerning antiderivatives, their connection with integrals, and the application of the latter to the cumulation of one variable quantity with regard to another variable quantity, cf. the author's book, *Calculus. A Modern Approach*, Boston, 1955.

2. *An inequality involving geometric and arithmetic means*, by Professor E. R. Johnston, Wisconsin State College, Whitewater.

It is well known that the geometric mean of a set of positive numbers is bounded above by their arithmetic mean; for example, see Pólya and Szego: *Aufgeben und Lehrsatze aus der Analysis*, Dover, vol. I, p. 50. This paper gives a strictly elementary proof of the following proposition, which is believed to be new, and which expresses a lower bound for the geometric mean in terms of the arithmetic mean.

If $0<P\leqq X_i \leqq Q$, $i=1, 2, 3, \cdots, N$, then

$$\left[\prod_{i=1}^{N} X_i\right]^{1/N} \geqq P^{(Q-\xi)/(Q-P)}Q^{(\xi-P)/(Q-P)},$$

where

$$\xi = \frac{\sum_{i=1}^{N} X_i}{N}.$$

3. *The use of high speed computers in solving problems*, by Professor P. C. Hammer, University of Wisconsin.

The high speed computing machines make it possible to utilize more fully the mathematical theories of the past in applications of various kinds. It is now possible to solve feasibly rather large linear equation systems, to invert matrices, to approximate numerically the solutions of ordinary differential equation systems and of certain partial differential equations. In part the high speed computing machine has made mathematics more useful and in part it has revealed a need for much sharper theories in some directions. Except for linear systems, no effective theory of roundoff errors has been given. Despite the advertising, it is very simple to pose problems not feasible for any existing computing machine.

4. *Mathematics in the social sciences*, by Mr. W. A. Golomski, Marquette University.

5. Panel Discussion: *Problems of the colleges in regard to the high school mathematics preparation of their students.*

Leaders: Professor J. J. McLaughlin, Wisconsin State College, River Falls; Professor E. G. Harrell, Wisconsin State College, Platteville; Professor H. P. Evans, University of Wisconsin.

Professor McLaughlin presented a brief report of the findings of the college section of the Wis-

consin State-wide Mathematics Curriculum Committee in their study of the mathematics curriculum in the secondary school for college preparatory students.

Professor Evans pointed out that the placement of entering freshmen at the University of Wisconsin in the mathematics sequence, based on the students' records and the result of a machine scored placement test, is considerably lower than it would be if the placement were based on the high school record of mathematical preparation only.

Professor Harrell called attention to the following problems: (a) since state colleges must accept any one presenting a high school diploma, the mathematics background of students varies greatly requiring a diversified college mathematics program; (b) the lack in the student of a will to learn something; (c) the difficulty with which the student changes from supervised to independent study habits; (d) lack of uniformity of curricula and pre-requisites.

The consensus of opinion seemed to be that the colleges should provide two or more track courses but not offer high school mathematics courses, do less coddling and provide more incentive.

SISTER MARY FELICE, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The ninth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of British Columbia, Vancouver, British Columbia, on June 17, 1955, in conjunction with the five hundred fifteenth meeting of the American Mathematical Society. Professor Ivan Niven, Chairman of the Section, presided at the meetings.

There were 71 persons in attendance, including the following 43 members of the Association:

C. B. Allendoerfer, J. P. Ballantine, R. A. Beaumont, J. L. Brenner, L. G. Butler, Harold Chatland, Paul Civin, P. A. Clement, D. F. Coulter, Jr., D. B. Dekker, Douglas Derry, W. H. Gage, H. M. Gelder, K. S. Ghent, S. G. Hacker, Mary E. Haller, Edwin Hewitt, E. Y. Hill, T. E. Hull, Burrowes Hunt, H. H. Irwin, D. H. Jones, E. S. Keeping, J. M. Kingston, V. L. Klee, Jr., M. S. Knebelman, A. E. Livingstone, A. T. Lonseth, J. E. Maxfield, Granville McCormick, L. V. Mead, A. F. Moursund, B. N. Moyls, D. C. Murdoch, Ivan Niven, T. G. Ostrom, C. A. Pursel, J. L. Simpson, P. O. Steen, C. A. Swanson, J. R. Vatnsdal, R. T. Wallace, R. M. Winger.

Following a joint dinner meeting with the American Mathematical Society a business meeting was held in the evening in Brock Hall. The following officers were elected: Chairman, Professor D. C. Murdoch, University of British Columbia; Vice-Chairman, Professor S. G. Hacker, Washington State College; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon. Reports of the High School Mathematics Contests in British Columbia, Washington and Oregon were received. A program committee consisting of Professor Arvid Lonseth, Professor Ostrom, Professor Murdoch and Professor Beaumont was appointed. Since the national meeting of the Association is to be held in Seattle in August, 1956, it was decided that no Pacific Northwest Section meeting will be scheduled for 1956.

The afternoon session consisted of two fifteen minute papers, an invited hour address and a symposium on freshman mathematics.

1. *An interpretation of differentials*, by Professor C. B. Allendoerfer, University of Washington.

Let $f(x)$ be a differentiable function of the n-variables x^i, and let $\xi^i(x)$ be an arbitrary contravariant vector field. Then the differential of $f(x)$ is defined to be $df(x) = (\partial f(x)/\partial x^i)\xi^i(x)$. This definition is invariant under coordinate transformations. The second differential is given by $d_{\eta\xi}f = (\partial^2 f/\partial x^i \partial x^j)\xi^j\eta^i + (\partial f/\partial x^j)(\partial \xi^j/\partial x^i)\eta^i$; this also is an invariant definition. When ξ^i and η^i are both constant in the x-coordinate system $d_{\eta\xi}f = d_{\xi\eta}f = (\partial^2 f/\partial x^i \partial x^j)\xi^j\eta^i$; this is not an invariant statement. If the first differentials of n independent functions are given, the vector $\xi^i(x)$ can be obtained. Hence vectors and differentials are equivalent ideas. The modern definition of a tangent vector introduced by Chevalley is pertinent here.

2. *S-numbers, an extension of Fermat numbers* (preliminary report), by J. E. and Margaret W. Maxfield, Naval Ordnance Test Station, China Lake, California, presented by Mrs. Maxfield.

Let s be a positive integer. An integer N will be called an s-number if N is greater than s and $a^{N-s} \equiv 1 \pmod{N}$ for every integer a prime to N. It is shown that an infinitude of s-numbers exist for each s.

The problem of s-numbers is linked to investigation of $\lambda(N-s)$ and to the relationship between λ and ϕ. It is shown that the highest power of an odd prime that can divide an s-number is one more than the highest power that divides s. It is shown that if an s-number is a product of two primes p, q, then $p+q \leqq s+1$. Conditions are found for s-numbers to have the form $p^k q^m$ when $s = p^i q^j$ and the form $2^k p^m$ when $s = 2^i p^j$.

3. *Statistical decisions*, by Professor E. S. Keeping, University of Alberta. (By invitation.)

Wald's decision theory ranks, along with Fisher's small-sample theory and the Neyman-Pearson theory of estimation, as one of the great advances in mathematical statistics. The relation of decision theory with the theory of games is discussed, the statistician being supposed engaged in a game with Nature, in which either side may employ pure or mixed strategies. Wald's minimax principle is compared with other decision criteria, including those of Laplace, Savage and Hurwicz.

Examples are given in which the minimax principle leads to a different estimate from that obtained by the customary procedures. A problem in sequential analysis is treated from the point of view of decision theory.

4. *Symposium on freshman mathematics*. Moderator: Professor M. S. Knebelman, Washington State College.

Speakers:

(1) Professor S. G. Hacker, Washington State College.

Two freshman mathematics populations were recognized: (1) students who plan to take only one year of mathematics and (2) students who will take non-elementary mathematics. A proposal was made that all member institutions of this Section consider giving a uniform test to determine capabilities of students for admittance to first year "college" mathematics. The training of all students should be mathematically sound with emphasis on (1) mathematical concepts of fundamental significance, (2) methods of mathematical proof with rigor commensurate with the student's capabilities and mathematical maturity and (3), less importantly, technique. No explicitly topical program of instruction was given, but illustrative ideas and proofs were cited.

(2) Professor Edwin Hewitt, University of Washington.

The following remarks are based on the writer's experience in teaching a liberal arts course in mathematics to very heterogeneous classes at the University of Washington. It was found that the calculus of propositions and simple applications to verbal problems are easily learned. The logic of "$\forall$" and "$\exists$" is grasped by some but not all. The axiomatic theory of fields, suitably disguised as a rigorous discussion of elementary algebra, excites great interest. Finite fields, however, offer serious difficulties to many. Deductive systems, with a rigorously defined language and axioms,

are definitely accessible. Curiously, the theory of finite groups seems hard, although in a class of 25 there are usually 2 or 3 students who can compute all groups of order 6, thus getting an automatic "A."

(3) Professor A. F. Moursund, University of Oregon.

The freshman mathematics course for liberal arts students should be reasonably satisfactory for: the large group of students who will take no further work in mathematics; those students, principally physical science majors, who will take calculus; an increasing number of students, principally social and biological science majors, who will take some work in statistics. The author proposes a general course, more of the traditional type than that included in the recent Allendoerfer and Oakley text. He advocates that the usual treatments of trigonometry and analytic geometry be greatly curtailed; that certain traditional algebraic topics be omitted and others simplified; that some work in the calculus and in elementary statistics be included; and that basic mathematical ideas, the nature of mathematical proof, and understanding be emphasized along with the attainment of fair manipulative skill.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA, South Dakota State College, Brookings, October 15, 1955.

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April, 1956.

OKLAHOMA, Oklahoma City University, October 28, 1955.

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

ROCKY MOUNTAIN

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

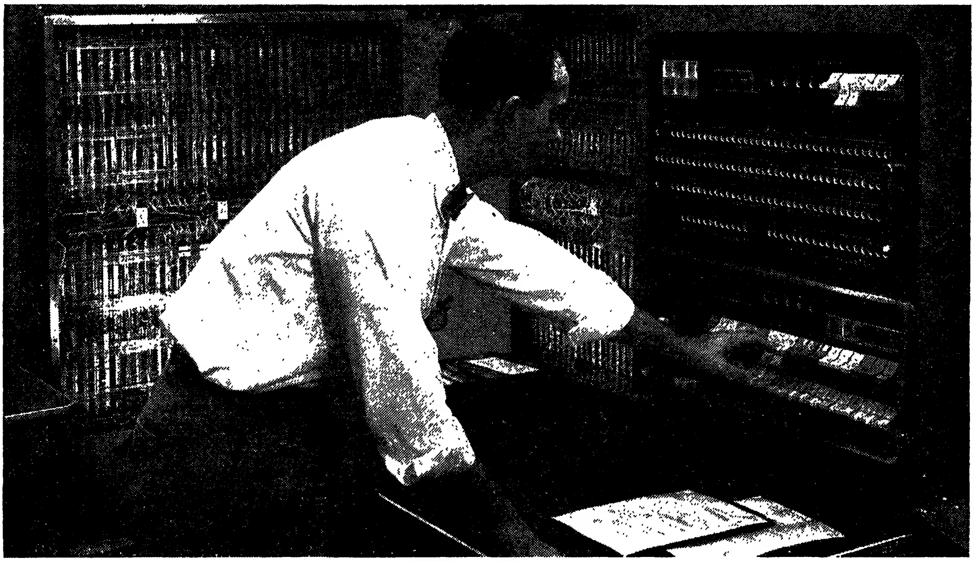
SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

TEXAS, Southwest Texas State Teachers College, San Marcos, April, 1956.

UPPER NEW YORK STATE, Alfred University, Alfred, April 28, 1956.

WISCONSIN, Marquette University, Milwaukee, May, 1956.



Mathematical Analyst *Keith Kersery loads jet transport flutter problem into one of Lockheed's two 701's. On order: two 704's to help keep Lockheed in forefront of numerical analysis and production control data processing.*

704's and 701's speed Lockheed research in numerical analysis

With two 701 digital computers already in operation, Lockheed has ordered two 704's to permit greater application of numerical analysis to complex aeronautical problems now being approached. Scheduled for delivery early next year, the 704's will replace the 701's.

Much of the work scheduled or in progress is classified. However, two features are significant to the career-minded Mathematical Analysts: 1) the wide variety of assignments created by Lockheed's diversified development program and 2) the advanced nature of the work, which falls largely into unexplored areas of numerical analysis.

Career positions for Mathematical Analysts. Lockheed's expanding development program in nuclear energy, turbo-prop and jet transports, radar search planes, extremely high-speed aircraft and other classified projects has created a number of openings for Mathematical Analysts to work on the 704's.

Lockheed offers you attractive salaries, generous travel and moving allowances which enable you and your family to move to Southern California at virtually no expense; and an extremely wide range of employe benefits which add approximately 14% to each engineer's salary in the form of insurance, retirement pension, etc.

Those interested in advanced work in this field are invited to write E. W. Des Lauriers, Dept. MA-33-9.

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A WAY TO SIMPLIFY TRUTH FUNCTIONS

W. V. QUINE, Harvard University

The quest is herewith resumed of a convenient technique for converting a truth-functional formula into its shortest equivalent in alternational normal form. We may, as before,* confine our attention to formulas given in alternational normal form. These may be described as comprising all *literals* (letters and negations of letters), and more generally all *fundamental formulas* (literals and conjunctions of literals, containing no letter twice), and more generally all alternations of fundamental formulas.

It will also be convenient, for purposes of the technique here to be developed, to exclude from consideration those formulas which are valid, or tautologous. A formula can be quickly tested for validity, and if found valid it can be rewritten in simplest form, '$p \vee \bar{p}$', out of hand. So let us assume hereafter that the formulas under investigation are in alternational normal form but not valid.

Where ϕ and ψ are fundamental formulas, we say that ϕ *subsumes* ψ if all the literals whereof ψ is a conjunction are among the literals whereof ϕ is a conjunction. We call ϕ a *prime implicant* of a formula Φ if ϕ implies Φ and subsumes no shorter formula which implies Φ. Now any shortest equivalent (in alternational normal form, as usual) of a formula Φ is an alternation of prime implicants of Φ (PSTF, p. 524); so a major part of the job of finding a shortest equivalent of Φ is the eliciting of all the prime implicants of Φ. A drawback of the procedure in PSTF, there remarked upon (p. 531), was that the prime implicants were exhausted only with help of a preliminary expansion into the cumbersome "developed normal form." Now, on the other hand, a speedy and direct method will be explained for getting all the prime implicants. Φ can be transformed into the alternation of all its prime implicants simply by continued use of the following operations (i) and (ii).**

(i) *Drop these obvious superfluities:* If one of the clauses of alternation subsumes another, drop the subsuming clause. Also supplant $\alpha \vee \bar{\alpha}\phi$ by $\alpha \vee \phi$ (and $\bar{\alpha} \vee \alpha\phi$ by $\bar{\alpha} \vee \phi$), where α is a single letter.

(ii) *Adjoin, as an additional clause of alternation, the* consensus *of two clauses.* Definition: The conjunction $\phi\psi$ (with any duplicate literals deleted) is called the *consensus* of $\alpha\phi$ and $\bar{\alpha}\psi$, provided that it contains no letter both affirmed and negated. The operation (ii) is to be regarded as not applying in case the

* *The problem of simplifying truth functions*, this MONTHLY, vol. 59, 1952, pp. 521–531; cited hereafter as PSTF, but not drastically presupposed.

** *Note added June 7, 1955:* It has today come to my attention, more than two months after submission of the present paper, that this result was anticipated in an Air Force memorandum of April 1954 by Edward W. Samson and Burton E. Mills, *Circuit minimization: algebra and algorithms for new Boolean canonical expressions*, AFCRC Technical Report 54-21. For my present paper I would still plead brevity and perspicuity, and certain novelties in the later portion; but to Samson and Mills belongs the credit for discovering that the alternation of prime implicants can be got by (i) and (ii).

consensus subsumes a clause already present; otherwise we could get an unending oscillation of (i) and (ii).

The two operations are to be performed as long as possible. (i), in particular, is to be performed as much as possible before and after each performance of (ii). When neither is applicable further, then, as will be proved, we have the alternation of all and only the prime implicants. First let us see an example.

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee prs \vee qr\bar{s} \vee pqrt. \tag{1}$$

By (i) we drop 'prs', which subsumes 'ps'. By (ii), next, we add on the consensus 'pqr' of 'ps' and '$qr\bar{s}$'. Result:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqrt \vee pqr. \tag{2}$$

By (i) now we drop '$pqrt$', which subsumes 'pqr'. Result:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr. \tag{3}$$

By (ii), finally, we add on the consensus '$r\bar{s}t$' of '$\bar{q}t$' and '$qr\bar{s}$', and also the consensus 'prt' of '$\bar{q}t$' and 'pqr', obtaining:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee r\bar{s}t \vee prt. \tag{4}$$

Here the process ends. No further pair of clauses has a consensus, except such as would subsume (indeed match) an existing clause.

Now to the proof that (i) and (ii) deliver all the prime implicants. This will be proved by proving that any (non-valid) formula Φ is still susceptible to an application of (i) or (ii) as long as there is a prime implicant χ of Φ which is not a clause of Φ.

Φ is implied by χ, yet not valid; so χ must have letters in common with Φ. Moreover, since χ is a *prime* implicant, it has no letters foreign to Φ; for, any such could be dropped without impairing the implication.* Moreover, since χ is a prime implicant of Φ, and each clause of Φ also implies Φ, no clause of Φ other than χ itself is subsumed by χ; hence none, since χ is not a clause of Φ. So there is at least one fundamental formula (χ itself, for one) fulfilling these three conditions: (a) it subsumes χ, (b) it subsumes no clause of Φ, and (c) it contains only letters of Φ. Let ψ be a longest fundamental formula fulfilling (a), (b), and (c). Still ψ will lack some letter α of Φ. (For, if ψ contained all letters of Φ, then, by (b), ψ would conflict with each clause of Φ in point of the affirming or negating of some letter or other; whereas we know rather, by (a), that ψ implies Φ.) Now since ψ is a longest formula fulfilling (a), (b), and (c), the longer formulas $\alpha\psi$ and $\bar{\alpha}\psi$ must fail to fulfill (b); for they do fulfill (a) and (c). So $\alpha\psi$ and $\bar{\alpha}\psi$ each subsumes a clause of Φ. These subsumed clauses must contain α and $\bar{\alpha}$ respectively, since they were not subsumed by ψ alone. But these clauses

* The reasoning in this sentence and the preceding one is an improvement on the proof of Theorem 2 in PSTF, where the case of prime implicants consisting of a single literal was overlooked. Strictly speaking, Theorem 2 fails for tautologies, since any literal is by definition a prime implicant of any tautology.

are not simply α and $\bar{\alpha}$, or Φ would be valid. So there are just three possible cases: the clauses are respectively $\alpha\phi$ and $\bar{\alpha}$, where ψ subsumes ϕ (Case 1); or they are α and $\bar{\alpha}\phi'$, where ψ subsumes ϕ' (Case 2); or they are $\alpha\phi$ and $\bar{\alpha}\phi'$ (Case 3). In Case 1, however, Φ contains $\bar{\alpha} \vee \alpha\phi$ and is accordingly susceptible to operation (i). Similarly for Case 2. In Case 3, finally, Φ is susceptible to an application of (ii), consisting in the adding on of the consensus of $\alpha\phi$ and $\bar{\alpha}\phi'$. This consensus, namely $\phi\phi'$ (minus any duplicate literals), is readily seen to meet the requirements of (ii): it contains no letter both affirmed and negated, since it is subsumed by a fundamental formula ψ; and it subsumes no clause of Φ, since ψ subsumed none.

This completes the proof that (i) and (ii) deliver *all* the prime implicants. The proof of the converse, namely that (i) and (ii) when continued as long as possible yield an alternation of prime implicants *only*, can now be added in a few words. Since $\alpha\phi \vee \bar{\alpha}\psi$ is equivalent to $\alpha\phi \vee \bar{\alpha}\psi \vee \phi\psi$, clearly (ii) is, like (i), an equivalence transformation. Accordingly the alternation obtained from a formula Φ by applying (i) and (ii) as long as possible can be depended upon to be an alternation of clauses each of which implies Φ. But, as we just finished proving in the preceding paragraph, every prime implicant is a clause. Accordingly any clause that is not a prime implicant subsumes another clause which is a prime implicant. But this cannot happen; (i) would be applicable again.

I shall discuss, for the remainder of the paper, the business of moving from the alternation of all prime implicants to a shortest equivalent (as always, in alternational normal form). This is wholly a matter of dropping dispensable clauses. There is, moreover, this *test of dispensability* of a single clause ϕ: see whether ϕ implies the remainder, Ψ, of the alternation. This may be quickly decided by testing Ψ for truth when the letters affirmed in ϕ are marked true and those negated in ϕ are marked false.

The dropping of one dispensable clause, however, can render another originally dispensable clause indispensable to the remaining alternation. We want rather to find the largest simultaneously dispensable combination of clauses. Certain aids to this end will now be noted.

We can get a head start by reviewing the applications of (ii) which were made in arriving at the alternation of all prime implicants. We bracket out, in a body, all clauses that were added by (ii) after the last application of (i). In our example we get:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s} \vee pqr \vee [r\bar{s}t \vee prt]. \tag{5}$$

These brackets serve as a reminder that the clauses enclosed are dispensable, and not only singly but jointly; for, the alternation which includes those bracketed clauses was originally got, from itself minus those clauses, by an equivalence transformation.

We must not, however, without further check, bracket a clause ϕ ('pqr', in the example) which was added by (ii) prior to a use of (i); for the subsequent use of (i) may, by banishing another clause, have rendered ϕ indispensable.

The next move, rather, is to subject each unbracketed clause to the test of dispensability formulated above, and bracket individually each clause which meets the test. The result, in the case of (5), is:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee [qr\bar{s}] \vee [pqr] \vee [r\bar{s}t \vee prt]. \tag{6}$$

This done, we can conclude a good deal about any shortest equivalent. It is bound to retain the unbracketed part. (This is what was called the *core* in PSTF, p. 527.) Its remaining clauses, if any, will be certain of those shown in brackets; and not all of them, since any one bracketed portion, at least, is dispensable. If, as commonly happens, there are no bracketings or just one, then the shortest equivalent is the core itself. It is only in cases such as (6), with an unusual lot of bracketing, that any serious exhaustion of possibilities remains to be done. An approach to such cases (doubtless susceptible, however, to streamlining) is the following.

A bracketed clause ϕ may or may not imply the core; all we know is that ϕ implies the whole alternation minus ϕ itself. If ϕ does imply the core, then ϕ should be cancelled for good. Any shortest equivalent, retaining the whole core as it does, is bound to omit any clause that implies the core. Such clauses are *absolutely* dispensable—independently of the omission or retention of other bracketed clauses.

So each bracketed clause should be tested individually (in the quick way lately mentioned) to see if it implies the core; and each which does should be deleted. In our example (6), none of the four bracketed clauses proves to imply the core. But here is an example where the phenomenon does occur:

$$pq \vee pr \vee \bar{p}s \vee \bar{r}t \vee [pt] \vee [qs].$$

Here, in fact, each of the bracketed prime implicants proves absolutely dispensable; only the core remains.

In general the remaining task, if any, in finding the shortest equivalents of a formula, is to test combinations of bracketed passages for joint dispensability. This can be done as follows. To see whether $\phi_1, \cdots, \phi_n$ are jointly dispensable in $\Psi \vee \phi_1 \vee \cdots \vee \phi_n$, hence whether $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ, check separately for each i (by the quick method noted earlier) to see whether ϕ_i implies Ψ. If ϕ_i implies Ψ for each i, then and only then $\phi_1 \vee \cdots \vee \phi_n$ implies Ψ and is dispensable *in toto*.

There is evident strategy in testing big combinations ahead of smaller ones. Only partial combinations want testing, however; the preceding search for absolutely dispensable clauses was *ipso facto* a test of whether the *whole* bracketed portion was dispensable.

In the case of (6), what we find is that '$qr\bar{s} \vee r\bar{s}t \vee prt$' implies the alternation of the remaining four clauses, and similarly for '$pqr \vee r\bar{s}t \vee prt$'; so we end up with two shortest equivalents:

$$ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee pqr, \qquad ps \vee \bar{p}\bar{s} \vee \bar{q}t \vee qr\bar{s}.$$

Despite the evident advantage of our new method over the method of developed normal forms and tables in PSTF, we should continue to exploit the separation expedient noted on page 529 of PSTF when we can.

ON SPHERICAL DRAWING AND COMPUTATION

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1. Introduction. Pictorial spherical diagrams (Figure 1) are tedious to draw by conventional methods and, as a result, are frequently found approximated or sketched, even in textbooks and treatises. Such diagrams will be recognized as axonometric projections of a sphere intersected by planes passing through its center. Besides their pictorial value, such diagrams provide a means for the geometrical solution of spherical triangles.

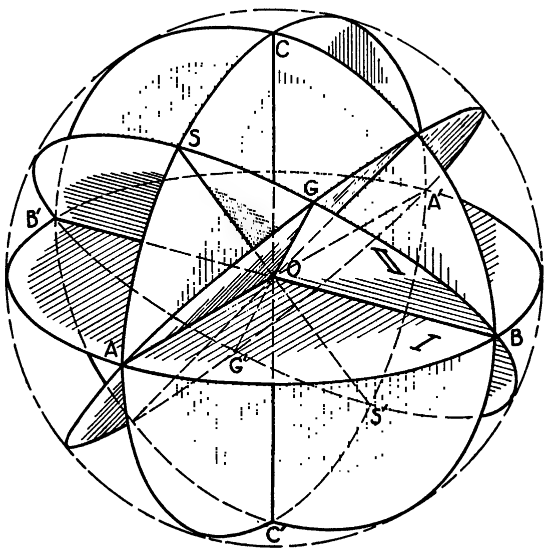

FIG. 1

The planes shown in Figure 1 intersect the sphere in great circles, and these appear on the drawing as *derived ellipses* of the respective great circles *primitive* to them. Orthogonal diameters of the great circles appear as *conjugate diameters* of the derived ellipses, for example, AA', CC'. The dihedral angle between two planes (I and II) having a given line of intersection (BB') is measured by the intercepted arc (AS) of the great circle ($ACA'C'$) normal to BB'.

Two characteristic problems are involved in drawing diagrams like Figure 1:

(1) Drawing an ellipse, given a pair of diameters. These are usually conjugate diameters, like AA', CC', but may be non-conjugate, like AA', GG'. The required ellipse will represent the great-circle intersection of the sphere with the plane determined by the given diameters. (2) Laying off on an ellipse an arc having a given angular measure on the primitive great circle. One important application of this problem is to normal great circles like $ACA'C'$, whereby a plane (II) making the given angle (AS) with a second plane (I) is determined.

Simple and rapid methods are described here for solving these problems using a grid of spherical coordinates slipped under the tracing paper (Figure 2). Conventional methods for solving problem 1 can do so only when the given diameters are conjugate, but the grid method can handle the general case.

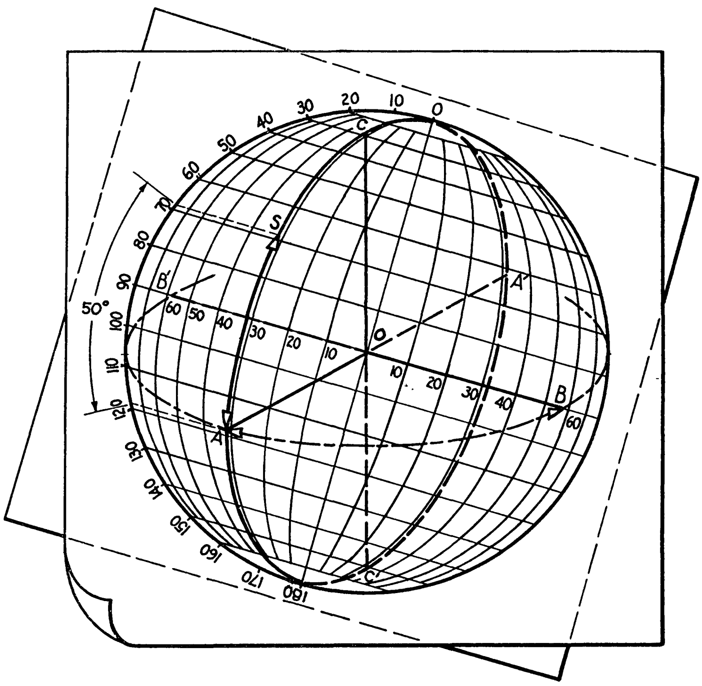

FIG. 2

2. Starting the drawing. The drawing is usually started with the projection of a unit sphere containing three orthogonal diameters (AA', BB', CC') representing three unit coordinate axes (Figure 3). The trimetric system shown is a convenient one for general use. Dotted lines show radii behind the plane of the circle representing the sphere outline (the so-called *meridian* plane). In the *isometric* system the axes are equally spaced (120°) about the center, with length of each equal to 0.815 of the diameter. Methods for computing such systems are properly part of the doctrine of axonometric scaling, and are discussed later. It is noteworthy that the grid shown in Figure 2 provides the simplest method

of computing axonometric scales.

3. Characteristic problem 1. To show the great-circle intersections of planes determined by the coordinate axes with the sphere brings up the first characteristic problem in the common form: Given the conjugate diameters AA', CC', draw the ellipse. To do this, the sphere is centered on a grid of the same size, and rotated until some one ellipse of the grid, interpolated if necessary, passes through all of the diametral end-points, A, C, A', C'. There are actually two ellipses that will do this on the drawing sheet, but one of them will be absurd so far as passing through the given points in space. (Monitor by considering visibility relative to the meridian plane.)

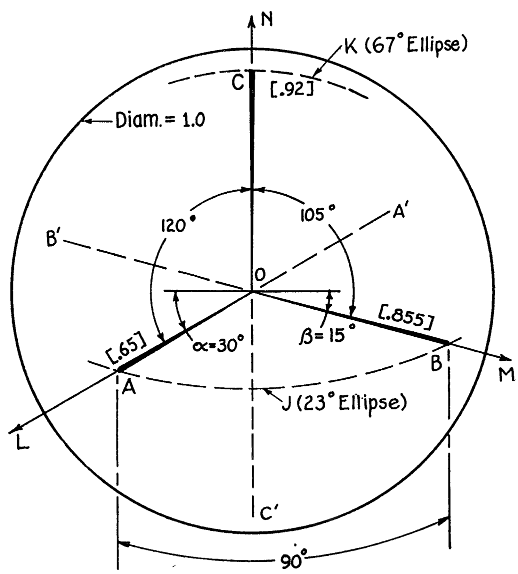

FIG. 3

Conventional methods for solving this problem will be found in books on drafting (see Bibliography at end of article). Two of these methods, the so-called "parallelogram" and "circle" methods, involve point-by-point plotting of the required ellipse. In a third, the "trammel" method, a triangle is constructed which is moved in a certain manner to produce the required ellipse as a locus.

The grid method will work equally well for any pair of diameters, whether or not conjugate. In Figure 1 is shown the ellipse $AGA'G'$ established from the non-conjugate diameters AA', GG'.

For later use, notice that the minor axis of the ellipse $ACA'C'$, lying along the equatorial line of the grid, is in the direction of the coordinate axis BB', thus representing perpendicularity in space relative to the primitive of the ellipse. (Do not confuse axes of the ellipse with coordinate axes of the diagram or with conjugate diameters.)

4. Characteristic problem 2. Given the ellipse $ACA'C'$, to lay off an arc AS representing 50° on the primitive great circle. When chords are drawn in the direction of the minor axis of an ellipse representing a great circle of a sphere, they cut arcs of equal angular measure from the ellipse and the sphere-outline circle. Accordingly, the sphere is centered on the grid and rotated until the ellipse $ACA'C'$ coincides with some one ellipse of the grid, interpolated if necessary. Using the parallels of the grid as projectors, the required arc on the ellipse is then established from the graduated outline.

The arrangement in Figure 2 illustrates, in passing, a second and faster method of laying out the particular value 90° if there is represented on the drawing a diameter perpendicular in space to the primitive of the given ellipse. The given ellipse is in this case $ABA'B'$ (not $ACA'C'$), the diameter through the initial point is AA', and the diameter perpendicular in space to the primitive of $ABA'B'$ is CC'. The grid and drawing are rotated until some one ellipse ($ACA'C'$) of the grid passes through all of the end points of the two orthogonal diameters AA', CC'. The equatorial line of the grid, being perpendicular in space to each of these diameters, then intersects the ellipse $ABA'B'$ at the required point B.

5. Computation of sphere-and-axes systems. The fundamental problem of axonometric scaling is to compute the relative projected lengths of unit axes orthogonal in space, given their directions on the drawing. The result is a set of *axonometric scales*. In drawing spherical diagrams, the relative true length of the unit is also needed, to establish the diameter of the sphere.

There is to be found in the literature a geometrical construction yielding true and projected lengths (see Bibliography), which will not be repeated here. Another method is to compute them from the following equations:

$$AA':BB':CC':\text{Diam.} = \sqrt{\sin 2\beta}:\sqrt{\sin 2\alpha}:\sqrt{\sin(2\alpha+2\beta)}$$
$$:\sqrt{\tfrac{1}{2}[\sin 2\beta+\sin 2\alpha+\sin(2\alpha+2\beta)]}.$$

If one starts with a set of axonometric scales, such as are found ready-made in drafting books, the relative true length of the unit is thus given by the square root of half the sum of squares of the given scale values.

The simplest method of computing sphere-and-axes systems, or axonometric scales alone, is with the grid, as shown in Figure 3. The grid is centered under the intersection (O) of the three given direction lines (OL, OM, ON) with its parallels vertical. An ellipse is located whose intersections (A, B) with the two non-vertical direction lines (OL, OM) are 90° apart as indicated by the parallels. These intersections establish the required unit lengths in these directions. The point establishing the unit length in the vertical direction is the intersection (C) of the vertical direction line (ON) with the ellipse (K) whose degree reading is the complement of that of the 23°-ellipse (J), that is, the 67°-ellipse.

6. Geometrical solution of spherical triangles. If the triangle is placed as shown in Figure 4, and only elements x, y, z, X, and Z are used, the parts can be measured or laid off directly using the grid. Element y is measured from the outline circle, elements z and X when the grid has its common major axis along UV, and elements x and Z when it is along $U'V'$. To avoid use of element Y may require working the problem a second time after reassignment of knowns and unknowns. The additional construction required to measure Y directly is shown in the figure, where DUP and $DU'T$ are 90°-arcs, and the ellipse $FPTE$ has its minor axis in the direction OD. The arc TP provides the required angular measure of Y.

When solving a triangle given the three angles, the three sides of the corresponding polar triangle are calculated as the respective supplements of the angles of the given triangle. The polar triangle is solved for its angles, using the method given, and the supplements of these are then calculated for the required sides of the given triangle.

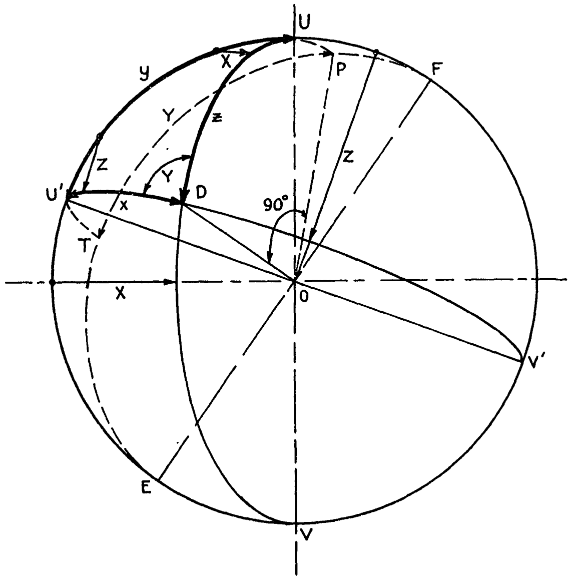

FIG. 4

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ON THE POSTULATES DEFINING A GROUP

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1. Introduction. Although a group is usually defined in terms of the operation $x+y$, when written additively, there are some concepts connected with a group, such as congruence and subgroup, which are more readily definable in terms of the inverse operation $x-y=x+(-y)$. It is possible to formulate the postulates for a group in terms of the latter operation and then to define the former operation in terms of the latter. The object of this paper is to carry out that formulation and to consider some of its consequences.

2. The postulates. Let G be a set of elements over which is defined a binary operation written $x-y$ and satisfying, for any elements x, y, and z of G, the following postulates:

I. $x-y\in G$,

II. *There is an element* 0 *of* G *such that* $x-y=0$ *if, and only if,* $x=y$,

III. $(x-z)-(y-z)=x-y$.

This operation will be called *subtraction* and $x-y$ the *difference* of x and y. A second operation, called *addition* and written $x+y$, will be defined as

$$x + y = x - (0 - y). \tag{1}$$

THEOREM 1. *If G satisfies* I, II, *and* III, *and if addition is defined by* (1), *then G is a group. Conversely, if G is a group, then* I, II, *and* III *are satisfied.*

Proof. It is evident that if, in a group, $x-y$ is defined to be $x+(-y)$, then I, II, and III are satisfied. Conversely, G is obviously closed under the operation of addition defined by (1). Setting $z=x$ in III,

$$0 - (y - x) = x - y. \tag{2}$$

From III can be derived the related identities

$$(x + z) - (y + z) = x - y, \tag{3}$$

$$(x - z) + (z - y) = x - y. \tag{4}$$

A cancellation law also follows from II and III:

$$x - z = y - z \quad \text{if, and only if,} \quad x = y. \tag{5}$$

Setting $y=0$, $z=0$ in III,

$$(x - 0) - 0 = x - 0;$$

then from (5),

$$x - 0 = x.$$

Setting $y=0$ in (2),

$$0 - (0 - x) = x.$$

Now the element 0 is a left identity for x, and $0-x$ a left inverse of x, for

$$0 + x = 0 - (0 - x) = x,$$
$$(0 - x) + x = (0 - x) - (0 - x) = 0.$$

Setting $y=0$, $z=y$ in (3),

$$(x + y) - y = x.$$

The associative law now follows:

$$\begin{aligned} x + (y + z) &= ((x + y) - y) + (y - (0 - z)) \\ &= (x + y) - (0 - z) = (x + y) + z. \end{aligned}$$

This completes the proof.

In considering the logical independence of I, II, and III, we shall show that I, III, and the statements

IIa. *There is an element* 0 *such that if* $x-y=0$, *then* $x=y$,

IIb. *There is an element* 0 *such that if* $x=y$, *then* $x-y=0$

are logically independent. In fact, table A fails to satisfy I, table B with $a=0$ does not fulfill IIa, with $b=0$ it does not fulfill IIb, and table C fails to satisfy III, while in each case the three remaining postulates hold. The independence of IIa and IIb is to be interpreted in the sense that G may have an element 0 satisfying one of these postulates but not the other.

$-$	a	b	c
a	a	c	b
b	b	a	c
c	c	b	a

Table A.

$-$	a	b
a	a	a
b	a	a

Table B.

$-$	a	b	c
a	a	b	c
b	b	a	c
c	c	b	a

Table C.

3. Postulates for a commutative group. The commutative law may be written in the form

IV. $x-(x-y)=y$,

for IV is obviously equivalent to $x+y=y+x$ in any group.

THEOREM 2. *A necessary and sufficient condition for* G *to be a commutative group is that it satisfy* I, III, *and* IV.

Proof. It only remains to show that I, III, and IV imply II. We first observe that

$$(x - (x - y)) - (x - (x - y)) = x - x = y - y;$$

this element we shall call 0. Setting $y=x$ in IV,

$$x - 0 = x.$$

Now if $x-y=0$, then

$$x - (x - y) = x = y.$$

Tables A and C show that I, III, and IV are also logically independent.

The condition

V. $(x-z)-(x-y)=y-z$

resembles III in form but is really much stronger, as the following theorem shows.

THEOREM 3. *A necessary and sufficient condition for G to be a commutative group is that it satisfy* I, II, *and* V.

Proof. The condition is evidently necessary. Conversely, setting $x=0$ in V, we conclude that

(6) $$0 - y = 0 - z \quad \text{if, and only if,} \quad y = z.$$

Setting $y=0$, $z=x$ in V,

$$0 - (x - 0) = 0 - x.$$

From (6) it follows that

$$x - 0 = x.$$

Setting $z=0$ in V,

(7) $$x - (x - y) = y.$$

Finally, replacing x by $x-z$, y by $x-y$ in (7),

$$(x - z) - (y - z) = x - y.$$

This proves the sufficiency of the condition. Tables A, B, and C suffice to prove the logical independence of I, II, and V.

4. Finite group tables. Let G be a finite set of elements $a_1, \cdots, a_n$ over which is defined a subtraction, and let the differences a_i-a_j be arranged in a table in which

$$a_i - a_j = a_{ij}, \qquad (i, j = 1, \cdots, n).$$

It will be convenient, in deciding whether or not the table represents a group, to see if it satisfies I, II, and III. Postulates I and II can be checked by inspection: the elements on the main diagonal must be the same and must appear no-

where else in the table. In order for III to be fulfilled, the differences of corresponding elements in any two given rows must equal the difference of the row headings, that is,

$$a_{ik} - a_{jk} = a_i - a_j, \qquad (i, j, k = 1, \cdots, n).$$

If the table is to represent a commutative group, then it must satisfy V in addition to I and II, that is, the differences of corresponding elements in any two given columns must equal the difference of the column headings reversed:

$$a_{ij} - a_{ik} = a_k - a_j, \qquad (i, j, k = 1, \cdots, n).$$

Checking III or V is somewhat simpler than checking the associative law in the addition table.

The postulate

IIc. *There is an element* 0 *of* G *such that* $0-0=0$ *and such that if* $x-y=0$, *then* $x=y$,

although weaker than II in general, is equally strong in the case of finite groups, as the following theorem demonstrates.

THEOREM 4. *A necessary and sufficient condition for a finite set* G *to be a group is that it satisfy* I, IIc, *and* III.

Proof. Let the elements of G be $a_1, \cdots, a_n$. It remains only to show that $a_{ii}=0$ for $1 \leqq i \leqq n$. Let the elements of G be so ordered that $a_{ii}=0$ implies $1 \leqq i \leqq m$, and suppose that $m<n$. If $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, then

$$a_{ij} - a_{ij} = a_i - a_i = 0,$$

so that $a_{ij}=a_k$, where $1 \leqq k \leqq m$. Since 0 is not among the elements $a_{i,m+1}$ for $1 \leqq i \leqq m$, two of them must be equal, say $a_{r,m+1}=a_{s,m+1}$. But then

$$a_{r,m+1} - a_{s,m+1} = a_r - a_s = 0,$$

or $a_r=a_s$, which is a contradiction. Hence, $m=n$.

5. Refinement of the postulates. Let G be a set over which are defined an equivalence relation, called congruence and written $x \equiv y$, and an operation, written $x-y$, which satisfy I, II, III, with equality replaced by congruence in II and III, and the postulate

VI. *If* $v \equiv x$ *and* $w \equiv y$, *then* $v-w \equiv x-y$.

Then the equivalence classes of G form a group, by Theorem 1. In this sense, we shall say that G itself is a group.

Postulate VI can be written as the conjunction of the statements

VIa. *If* $v \equiv x$, *then* $v-w \equiv x-w$,

VIb. *If* $w \equiv y$, *then* $x-w \equiv x-y$,

LINEAR DIFFERENCE EQUATIONS AND THE DIRICHLET SERIES TRANSFORM

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The solution of linear differential equations by means of the Laplace transform is now commonplace. Linear difference equations with constant coefficients also can usually be solved by means of the Laplace transform. The Dirichlet series transform, however, is usually simpler and seems a much more natural instrument for the solution of difference equations. Apparently it has been overlooked.

We let t be an integer and then let

$$f(s) = D\{a(t)\} = \sum_{t=0}^{\infty} m^{-st}a(t), \qquad m > 1,\ s > 1. \tag{1}$$

This is the basic transformation of the paper. We assume the series convergent. We note in passing that the transformation is linear.

Let

$$\Delta F(t) = F(t+1) - F(t). \tag{2}$$

We call attention to the summation by parts formula

$$\sum_{t=a}^{b} u(t)\Delta v(t) = u(t)v(t)\Big]_a^{b+1} - \sum_{t=a}^{b} v(t+1)\Delta u(t). \tag{3}$$

We apply this formula to Dirichlet series (1). Let $u(t) = a(t)$,

$$\Delta v(t) = m^{-st}, \qquad v(t) = \frac{m^{-st}}{m^{-s}-1}\cdot$$

We get equation (4) below

$$f(s) = D\{a(t)\} = a(t)\frac{m^{-st}}{m^{-s}-1}\Big]_0^{\infty} - \sum_{t=0}^{\infty}\frac{m^{-s(t+1)}}{m^{-s}-1}\Delta a(t). \tag{4}$$

$$= -\,a(0)\frac{m^s}{1-m^s} - \frac{1}{1-m^s}\sum_{t=0}^{\infty} m^{-st}\Delta a(t).$$

We now repeat the operation of summation by parts on the final sum in formula (4). We notice that this sum is the same as that in (1) with $a(t)$ replaced by $\Delta a(t)$. We get the following result:

$$f(s) = -\,a(0)\frac{m^s}{1-m^s} + [\Delta a(0)]\frac{m^s}{(1-m^s)^2} + \frac{1}{(1-m^s)^2}\sum_{t=0}^{\infty} m^{-st}\Delta^2 a(t). \tag{5}$$

The operation can be repeated indefinitely, and we get the following formula:

$$(6)\quad \begin{aligned} f(s) = & -\frac{m^s}{1-m^s}a(0) + \frac{m^s}{(1-m^s)^2}\Delta a(0) - \frac{m^s}{(1-m^s)^3}\Delta^2 a(0) \\ & + \cdots + (-1)^n \frac{m^s}{(1-m^s)^n}\Delta^{n-1}a(0) + (-1)^n \frac{1}{(1-m^s)^n}\sum_{t=0}^{\infty} m^{-st}\Delta^n a(t). \end{aligned}$$

The next step is to solve for the respective transforms:

(7) $D\{a(t)\} = f(s)$

(8) $D\{\Delta a(t)\} = -m^s a(0) - (1-m^s)f(s)$

(9) $D\{\Delta^2 a(t)\} = -m^s\Delta a(0) + m^s(1-m^s)a(0) + (1-m^s)^2 f(s)$

. . . .

(10) $$\begin{aligned} D\{\Delta^n a(t)\} = & -m^s\Delta^{n-1}a(0) + m^s(1-m^s)\Delta^{n-2}a(0) \\ & + \cdots + (-1)^{n-2}m^s(1-m^s)^{n-3}\Delta^2 a(0) + (-1)^{n-1}m^s(1-m^s)^{n-2}\Delta a(0) \\ & + (-1)^n m^s(1-m^s)^{n-1}a(0) + (-1)^n(1-m^s)^n f(s). \end{aligned}$$

Even easier are the transforms of successive values. Let j be a positive integer;

$$(11)\quad \begin{aligned} D\{a(t+j)\} & = m^{js}\sum_{t=0}^{\infty} m^{-s(t+j)}a(t+j) \\ & = m^{js}\sum_{t=0}^{\infty} m^{-st}a(t) - m^{js}\sum_{t=0}^{j-1} m^{-st}a(t) \\ & = m^{js}D\{a(t)\} - m^{js}\sum_{t=0}^{j-1} m^{-st}a(t). \end{aligned}$$

We now prepare a table of transforms for the most frequently occurring functions.

I. $a(t) = 1$

$$D\{a(t)\} = \sum_{t=0}^{\infty} m^{-st} = \frac{m^s}{m^s-1}\cdot$$

II. $a(t) = t$

$$\begin{aligned} D\{a(t)\} & = \sum_{t=0}^{\infty} m^{-st}t = t\frac{m^{-st}}{m^{-s}-1}\Bigg]_0^{\infty} - \sum_{t=0}^{\infty}\frac{m^{-st}(t+1)}{m^{-s}-1} \\ & = -\frac{m^{-s}}{m^{-s}-1}\sum_{t=0}^{\infty} m^{-st} = \frac{1}{m^s-1}D\{1\} = \frac{m^{-s}}{(m^s-1)^2}\cdot \end{aligned}$$

III. $a(t) = t(t-1) = t^{(2)}$

$$D\{a(t)\} = \sum_{t=0}^{\infty} m^{-st}t^{(2)} = -\frac{1}{m^{-s}-1}\sum_{t=0}^{\infty} m^{-s(t+1)}2t$$

$$= -\frac{2m^{-s}}{m^{-s}-1}\sum_{t=0}^{\infty} m^{-st}t = \frac{2}{m^{s}-1}D\{t\} = \frac{2m^{s}}{(m^{s}-1)^{3}}.$$

. . . .

IV. $a(t) = t(t-1)\cdots(t-r+1) = t^{(r)}$

By the use of formula (3) as under III,

$$D\{a(t)\} = \frac{r}{m^{s}-1}D\{t^{(r-1)}\} = \frac{r!m^{s}}{(m^{s}-1)^{r+1}}.$$

V. $a(t) = t^{r}$ where r is a positive integer

Write t^{r} as a linear function of $t^{(r)}, t^{(r-1)}, \cdots, 1$ and use IV.

VI. $a(t) = k^{t}$ where $|k| < m^{s}$

$$D\{a(t)\} = \sum_{t=0}^{\infty}(m^{-s}k)^{t} = \frac{m^{s}}{m^{s}-k}.$$

VII. $a(t) = t^{(r)}k^{t}$

$$D\{a(t)\} = \sum_{t=0}^{\infty}(m^{-s}k)^{t}\cdot t^{(r)}.$$

Replace m^{-s} in IV by $m^{-s}k$. We get

$$D\{a(t)\} = \frac{r!m^{s}k^{r}}{(m^{s}-k)^{r+1}}.$$

VIII. $a(t) = \sin qt$

$$D\{a(t)\} = \frac{1}{2i}\sum_{t=0}^{\infty}m^{-st}e^{qit} - \frac{1}{2i}\sum_{t=0}^{\infty}m^{-st}e^{-qit}$$

$$= \frac{1}{2i}\sum_{t=0}^{\infty}(m^{-s}e^{qi})^{t} - \frac{1}{2i}\sum_{t=0}^{\infty}(m^{-s}e^{-qi})^{t}$$

$$= -\frac{1}{2i}\left[\frac{1}{m^{-s}e^{qi}-1} - \frac{1}{m^{-s}e^{qi}-1}\right] = \frac{m^{s}\sin q}{1+m^{2s}-2m^{s}\cos q}.$$

IX. $a(t) = \cos qt$

$$D\{a(t)\} = -\frac{1}{2}\left[\frac{1}{m^{-s}e^{qi}-1} + \frac{1}{m^{-s}e^{-qi}-1}\right] = \frac{m^{2s}-m^{s}\cos q}{1+m^{2s}-2m^{s}\cos q}.$$

X. $a(t) = t^{(r)}\sin qt$

This transformation is worked out by means of exponentials and VII. If

$r>1$ it is advised to replace $\sin qt$ by exponentials in the beginning and to carry the work through to the desired solution. A return can then be made to trigonometric functions if desired.

However, let

$$(m^s - \cos q + i \sin q)^{r+1} = X + Yi.$$

We then find

$$D\{a(t)\} = r!m^s \frac{Y \cos rq + X \sin rq}{(m^{2s} + 1 - 2m^s \cos q)^{r+1}}.$$

In particular,

$$D\{t \sin qt\} = \frac{m^{3s} \sin q - m^s \sin q}{(m^{2s} + 1 - 2m^s \cos q)^2}.$$

XI. $a(t) = t^{(r)} \cos qt$

$$D\{a(t)\} = r!m^s \frac{X \cos rq - Y \sin rq}{(m^{2s} + 1 - 2m^s \cos q)^{r+1}}.$$

In particular,

$$D\{t \cos qt\} = \frac{(m^{3s} + m^s) \cos q - 2m^{2s}}{(m^{2s} + 1 - 2m^s \cos q)^2}.$$

XII. $a(t) = k^t \sin qt,\ |k| < m^s$

$$D\{a(t)\} = \sum_{t=0}^{\infty} (m^{-s}k)^t \sin qt = \frac{m^s k^{-1} \sin q}{1 + m^{2s}k^{-2} - 2m^s k^{-1} \cos q}.$$

XIII. $a(t) = k^t \cos qt,\ |k| < m^s$

$$D\{a(t)\} = \frac{m^{2s}k^{-2} - m^s k^{-1} \cos q}{1 + m^{2s}k^{-2} - 2m^s k^{-1} \cos q}.$$

XIV. $a(t) = t^{(r)}k^t \sin qt,\ |k| < m^s$

Replace m^s in X by $m^s k^{-1}$.

In particular,

$$D\{tk^t \sin qt\} = \frac{m^{3s}k^{-3} \sin q - m^s k^{-1} \sin q}{(m^{2s}k^{-2} - 2m^s k^{-1} \cos q + 1)^2}.$$

XV. $a(t) = t^{(r)}k^t \cos qt,\ |k| < m^s$

Replace m^s in XI by $m^s k^{-1}$. In particular,

$$D\{tk^t \cos qt\} = \frac{(m^{3s}k^{-3} + m^s k^{-1}) \cos q - 2m^{2s}k^{-2}}{(m^{2s}k^{-2} - 2m^s k^{-1} \cos q + 1)^2}.$$

Example.

$$\Delta y(t) - y(t) = t.$$

Apply the Dirichlet series transformation to both sides of this equation and we get the following equation:

$$-y(0)m^s + (m^s - 1)f(s) - f(s) = \frac{m^s}{(m^s - 1)^2}\cdot$$

Solve this for $f(s)$. We get

$$\begin{aligned} f(s) &= \frac{m^s}{(m^s - 1)^2(m^s - 2)} + y(0)\frac{m^s}{m^s - 2} \\ &= -\frac{m^s}{m^s - 1} - \frac{m^s}{(m^s - 1)^2} + \frac{m^s}{m^s - 2} + y(0)\frac{m^s}{m^s - 2}\cdot \end{aligned}$$

We now refer to our tabulated transforms and obtain the solution of our equation:

$$y(t) = -1 - t + 2^t + y(0)2^t.$$

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

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NOTE ON A CONVERGENCE PROBLEM

MARSHALL FREIMER, Harvard University

In [1], A. Shields suggested the following as a Research Problem:

If $a_1 < a_2 < \cdots$ are positive integers, if C is compact, and if $\sin a_n x \to 0$ for all x in C, prove that the convergence must actually be uniform.

A simple counterexample to this assertion is afforded by taking $a_n = n!$ $(n = 1, 2, 3, \cdots)$ and C to be the set of points $0, \pi, \pi/2, \pi/3, \pi/4, \cdots$. Since $n!/m$ is an integer for $n > m$, $\sin a_n x = 0$ for all $n > N_x$ and $\sin a_n x \to 0$ for all x in C. Yet for any fixed n there is an x in C for which $\sin a_n x = 1$, namely $x = \pi/(2n!)$. Convergence is therefore not uniform.

Reference

1. Bull. Amer. Math. Soc., vol. 60, 1954, p. 589.

A NOTE ON HERMITE POLYNOMIALS

L. Carlitz, Duke University

Feldheim [2] has proved the formula

$$H_m(x)H_n(x) = \sum_{r=0}^{\min(n,n)} 2^r r! \binom{m}{r}\binom{n}{r} H_{m+n-2r}(x), \tag{1}$$

where $H_m(x)$ is the Hermite polynomial defined by

$$H_m(x) = \sum_{2s \leqq m} (-1)^s (2x)^{m-2s} \frac{m!}{s!(m-2s)!}. \tag{2}$$

For other proofs of (1) see Watson [3] and Burchnall [1]. We wish to point out that (1) can be proved rapidly by induction. Indeed if $U_\mu(x)$ denotes an arbitrary solution of the equation

$$U_{\mu+1} = 2xU_\mu - 2\mu U_{\mu-1}, \tag{3}$$

where μ is an arbitrary complex number, then we shall prove

$$H_m(x)U_\mu(x) = \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-2r}(x). \tag{4}$$

Since $H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x)$ it is clear that (4) includes (1).

Since the parabolic cylinder function $D_\mu(x)$ [4; Chapter 16] satisfies $D_{\mu+1}(x) = xD_\mu(x) - \mu D_{\mu-1}(x)$, it follows that $U_\mu = 2^{\mu/2} D_\mu(2^{1/2}x)$ satisfies (3).

We now prove (4). For $m=0$, (4) is obvious. For $m=1$, (4) reduces to (3). We accordingly assume the truth of (4) for the value m; then

$$H_{m+1}(x)U_\mu(x) = \{2xH_m(x) - 2mH_{m-1}(x)\} U_\mu(x)$$

$$= 2x\sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-2r}(x) - 2m \sum_{r=0}^{m-1} 2^r r! \binom{m-1}{r}\binom{\mu}{r} U_{m+\mu-1-2r}(x)$$

$$= \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} \{2xU_{m+\mu-2r}(x) - 2(m+\mu-2r)U_{m+\mu-1-2r}(x)\}$$

$$+ 2\sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} (\mu - r) U_{m+\mu-1-2r}(x)$$

$$= \sum_{r=1}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu+1-2r}(x) + \sum_{r=1}^{m+1} 2^r r! \binom{m}{r-1}\binom{\mu}{r} U_{m+\mu+1-2r}(x)$$

$$= \sum_{r=0}^{m+1} 2^r r! \binom{m+1}{r}\binom{\mu}{r} U_{m+\mu+1-2r}(x).$$

This evidently completes the induction.

In a similar way we can prove the companion formula

$$(5) \qquad U_{m+\mu}(x) = \sum_{r=0}^{m} (-1)^r 2^r r! \binom{m}{r}\binom{\mu}{r} H_{m-r}(x) U_{\mu-r}(x).$$

It follows from (4) that

$$(6) \qquad \frac{H_m(x)U_{\mu+1}(x) - H_{m+1}(x)U_\mu(x)}{2(m-\mu)} = \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-1-2r}(x),$$

$$\frac{(m-\mu+1)H_m(x)U_{\mu+2}(x) - 2(m-\mu)H_{m+1}(x)U_{\mu+1}(x) + (m-\mu-1)H_{m+2}(x)U_\mu(x)}{4(m-\mu+1)(m-\mu)(m-\mu-1)}$$

$$= \sum_{r=0}^{m} 2^r r! \binom{m}{r}\binom{\mu}{r} U_{m+\mu-2-2r}(x),$$

and so on.

Added in proof. The formulas (1), (4) and (5) occur in slightly different notation in a paper by N. Nielsen, *Recherches sur les polynomes d'Hermite*, Det kgl. Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser, I, 6, pp. 1–78, in particular pp. 31–33.

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1. J. L. Burchnall, A note on the polynomials of Hermite, Quart. J. Math. (Oxford), vol. 12, 1941, pp. 9–11.
2. E. Feldheim, Quelques nouvelles relations pour les polynomes d'Hermite, J. London Math. Soc., vol. 13, 1938, pp. 22–29.
3. G. N. Watson, A note on the polynomials of Hermite and Laguerre, J. London Math. Soc. vol. 13, 1938, pp. 29–32.
4. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th Edition, Cambridge, 1927.

NOTES ON MATRIX THEORY—VII

RICHARD BELLMAN, The Rand Corporation

1. Introduction. There is a meta-mathematical theorem to the effect that to every multiplicative theorem in analysis, there corresponds an additive one obtained by a limiting process. The purpose of this note is to show how this principle may be used to derive one inequality of Ky Fan's from another inequality of multiplicative type.

Let A, B, and $C=\lambda A+(1-\lambda)B$, for $0\leqq\lambda\leqq 1$, be three positive definite matrices with the respective sets of characteristic values (which we take to be arranged in increasing order), x_i, y_i, z_i. Let us define

$$(1.1) \qquad |A|_k = \prod_{i=1}^{k} x_i, \qquad |B|_k = \prod_{i=1}^{k} y_i, \qquad |C|_k = \prod_{i=1}^{k} z_i.$$

Then Ky Fan showed, [1] (see also [2], [3] for an alternate proof), that

$$(1.2) \qquad |C|_k \geqq |A|_k^{\lambda} |B|_k^{(1-\lambda)}.$$

In another paper, [4], he showed that the following inequality is also valid:

$$\sum_{i=1}^{k} z_i \geqq \lambda \sum_{i=1}^{k} x_i + (1 - \lambda) \sum_{i=1}^{k} y_i. \tag{1.3}$$

Let us now show that (1.3) is a consequence of (1.2).

2. Proof. Let us define three new positive definite matrices by means of the relations

$$X = I + \epsilon A, \qquad Y = I + \epsilon B, \qquad Z = I + \epsilon C, \tag{2.1}$$

where ϵ is a small positive quantity. The characteristic roots of X are $1+\epsilon x_i$, those of Y are $1+\epsilon y_i$, while those of Z are $1+\epsilon z_i$. Substituting in the relation in (2), we have, keeping only terms of the zero-th and first degree in ϵ,

$$1 + \epsilon\left(\sum_{i=1}^{k} z_i\right) \geqq \left(1 + \epsilon \sum_{i=1}^{k} x_i\right)^{\lambda}\left(1 + \epsilon \sum_{i=1}^{k} y_i\right)^{(1-\lambda)} + o(\epsilon) \tag{2.2}$$

or

$$1 + \epsilon\left(\sum_{i=1}^{k} z_i\right) \geqq \left(1 + \epsilon\lambda \sum_{i=1}^{k} x_i\right)\left(1 + \epsilon(1 - \lambda) \sum_{i=1}^{k} y_i\right) + o(\epsilon) \tag{2.3}$$

or

$$\sum_{i=1}^{k} z_i \geqq \lambda \sum_{i=1}^{k} x_i + (1 - \lambda) \sum_{i=1}^{k} y_i + o(1). \tag{2.4}$$

Allowing ϵ to approach zero, we obtain (1.3).

3. Remarks (suggested by referee). The inequality above is true for arbitrary symmetric matrices since we can always find a positive scalar c with the property that $A+cI$ and $B+cI$ are positive definite. From this it follows that we can formulate the result in (1.3) in a manner which is independent of λ: If A, B, $C=A+B$ are three symmetric matrices with the respective sets of characteristic values, arranged in increasing order, x_i, y_i, z_i, then

$$\sum_{i=1}^{k} z_i \geqq \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i. \tag{3.1}$$

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3. Ky Fan, Problem 4430, this MONTHLY, vol. 58, 1951, pp. 194; solution in vol. 60, 1953, p. 50.
4. ———, On a theorem of Weyl concerning eigenvalues of linear transformations—I, *Proc. Nat. Acad. Sci.*, vol. 35, 1949, p. 65.

THE BLASIUS FORMULAE

G. POWER, University of Nottingham

In recent years, a great deal of consideration has been given to flow problems, and in particular to the force experienced by a cylinder placed in a two-dimensional field. Emphasis has been placed on the well-known Blasius formulae when a complex potential function exists (as a function of z only), especially in relation to the cases of a cylinder in an incompressible inviscid fluid, and a cylindrical conductor in an electrical field. It is perhaps not generally realized that formulae of this type can be obtained for the force caused by the refraction of the lines of flow at a surface of discontinuity separating different dielectric media, without assuming the existence of a complex potential function for the electric field.

Consider a nested system of homogeneous isotropic dielectric cylinders having a cross-section of general shape, and let us examine the stress on the contour of the interface separating the medium (s) of dielectric constant k_s from the medium $(s+1)$ of dielectric constant k_{s+1}. Take the origin of coordinates inside the contour and let a pair of rectangular axes define the complex z-plane in the usual manner.

If X, Y are the components of electric intensity due to the field in the medium $(s+1)$ then the Maxwell stress components are:

$$P_{xx} = \frac{k_{s+1}}{8\pi}\{X^2 - Y^2\}, \qquad P_{yy} = \frac{k_{s+1}}{8\pi}\{Y^2 - X^2\}, \qquad P_{xy} = P_{yx} = \frac{k_{s+1}}{4\pi}XY.$$

The effects of electrostriction have been ignored. The interface C will experience a force whose components per unit length are

$$X_{s+1} = \int_C (P_{xx}dy - P_{xy}dx), \qquad Y_{s+1} = \int_C (P_{yx}dy - P_{yy}dx),$$

so that the complex force is

$$Z_{s+1} = Y_{s+1} + iX_{s+1} = \frac{k_{s+1}}{8\pi}\int_C (X - iY)^2 dz. \tag{1}$$

In a similar manner Γ_{s+1}, the anticlockwise moment about the origin, is given by

$$\Gamma_{s+1} = Re\frac{k_{s+1}}{8\pi}\int_C (X - iY)^2 z dz. \tag{2$'$}$$

There are similar expressions for the complex force Z_s and couple Γ_s due to the field in medium (s). The total mechanical force and couple on the interface are thus given respectively by

$$Z_{s,s+1} = Z_{s+1} - Z_s, \qquad \Gamma_{s,s+1} = \Gamma_{s+1} - \Gamma_s. \tag{3}$$

These formulae do not take into account the tendency of the dielectric material to change its shape. In general it is impossible to include this effect for two solid media, but it can be demonstrated that if the medium $(s+1)$ is fluid, electrostriction is accounted for by omitting the last term on the right-hand side of each of the equations (3).

The corresponding results for a cylinder in a steady incompressible inviscid fluid flow are

$$(4) \qquad Z = -\frac{\rho}{2}\int_C (u - iv)^2 dz, \qquad \Gamma = Re - \frac{\rho}{2}\int_C (u - iv)^2 z dz,$$

where u, v are the fluid velocity components. We note that the motion need not be irrotational. It should be noted that in the above results the integrands are not analytic functions of z, but this difficulty disappears since on the interface C $\bar{z}$, the complex conjugate of z, is known in terms of z.

Of course, when there is a complex potential as a function of z only, all the above formulae take the normal Blasius form, which is obviously much more useful mathematically.

CLASSROOM NOTES

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REMARK ON EQUIVALENCE RELATIONS

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Students are sometimes bothered by the following pseudo-proof that symmetry and transitivity together imply reflexivity:

Suppose that, for arbitrary x, $x\sim y$. If $y=x$, there is nothing to prove. If $y\neq x$, by symmetry, $y\sim x$. Hence, by transitivity, $x\sim x$.

The difficulty, of course, lies in the assumption of the existence of y such that $x\sim y$. The following example may be of use in clarifying the situation:

Let $S=\{1, 2, 4, 6, 10\}$, and let $p\sim q$ be interpreted as meaning $p+q\equiv 0$ (mod 4). It is immediate from the definition that the relation is symmetric and it is easy to verify that it is transitive. We observe that $2\sim 6$, and hence expect, by the argument of the pseudo-proof, that $2\sim 2$, which is the case. We note that there is no $x\neq 1$ such that $x\sim 1$. Therefore, we cannot prove that $1\sim 1$; and, indeed $1\not\sim 1$. Finally, although there is no $x\neq 4$ such that $x\sim 4$, it is true that $4\sim 4$.

THE CALCULUS OF ABSOLUTE VALUES

KENNETH O. MAY, Carleton College

The customary way of dealing with functions involving absolute values is to consider separately the cases in which the expressions within absolute value symbols are positive and negative. This is clumsy, particularly where differentiation or integration is involved. It may be responsible for the neglect of such functions, even when they would be appropriate in applications. The fact is that the function defined by $y=|x|$ is a very well behaved continuous function with a continuous derivative everywhere except at the origin. There exist simple formulas by which it may be differentiated and integrated with no more attention to discontinuities or existence than is required for such functions as the negative powers of x.

A useful identity is

$$|x| = x\,\mathrm{sg}x \tag{1}$$

where sg is the signum function defined by $\mathrm{sg}x=-1, 0, 1$ according as $x<, =, >0$. Evidently $D\mathrm{sg}x=0$ everywhere except at the origin, where $\mathrm{sg}x$ has a jump of 2. We note also that $\mathrm{sg}x=|x|/x=x/|x|$ for $x\neq 0$. Then it is easy to establish that

$$D|x| = |x|/x. \tag{2}$$

We have also, for $x\neq 0$,

$$D|x| = \mathrm{sg}x. \tag{3}$$

However, (2) has the advantage of calling attention to the non-existence of the derivative for $x=0$.

With the aid of (2) and the chain rule we can find derivatives of composite functions involving absolute values. By this means it is easy to show that

$$D|x^n| = nx^{n-1}\mathrm{sg}x^n. \tag{4}$$

This holds for all x if $n>1$. If $n\leqq 1$, the derivative fails to exist at the origin. More generally,

$$D^n|f(x)| = f^{(n)}(x)\mathrm{sg}f(x) \tag{5}$$

wherever the derivative on the left exists.

The maxima and minima of $|f(x)|$ and $f(x)$ are related in a simple way. Obviously $|f(x)|$ has an absolute minimum at any zero of $f(x)$. Since in any interval where $f(x)>0$ the graphs of $|f(x)|$ and $f(x)$ coincide, they have relative maxima and minima at the same points in such intervals. In intervals where $f(x)<0$ the graphs of $|f(x)|$ and $f(x)$ are each other's mirror images in the x-axis, and $|f(x)|$ has a relative maximum (minimum) where $f(x)$ has a relative minimum (maximum). These remarks hold even if derivatives do not exist at the points in question. However, if the derivative tests are applicable, formula (5)

gives the same results.

Integration by parts yields the following formula, and it can easily be verified by differentiating the right member.

$$\int |x|\,dx = \tfrac{1}{2}x|x| = \tfrac{1}{2}x^2\operatorname{sg}x. \tag{6}$$

Similarly one can find and verify

$$\int |x^n|\,dx = \frac{1}{n+1}|x^n|\,x = \frac{1}{n+1}x^{n+1}\operatorname{sg}x^n \qquad (n \neq -1). \tag{7}$$

As an application of some of the above ideas, consider the problem of proving that a median of a probability distribution is a point from which the mean of the absolute deviations is a minimum. We consider only the case of a continuous distribution, though the discrete case or the general case may be treated by the same methods. The problem is to minimize the integral

$$F(t) = \int_{-\infty}^{\infty} |x - t|\,f(x)dx \tag{8}$$

where $f(x)$ is non-negative and the integral of $f(x)$ from $-\infty$ to ∞ is 1. The usual proof is rather clumsy. However, using the above formulas, we have

$$F'(t) = \int_{-\infty}^{\infty} \operatorname{sg}(t - x)f(x)dx \tag{9}$$

$$= \int_{-\infty}^{t} f(x)dx - \int_{t}^{\infty} f(x)dx \tag{10}$$

$$= 2\int_{-\infty}^{t} f(x)dx - 1. \tag{11}$$

But a median is defined as a value of t that makes the right member of (11) zero. Moreover, since

$$F''(t) = 2f(t) \geqq 0 \tag{12}$$

for all values of t, $F'(t)$ is monotonically increasing, and equation (11) does yield an absolute minimum of $F(t)$.

As a second example, consider the following problem given in Kaplan's *Advanced Calculus* on page 515. Where is the function $|x^2-y^2|+i|2xy|$ analytic? Since $u=|x^2-y^2|$ and $v=|2xy|$ are continuous functions with continuous derivatives in any domain not including points on the axes or on the lines defined by $x^2=y^2$, we need consider only the Cauchy-Riemann conditions. We have

$$\frac{\partial u}{\partial x} = 2x\operatorname{sg}(x^2 - y^2) \qquad \frac{\partial v}{\partial y} = 2x\operatorname{sg}(xy)$$

$$\frac{\partial u}{\partial y} = -2y\,\mathrm{sg}(x^2 - y^2) \quad -\frac{\partial v}{\partial x} = -2y\,\mathrm{sg}(xy).$$

The condition for analyticity is evidently $\mathrm{sg}(x^2-y^2)=\mathrm{sg}(xy)$, and hence the domains are defined by $xy(x^2-y^2)>0$. The procedure is much simpler than considering the combinations of cases for different signs of x^2-y^2 and $2xy$.

EXPANSIONS FOR π AND π^2

L. L. PENNISI, University of Illinois

By establishing the following two formulas, we shall be able to give several expansions for π and π^2 in terms of infinite series.

If $|u|<1$, then

$$\sum_{n=0}^{\infty} \frac{n!2^n u^{2n}}{1\cdot 3\cdot 5 \cdots (2n+1)} = \frac{\arcsin u}{u\sqrt{1-u^2}} \tag{1}$$

and

$$\frac{u^2}{2} + \sum_{n=1}^{\infty} \frac{2\cdot 4 \cdots 2n u^{2n+2}}{[1\cdot 3\cdot 5 \cdots (2n+1)](2n+2)} = \tfrac{1}{2}[\arcsin u]^2. \tag{2}$$

Proof. By a well known formula (see, A. Zygmund, Trigonometrical Series, p. 222), we have

$$\frac{1}{n!}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2t)^{-1/2}dt = \int_0^{1/2} dx_{n+1}\int_0^{x_{n+1}} dx_n \cdots \int_0^{x_3} dx_2 \int_0^{x_2} (2x_1)^{-1/2}dx_1$$
$$= \frac{1}{1\cdot 3\cdot 5 \cdots (2n+1)}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{n!(2u^2)^n}{1\cdot 3\cdot 5 \cdots (2n+1)} = \sum_{n=0}^{\infty}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2u^2)^n(2t)^{-1/2}dt.$$

As a consequence of the Lebesgue Monotone Convergence Theorem (see, M. E. Munroe, Introduction to Measure and Integration, pp. 186–187), we have

$$\sum_{n=0}^{\infty}\int_0^{1/2}\left(\frac{1}{2}-t\right)^n (2u^2)^n(2t)^{-1/2}dt$$
$$= \int_0^{1/2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{2}-t\right)2u^2\right]^n (2t)^{-1/2}dt$$
$$= \int_0^{1/2}\frac{(2t)^{-1/2}dt}{(1-u^2)+2u^2t} = \frac{1}{u\sqrt{1-u^2}}\arctan\frac{u}{\sqrt{1-u^2}} = \frac{\arcsin u}{u\sqrt{1-u^2}},$$

thus establishing (1). By integrating (1), we obtain (2).

Example 1. Substituting $u=1/\sqrt{2}$ into (1), we obtain

$$1+\frac{1}{3}+\frac{2!}{3\cdot 5}+\frac{3!}{3\cdot 5\cdot 7}+\cdots=\frac{\pi}{2}\cdot$$

Example 2. Substituting $u=\frac{1}{2}$ into (2), we obtain

$$\frac{1}{8}+\sum_{n=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)(2n+2)2^{2n+2}}=\frac{\pi^2}{72}\cdot$$

A THEOREM ON THE TAYLOR EXPANSION

C. S. Ogilvy, Hamilton College

Consider the Taylor expansion of $f(x+h, y+k)$ around the point (x, y):

$$f(x+h, y+k)=f(x, y)+\sum_{n=1}^{\infty}\phi_n,$$

where

$$\phi_n = 1/n!\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x, y),$$

and

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

means

$$h^2\frac{\partial^2 f(x, y)}{\partial x^2}+2hk\frac{\partial^2 f(x, y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x, y)}{\partial y^2}$$

and similarly with the higher order terms.

Suppose we set $\phi_n=0$, $n=2, 3, \cdots$. Then we have the

THEOREM: *If there exists an expression (in x, y) for h/k which annuls identically both ϕ_2 and ϕ_3, then it annuls identically also all ϕ_n, $n>3$.*

A geometric proof is easy. The elimination of h/k between $\phi_2=0$ and $\phi_3=0$ results in the well known partial differential equation of the general *ruled surface*.* This is another way of saying that if there exists a function h/k which satisfies both $\phi_2=0$ and $\phi_3=0$, then $f(x, y)$ represents a ruled surface. We have previously shown† that the existence of an h/k such that $f(x+h, y+k)=f(x, y)+\phi_1$ is a necessary and sufficient condition for the surface $f(x, y)$ to be ruled. This is obviously the same h/k which annuls ϕ_2 and ϕ_3; and hence it annuls all higher order terms also.

The theorem may be of value in non-geometric applications. A purely analytic proof is feasible, but the required induction appears to be tedious.

* Goursat-Hedrick, Vol. II, Part II, p. 281, equations (A) and (B).

† C. S. Ogilvy, This Monthly, vol. 59, 1952, p. 548, equation (1).

SOLUTIONS

A Minimal Collection of Positive Integers

E 1157 [1955, 181]. *Proposed by T. F. Mulcrone, St. Charles College, Grand Coteau, La.*

Determine the smallest collection of positive integers such that for any integer k, $0<k\leqq c$, where c is a fixed integer, there is a subcollection whose sum is k.

Solution by L. A. Ringenberg, Eastern Illinois State College. Since c different sums of subcollections are required, it follows that the required collection has at least c distinct subcollections (excluding the null set as a subcollection). If n is the cardinal number of the required smallest collection and if m is the smallest integer such that $2^m-1\geqq c$, then $n\geqq m$. But every k, $0<k\leqq c$, is the sum of a subcollection of the collection $1, 2, 4, \cdots, 2^{m-1}$. Hence $n\leqq m$ and, finally, $n=m$.

Also solved by Hüseyin Demir, A. J. Goldman, Virginia Hanly, A. R. Hyde, J. M. Kingston, M. S. Klamkin, D. C. B. Marsh, J. V. Pennington, Azriel Rosenfeld, Michael Skalskyj, D. D. Strebe, Alan Wayne, and the proposer.

Several solvers saw in this the Bachet weights problem for the "one-pan" case. Wayne called attention to MacMahon's article in *Quarterly Journal of Mathematics*, vol. 21, 1886, pp. 367–373.

Roots of a Polynomial

E 1158 [1955, 181]. *Proposed by J. E. Hanson, Johns Hopkins University*

Find the roots of

$$\sum_{i=0}^{n}\binom{n+i+1}{2i+1}x^i = 0, \qquad n \geqq 1.$$

Solution by James Singer, Brooklyn College. Put

$$p_0(x) = 1,$$
$$p_n(x) = \sum_{i=1}^{n}\binom{n+i+1}{2i+1}x^i, \qquad n \geqq 1.$$

It is readily proved that

$$p_{n+2}(x) - (2+x)p_{n+1}(x) + p_n(x) \equiv 0, \qquad n \geqq 0.$$

By use of the method of difference equations, we find

$$p_n(x) = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta),$$

where α and β are the roots of $y^2-(x+2)y+1=0$. Hence a root of $\alpha=\omega\beta$, where $\omega^{n+1}=1$, $\omega\neq 1$, is a root of $p_n(x)=0$. A calculation yields the n (distinct, real, negative) roots of $p_n(x)=0$, namely

$$x = 2\left(\cos\frac{k\pi}{n+1} - 1\right), \qquad k = 1, 2, \cdots, n.$$

Also solved by W. J. Blundon, Leonard Carlitz, Chih-yi Wang, and the proposer.

Concerning the Euclidean Algorithm

E 1159 [1955, 181]. *Proposed by Albert Newhouse, University of Houston*

In finding the greatest common divisor (a, b) of a and b (either integers or polynomials) by the Euclidean algorithm there appear, in the successive steps, quotients q_i and remainders r_i. Show that $(a, b)=as+bt$, where

$$s = (-1)^{k-1}\begin{vmatrix} q_2 & 1 & 0 \cdots 0 \\ -1 & q_3 & 1 \cdots 0 \\ 0 & -1 & q_4 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}, \quad t = (-1)^{k}\begin{vmatrix} q_1 & 1 & 0 \cdots 0 \\ -1 & q_2 & 1 \cdots 0 \\ 0 & -1 & q_3 \cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 0 & 0 & 0 \cdots q_k \end{vmatrix}.$$

I. *Solution by A. R. Hyde, West Hartford, Conn.* It can be shown that each $r_i=as_i+bt_i$, where s_i and t_i are correspondingly defined. This is easily verified for $i=1$ (if s_1 is defined to be 1) and for $i=2$. Suppose it to hold for $i-1$ and i. By the algorithm,

$$r_{i+1} = r_{i-1} - q_{i+1}r_i = a(s_{i-1} - q_{i+1}s_i) + b(t_{i-1} - q_{i+1}t_i).$$

If the s_{i+1} determinant is expanded with respect to the last row (or column), it is seen that $s_{i+1}=s_{i-1}-q_{i+1}s_i$; similarly $t_{i+1}=t_{i-1}-q_{i+1}t_i$. Hence $r_{i+1}=as_{i+1}+bt_{i+1}$, and the desired relation holds for $i+1$. Therefore the relation holds to the end of the algorithm.

II. *Solution by Chih-yi Wang and David Zeitlin, University of Minnesota.* The successive steps in the Euclidean algorithm can be considered as a system of $k+1$ linear equations in the k unknowns $r_1, r_2, \cdots, r_k$. Since the system is consistent we may consider only the first k equations. Solving these for r_k by Cramer's rule easily yields the desired result.

Also solved by Hüseyin Demir, A. J. Goldman, William Googe, M. S. Klamkin, D. C. B. Marsh, Azriel Rosenfeld, Michael Skalskyj, D. D. Strebe, Lincoln Teng, and the proposer.

Editorial Note. It is interesting to note, using the notation of the first solution, that $as_{k+1}+bt_{k+1}=0$ and that

$$(a, b) = a/(-1)^k s_{k+1} = b/(-1)^{k+1}t_{k+1}.$$

In this connection see the following papers by E. Frank: (1) *On the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 144–158, (2) *The location of the zeros of polynomials with complex coefficients*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 890–898, (3) *On certain determinantal equations*, this MONTHLY, vol. 59, 1952, pp. 300–309.

A Property of the Newton Line of a Complete Quadrilateral

E 1160 [1955, 182]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that in a complete quadrilateral the isotomic line of any side with respect to the triangle formed by the other three is parallel to the Newton line of the quadrilateral.

I. *Solution by the Proposer.* Let d be one of the four sides of the quadrilateral and let ABC be the corresponding triangle. Denote the intersections of d with the sides BC, CA, AB of triangle ABC by α, β, γ. The isotomic line IJK of $\alpha\beta\gamma$ with respect to triangle ABC is obtained by taking the symmetrics I, J, K of the points α, β, γ with respect to the midpoints A', B', C' of the sides BC, CA, AB of triangle ABC. Let the midpoints of $A\alpha$, $B\beta$, $C\gamma$ be denoted by I', J', K'. These points of the Newton line of the quadrilateral are evidently on the sides of the medial triangle $A'B'C'$ of triangle ABC. It is easy to see that the complete quadrilateral formed by triangle ABC and line IJK is similar to that formed by triangle $A'B'C'$ and line $I'J'K'$, for, firstly, triangles ABC and $A'B'C'$ are similar, and are in the ratio 2:1, and secondly,

$$BI = C\alpha = 2(B'I'), \qquad AJ = C\beta = 2(A'J'), \qquad AK = B\gamma = 2(A'K').$$

This proves that the lines IJK and $I'J'K'$ are parallel.

II. *Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N.J.* Since there is one and only one parabola tangent to four lines, let us consider the complete quadrilateral as tangent to the parabola (referred to rectangular coordinates) $y^2=4ax$. Then $y=m_ix+a/m_i$, $i=1, 2, 3, 4$, may be taken as the equations of the four sides 1, 2, 3, 4 of the quadrilateral. Point

$$(a/m_1m_2,\ a/m_1 + a/m_2)$$

is the intersection of sides 1 and 2; other intersections are similarly given. The midpoint of the side 2 of triangle 123 has coordinates

$$(a/2)\left[(m_1 + m_3)/m_1m_2m_2,\ 2/m_2 + 1/m_1 + 1/m_3\right].$$

If (x, y) is the point on side 2 isotomic to the intersection of side 4 with side 2, then

$$y + a/m_2 + a/m_4 = 2a/m_2 + a/m_1 + a/m_3,$$

whence

$$y = a(1/m_1 + 1/m_2 + 1/m_3 - 1/m_4),$$

a result which is symmetric in m_1, m_2, m_3. This proves that the isotomic line of side 4 with respect to triangle 123 is parallel to the axis of the parabola. But the Newton line is also parallel to the axis of the parabola, for it is the locus of centers of all conics inscribed in the quadrilateral, and this locus contains the center of the parabola.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4658. *Proposed by P. R. Halmos, University of Chicago*

If X is a set, let X^2 be the Cartesian product of X with itself. Call a subset D of X^2 a *diagonal* if for every x in X there exists a unique y in X and there exists a unique z in X such that $(x, y)\in D$ and $(z, x)\in D$. Prove that there exists a mapping from X^2 onto X such that the inverse image of every point is a diagonal.

4659. *Proposed by R. C. Lyness, Preston, England*

A solid of uniform density is bounded by those halves of the surfaces $x^2+y^2+z^2=a^2$ and $x^2+y^2+c^2z^2=a^2$ for which $z\geqq 0$. Find c so that under an inverse square law the solid attracts a particle at the origin as if its mass were concentrated at a point on its surface.

4660. *Proposed by E. M. Wright, University of Aberdeen, Scotland*

For all $x\geqq 1, f(x)$ and $\phi(x)$ are non-negative functions, bounded and integrable in any finite interval. They satisfy the inequality

$$xf(x) \leqq \int_1^x f(t)dt + \phi(x).$$

(i) If $\int^\infty \phi(t)t^{-2}dt=\infty$, find an $f(x)$ such that $f(x)\to\infty$ as $x\to\infty$.
(ii) If $\phi(x)/x\to 0$ and $\int^\infty\phi(t)t^{-2}dt<\infty$. Show that

$$g(x) = \frac{1}{x}\int_1^x f(t)dt$$

tends to a limit as $x\to\infty$ and that $\lim g(x)=\overline{\lim} f(x)$.

(iii) Show that, whatever restriction we may impose on the order of $\phi(x)$ as $x\to\infty$, we cannot thereby ensure that $f(x)$ tends to a limit.

4661. *Proposed by J. B. Kelly, Michigan State University*

From an urn containing m black balls and m white balls, balls are drawn one at a time without replacement. An observer guesses the outcome (black or white) of each drawing. It is assumed that at any stage he will guess black if there remain more black balls than white ones and vice versa. Determine $E(m)$

the expected value of the number of correct guesses made during the entire procedure until all the balls have been withdrawn. Give an asymptotic formula for $E(m)-m$.

4662. *Proposed by J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, N. Y.*

If X is a measure space and if $\phi_i(x)$ and $\psi_j(x)$ are two sequences of functions in $L_2(X)$ which are complete in $L_2(X)$, then the sequence $\phi_i(x)\psi_j(y)$ is complete in $L_2(X, X)$.

SOLUTIONS

Stirling Numbers of the Second Kind

4595 [1954, 427]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

If

$$[xD]^n = \sum_{r=1}^{n} A_{r,n} x^{n-r+1} D^{n-r+1},$$

where D is the differential operator, determine $A_{r,n}$.

Solution by A. S. Hendler, Rensselaer Polytechnic Institute, Troy, N. Y. On writing $[xD]^n = x \cdot D[xD]^{n-1}$, we see that the $A_{r,n}$ must satisfy the equation of partial differences

$$A_{r,n} = A_{r,n-1} + (n - r + 1)A_{r-1,n-1},$$

and the initial conditions are $A_{1,n} = A_{n,n} = 1$. Thus the $A_{r,n}$ are completely determined.

However, if we take $A_{n-k+1,n} = B_{k,n} (k = 1, 2, \cdots, n)$ the equation of partial differences may be written

$$B_{k,n} = B_{k-1,n-1} + kB_{k,n-1}$$

with the initial conditions $B_{n,n} = B_{1,n} = 1$. But $\mathfrak{S}_n^1 = \mathfrak{S}_n^n = 1$, where $\mathfrak{S}_n^k$ are the Stirling numbers of the second kind defined by

$$\mathfrak{S}_n^k = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} {}_kC_i i^n.$$

Also, an easy calculation will verify that

$$\mathfrak{S}_{n-1}^{k-1} + k\mathfrak{S}_{n-1}^{k} = \mathfrak{S}_n^k.$$

Hence $B_{k,n} = \mathfrak{S}_n^k$ and

$$A_{r,n} = \mathfrak{S}_n^{n-r+1} = \frac{1}{(n - r + 1)!} \sum_{i=1}^{n-r+1} (-1)^{n-r+1-i} {}_{n-r+1}C_i i^n.$$

Also solved by N. Balasubramanian, W. J. Blundon, L. Carlitz, H. W. Gould, D. S. Greenstein, A. T. Hind, Jr., A. R. Hyde, D. C. B. Marsh, Norman Miller, Leo Moser, S. Parameswaran, Michael Skalskyj, M. R. Spiegel, Chih-yi Wang, and the Proposer.

Editorial Note. The problem has been treated in a number of places and the following references were cited by our correspondents: Jordan, *Calculus of Finite Differences*, pp. 195–196, (See also pp. 168–170 for the Stirling numbers), Schwatt, *Introduction of Operations with Series* (1924), pp. 86 ff., and an article by L. Carlitz, *On a class of finite sums*, this MONTHLY, (1930) pp. 473–479.

Almost Absolutely Convergent Sequences

4596 [1954, 428]. *Proposed by G. G. Lorentz, Wayne University, and M. S. Macphail, Carleton College, Ottawa*

Given a sequence s_n, let

$$\sigma_{n,p} = \frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p}. \tag{1}$$

We call s_n almost absolutely convergent (a.a.c.) if the series

$$\sum_{p=1}^{\infty} \left| \sigma_{n,p} - \sigma_{n,p+1} \right|$$

converges uniformly for $n=1, 2, \cdots$. Prove that:

1. A sequence s_n is a.a.c. if s_n is absolutely convergent, that is if $\sum |s_n - s_{n+1}| < \infty$.

2. If each a.a.c. sequence is absolutely summable by a method of summation $A = (a_{mn})$, then there is a bounded sequence which is absolutely A summable but is not a.a.c.

Solution by the Proposers.

1. From (1) we have

$$\sigma_{n,p} - \sigma_{n,p+1} = \frac{1}{p(p+1)} \left[(s_n - s_{n+1}) + 2(s_{n+1} - s_{n+2}) + \cdots + p(s_{n+p-1} - s_{n+p})\right].$$

Taking absolute values and summing with respect to p,

$$\sum_{p \geqq p_0} \left| \sigma_{n,p} - \sigma_{n,p+1} \right| \leqq \sum_{\nu=1}^{\infty} \left| s_{n+\nu} - s_{n+\nu+1} \right| = R_n.$$

It follows that all series

$$\sum_{p=1}^{\infty} \left| \sigma_{n,p} - \sigma_{n,p+1} \right|, \qquad n = 1, 2, \cdots \tag{2}$$

are convergent. Let $\epsilon > 0$ be arbitrary, and take n_0 so large that $R_n < \epsilon$ for $n \geqq n_0$. Then we take p_0 such that

$$\sum_{p \geqq p_0} \left| \sigma_{n,p} - \sigma_{n,p+1} \right| < \epsilon \qquad \text{for n} = 1, 2, \cdots, n_0. \tag{3}$$

The preceding inequalities then show that (3) holds for all n.

2. We need a Lemma. Let $\Omega(n)$ be a positive increasing function with $\Omega(n) \to \infty$ and

$$\sum_{n=1}^{\infty} \Omega(n) n^{-2} < \infty \tag{4}$$

(for instance, let $\Omega(n) = n^{1/2}$.) Let the sequence $n_1 < n_2 < \cdots$ of integers increase so rapidly that its counting function $\omega(n)$ ($\omega(n)$ is the number of ν for which $n_\nu \leqq n$) satisfies

$$\omega(n + m) - \omega(n) \leqq \Omega(m) \qquad n, m = 1, 2, \cdots. \tag{5}$$

Let s_n be a bounded sequence such that $s_n = 0$ if n is not one of the indices n_ν. Then s_n is a.a.c.

For the proof we show that the series $\sum_{n_\nu > n} (n_\nu - n)^{-1}$, $n = 1, 2, \cdots$, are uniformly convergent for all n. In fact, the rest of the above series corresponding to $\nu \geqq \nu_0$, is

$$\begin{aligned} \sum_{\nu=\nu_0}^{\infty} (n_\nu - n)^{-1} &\leqq C \sum_{\nu=\nu_0}^{\infty} \sum_{m \geqq n_\nu - n} m^{-2} = C \sum_{m=n_{\nu_0}}^{\infty} m^{-2} \sum_{n < n_\nu \leqq n+m} 1 \\ &= C \sum_{m=n_{\nu_0}}^{\infty} m^{-2} [\omega(n + m) - \omega(n)] \\ &\leqq C \sum_{m=n_{\nu_0}}^{\infty} \Omega(m) m^{-2}, \end{aligned}$$

and the last expression is small for large ν_0 and does not depend on n.

Returning to s_n, assume first that $n + p = n_\nu$, then if $|s_n| \leqq M$,

$$\begin{aligned} \left| \sigma_{n,p} - \sigma_{n,p+1} \right| &= \left| \frac{1}{p(p+1)} (s_n + \cdots + s_{n+p-1}) - \frac{1}{p+1} s_{n+p} \right| \\ &\leqq \frac{1}{p(p+1)} \Omega(p) M + \frac{1}{p+1} M \leqq M(p^{-2}\Omega(p) + (n_\nu - n)^{-1}). \end{aligned} \tag{6}$$

If $n + p$ is not a n_ν, we obtain in the same way

$$\left| \sigma_{n,p} - \sigma_{n,p+1} \right| \leqq M p^{-2} \Omega(p). \tag{7}$$

Relations (6) and (7) prove the Lemma.

Considering now the method $A' = (a_{m,n_\nu})$, $m, \nu = 1, 2, \cdots$, we see that A' sums absolutely each bounded sequence and hence (by G. G. Lorentz, *Direct*

theorems on methods of summability II, Canadian Journ. Math. 3 (1951), pp. 236–256, Theorem 7)

$$(8) \qquad V(n_\nu) = \operatorname*{Var}_m a_{m,n_\nu} = \sum_{m=1}^{\infty} \left| a_{m,n_\nu} - a_{m+1,n_\nu} \right| \to 0 \qquad \text{for } \nu \to \infty.$$

Since n_ν was an arbitrary sequence increasing sufficiently rapidly, (8) implies that

$$V(n) = \operatorname*{Var}_m a_{mn} \to 0 \qquad \text{for } n \to \infty.$$

Now it is sufficient to take a sequence s_n of the form

$$1, 0, \cdots, 0, 1, 1, 0, \cdots, 0, 1, 1, 1, 0, \cdots$$

with arbitrary long blocks of 1's separated by long blocks of 0's such that $\sum V(n_\nu) < \infty$, where the n_ν correspond to the values 1 of s_n. Then s_n is absolutely A summable, but is not a.a.c., which proves the assertion.

Representations in Terms of a Given Sequence

4597 [1954, 428]. *Proposed by Paul Erdös, University of Notre Dame*

Let $1 = a_1 < a_2 < \cdots$ be an infinite sequence of integers. Let n be any integer and write $n = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ where a_{i_k} is the greatest $a \leqq n$, a_{i_2} the greatest $a \leqq n - a_{i_1}$, *etc.* (Since $a_1 = 1$ this representation is always possible.) Put $f(n) = k$ (*i.e.*, the number of summands representing n). Prove that if the upper density of the a's is 0, then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=1}^{x} f(k) = \infty,$$

and if the lower density of the a's is > 0, then

$$\overline{\lim_{x\to\infty}} \frac{1}{x} \sum_{k=1}^{x} f(k) < \infty.$$

(If $N(y)$ denotes the number of a's $\leqq y$, the upper density of the a's is $\overline{\lim}\, N(y)/y$, and the lower density is $\underline{\lim}\, N(y)/y$.)

Solution by Howard Robbins, Mantorville, Minnesota. An identity for $S(x) = \sum_{r=1}^{x} f(r)$ can be derived by separating out the first term in the a-expansion of each integer r:

$$S(x) = x + S(x - a_n) + \sum_{k=2}^{n} S(a_k - a_{k-1} - 1).$$

Here a_n is the largest $a \leqq x$. There are x terms separated out, $S(a_k - a_{k-1} - 1)$ gives the additional terms needed to expand the integers $a_{k-1}+1$ through $a_k - 1$, and $S(x - a_n)$ gives the additional terms needed for integers $a_n + 1$ through x.

If the lower density of the a's is not zero, a number $M>1$ can be found such that $M\cdot N(x)\geqq x$ for all x. Then by induction it can be shown that $S(x)<Mx$ for all x, establishing one of the two desired results: The relation holds for $x=0$ or 1, and if it holds for all $x<z$, the equation gives $S(z)<z+M(z-N(z))\leqq Mz$. Hence it holds for all x, and $\overline{\lim}\ S(x)/x$ is finite.

If the upper density is zero, choose an x_0 and a K such that $S(x)>Kx$ for $x\geqq x_0$ (choose $K=0$ if necessary). Then the equation gives $S(x)>x+(x-x_0N(x))K$. For the arguments of all the S's on the right side of the equation add up to $x-n$, and discarding those S's whose arguments are less than x_0 can diminish this sum by at most $(x_0-1)N(x)$, leaving $x-x_0N(x)$ as a lower bound for the sum of the arguments of the rest, for each of which the function is at least K times the argument. Since the upper density is zero, an x_1 can be found so large that for $x\geqq x_1$, $x_0N(x)K<x/2$. This gives $S(x)>(K+\frac{1}{2})x$ for $x\geqq x_1$. Hence any lower bound for $\underline{\lim}\ S(x)/x$ can be increased by $\frac{1}{2}$, and $\underline{\lim}\ S(x)/x$ must be infinite.

Also solved by P. J. Owens.

Sequence Related to Laurent Series

4598 [1954, 476]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad (r < |z| < r^{-1}),$$

converge in an open ring containing the circle $|z|=1$. Show that a sequence $\{b_n\}$ exists satisfying the conditions

(i) $|b_n|<K$, for some K, for all $n=0, \pm 1, \pm 2, \cdots$,

(ii) $\sum_{n=-\infty}^{\infty} a_n b_{m-n}=0$ for all m,

(iii) $b_n\neq 0$ for some n,

if and only if $f(z)$ vanishes somewhere on the circle $|z|=1$.

Solution by Bruce Kellogg, University of Chicago. If a sequence satisfying the required conditions exists for the Laurent series $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$, convergent in a neighborhood of $|z|=1$, and if $g(z)=\sum_{n=-\infty}^{\infty} c_n z^n$ is convergent in a neighborhood of $|z|=1$, then the sequence $\{b_n\}$ also satisfies the required conditions for the Laurent series $f(z)\cdot g(z)$. For if $\sum_n a_n b_{m-n}=0$ for all m, then

$$\sum_n \left(\sum_p a_p c_{n-p}\right) b_{m-n} = \sum_n \left(\sum_p a_p b_{n-p}\right) c_{m-n} = 0$$

for all m. Hence if $f(z)$ does not vanish on $|z|=1$, $1/f(z)$ may be taken for $g(z)$ and the above result implies that: if a sequence $\{b_n\}$ exists for $f(z)$, it exists for the function $f(z)\cdot(1/f(z))=1$, which is a contradiction.

On the other hand, suppose $f(z)$ has a zero, $z=z_0$, $|z_0|=1$. Since there is a sequence $\{b_n\}$ for the function $(z-z_0)$, and

$$f(z) \equiv (z - z_0)\cdot\{f(z)/(z - z_0)\},$$

then there is a sequence $\{b_n\}$ for $f(z)$.

Also solved by the Proposer.

Number of Solutions of a Congruence

4599 [1954, 476]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2, $a\not\equiv 0 \pmod p$. Show that the number of solutions of the congruence

$$(x + y + z)^2 \equiv 2axyz \pmod p$$

is p^2+1.

Solution by Emma Lehmer, Berkeley, California. First of all we have the trivial solution (0, 0, 0); next, if two of the variables are zero this leads again to (0, 0, 0). If only one variable, say x, is zero, then we have p-1 solutions of the form $(0, y, -y)$, so that altogether we have $3p-2$ solutions in which one or more of the variables is zero.

If none of the variables is zero we can let $y=ux$, $z=vx$, so that the original congruence becomes

$$(1 + u + v)^2 \equiv 2auvx \pmod p.$$

Solving for x, y, and z in terms of u and v we have

$$x \equiv \frac{(u + v + 1)^2}{2auv}, \qquad y \equiv \frac{(u + v + 1)^2}{2av}, \qquad z \equiv \frac{(u + v + 1)^2}{2au},$$

with the only conditions on u and v that $u\not\equiv 0$, $v\not\equiv 0$, $u+v\not\equiv -1 \pmod p$. Hence there are $p-1$ possible values of u, and for each value of u, except for $u=p-1$, there are $p-2$ corresponding values of v satisfying these conditions. For $u=p-1$ there are $p-1$ such values of v, so that altogether we have $(p-2)^2+(p-1)$ $=p^2-3p+3$ non-zero solutions. Adding this to the $3p-2$ solutions which contain zero we get the required p^2+1 solutions altogether.

Also solved by Bruce Kellogg, D. C. B. Marsh, H. S. Zuckerman, and the Proposer.

Editorial Note. Zuckerman proves the somewhat more general theorem: If p is prime, $a\not\equiv 0 \pmod p$, $m\geqq 1$, $n\geqq 1$, then the congruence

$$(x_1 + x_2 + \cdots + x_n)^m \equiv ax_1x_2 \cdots x_n \pmod p$$

has $p^{n-1}+(-1)^{n-1}$ solutions if $n-m$ is relatively prime to $p-1$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

An Introduction to Deductive Logic. By Hughes Leblanc. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1955. xii+244. $4.75.

This book should be a useful text for an undergraduate course in symbolic logic. The author has managed to combine an elementary presentation of his material with many side glances at some of the manifold problems of modern logic. The book is provided with a good collection of problems.

Chapter one presents an informal introduction to the statement calculus. Truth tables are explained. The chapter contains two final sections introducing the ideas of many valued and modal logics. The second chapter gives a similar informal introduction to quantificational logic. A terminal section provides some information about intuitionistic logic.

Formalization of the sentential and quantificational calculi is carried through in chapter three. The sentential calculus is formalized twice, once with substitutional rules for sentence variables and a finite number of axioms, and once with axiom schemes instead of axioms. Axiom schemes are used for the formalization of quantification theory, but an alternative formalization making use of predicate variables is outlined. It is a useful feature of the book that it familiarizes students with both of these important approaches at an early stage. The chapter concludes with a short discussion of natural deduction.

The fourth chapter introduces the notion of identity. After axioms for equality have been added operators $(\underset{n}{E}Lx)$, $(\underset{n}{E}Ex)$, and $(\underset{n}{E}Mx)$, standing for "there are at least nx," "there are exactly nx," and "there are at most nx," respectively, are defined and shown to have formal properties consistent with their intended interpretation. The remainder of the chapter is devoted to a discussion of classes and relations, and Boolean algebras. The concept of a Boolean algebra is never defined, and the discussion of the relationship between certain logical calculi and Boolean algebras is not as clear or precise as it might be.

The last chapter is of a more advanced nature. The consistency, completeness, and independence of the axioms of the sentential calculus are proved. The quantificational calculus is shown to be consistent and Gödel's proof of the completeness of the quantificational calculus is sketched. A proof that the monadic quantificational calculus is decidable is also included. The book concludes with a few remarks about the identity calculus.

The following misprints were noted: p. 166, line 29, insert n under the first E; p. 197, line 35 change ζ_1 to ζ_i, and next line down change the last ψ in line to ζ; p. 213, line 1, replace ζ by ψ; p. 219, line 12, the subscript under the A should read 2^k, not $2k$.

STEVEN OREY
University of Minnesota

Engineering Cybernetics. By H. S. Tsien. New York, McGraw-Hill Book Co. 1954. xii+289 pages. $6.50.

This valuable book, written for (actual or potential) research engineers and applied mathematicians, deals with the science of control and communication, which N. Wiener has christened cybernetics. The author deliberately gives a purely theoretical treatment of highly practical topics, as is usually done in fluid mechanics. For example, he recognizes but is not afraid of the fact that some engineers will think the book highbrow, while mathematicians will be disappointed if they expect the rigor and elegance that are customary in abstract mathematics (and in some that is called applied). The principal mathematical prerequisites are of course ordinary differential equations and complex integration; variational methods are also used on occasion.

The author has not extended the scope of his work by means of bibliographic material or collections of exercises, but the text itself covers an unusually wide range of topics. This may be indicated by chapter as follows: 1. Introduction. 2. Laplace transform. 3. Transfer function. 4. Feedback servomechanism. 5. Noninteracting controls. 6. Alternating-current and oscillating-control servos. 7. Sampling servos. 8. Time lag. 9. Stationary random inputs. 10. Relay servos. 11. Nonlinear systems. 12. Linear system with variable coefficients. 13. Perturbation theory. 14. Control design with specified criteria (*e.g.*, minimize the time-average of the error-squared). 15. Controls that automatically seek to maximize a criterion of good performance. 16. Filtering of noise (sketches of the Wiener-Kolmogoroff and other theories). 17. Ultrastability and multistability (W. R. Ashby's conception of a system capable of learning in the sense of responding to environmental changes by searching out a stable pattern of behavior for itself). 18. Error control (J. von Neumann's theory of improving the reliability of a system by duplicating or multiplexing its elements). This last study has been extended (to nets of relays) by C. E. Shannon and E. F. Moore in some as yet unpublished work.

In Chapter 9 it appears that the author should have had the advice of someone familiar with statistics and probability. The distribution (9.49), often called exponential, is never called Poisson's distribution. Above (9.7), the terms variance and mean deviation should not be used for the standard deviation σ. The most serious lapse noted concerns the concept of ergodicity or metrical transitivity; it not only goes unnamed, but is described as if it were identical with stationarity, or were a consequence thereof.

In general, however, the book appears to be adequately clear and accurate. Also, the author has done his duty by a new and growing subject, by including much material (from recent research papers) which has not previously appeared in book form.

E. L. Kaplan
Bell Telephone Laboratories, Inc.

Existence Theorems for Ordinary Differential Equations. By F. J. Murray and K. S. Miller. New York University Press, 1955. x+153 pages. $5.00.

This book is an unhurried development of classical existence and uniqueness theorems for ordinary differential equations.

The leisurely pace of the book is set in the first two chapters of 32 pages devoted to six standard existence theorems. In Chapter 3 hypotheses are added to theorems of the first two chapters to secure sufficient conditions for unique solutions. Chapter 4 treats the Picard iterants. Properties of solutions, such as differentiability of solutions with respect to parameters, are discussed in Chapter 5. The final, and longest, chapter is devoted to linear differential equations.

In most instances the theorems are stated for real functions and, except for the linear equations, the theorems are usually for solutions in the small. Riemann integration is used throughout. The book gives the impression of being a good set of lectures rather than a polished exhibition prepared especially for publication. Thus the notation is modern but not condensed and the theorems, proofs, and examples are thoroughly discussed.

Students possessing some background in function theory will find this book a readable and reliable account of an important topic in analysis.

W. R. UTZ
University of Missouri

General Topology. By J. L. Kelley, New York, D. Van Nostrand Co., Inc., 1955. xiv+298 pages. $8.75.

This book is intended as "a systematic exposition of general topology which has proven useful in several branches of mathematics" especially analysis. As such it is thoroughly successful and should provide an excellent reference volume for those mathematicians who are interested in this subject as a *Gebrauchtopologie.*

The book begins with Chapter 0 on set theory, relations, orderings and the like, culminating in a theorem which proves eight statements of the extremum law = Hausdorff maximal principle = Zorn's lemma *etc.* to be equivalent. Chapter 1 introduces the notion of a topology and explores some of its implications; Chapter 2 deals with Moore-Smith convergence. Chapter 3 covers product and quotient spaces, while Chapter 4 on embedding and metrization proves the standard metrization theorems. The latter chapter also investigates some of the properties of pseudometric spaces. In Chapter 5 on compact spaces the productive and divisible properties of compact spaces are discussed, the usual compactification theorems are presented, and finally paracompactness is briefly presented. Chapter 6 concerns uniform spaces which are presented *à la* Bourbaki. The pseudo-metrization theorem for uniform spaces is proved as well as the uniqueness of the uniform topology on a compact space. A form of the Baire category theorem for pseudo-metric spaces closes the chapter. Chapter 7 on function spaces lies somewhere between general topology and analysis. It in-

vestigates various topologies that may be introduced in spaces of functions and ends with a proof of the Ascoli theorem. This is presented first in the usual form for equicontinuous families and then in the author's form for evenly continuous families. Finally there is an extensive appendix on elementary set theory, which develops this theory axiomatically.

The problems at the end of each chapter contain a large part of the theory. Some are simple exercises, examples and counter examples which form, as the author indicates, a necessary adjunct for the understanding of the theory. Others are major theorems and theories; for example the Tietze extension theorem and the Stone-Weierstrass theorem occur as exercises. Suggestions for the proof of the more difficult of such theorems is given by means of hints or a suggested sequence of steps (lemmas) so that the mature student should be able to carry out the details with only minor difficulties.

This volume has considerable value as a reference book. On the other hand, the reviewer feels that it has only limited utility as a text. For a course in topology *per se* it is unsuitable since it does not prove many of the purely topological results. As an example may be cited that the topological characterizations of the arc and simple closed curve do not occur either in the text or as exercises. Its usefulness as a text would come in a course on topology for analysts. Such a course could cover the major part of the book in one semester if a knowledge of Chapter 0 were presupposed.

The book has an attractive format and the reviewer found but few typographical errors. The style is quite readable and, if this may be said of mathematical writing, sprightly.

J. D. BAUM
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OBITUARY

JULIAN LOWELL COOLIDGE

IN MEMORIAM

I

There have been Coolidges in America ever since John Coolidge came to Massachusetts Bay with the first settlers and made himself a home in Watertown. His descendants have spread and formed their own families, of which members have made a name in public life, in the law, in the arts and in the sciences. Many of them have remained faithful to New England, its institutions and traditions. Here also Julian Lowell Coolidge grew up and worked, not only a New Englander, but in particular a Harvard man. With Harvard he was

associated through family associations and through intimate personal connections, which began with his undergraduate years in the nineties and ended only with his death.

He was born in Brookline, Massachusetts, September 28, 1873, the son of Joseph Randolph Coolidge and his wife Julia Gardner Coolidge. The father, who was a member of the Massachusetts bar, provided his children with a good education which included travel. Julian, after a year at Exeter, entered Harvard College in 1892, where he received his B.A. in 1895, *summa cum laude*. Although at that time he did not yet think of a career in mathematics, there was already enough activity at Harvard to stimulate a budding scientist of exact leanings, and the young student knew how to profit by it.

Coolidge himself has later described the scientific climate of the Harvard of his early days: "The great museum bore the name of the elder Agassiz, and was ably managed by the younger. Botany brought to mind the names of Goodale and Farlow; psychology, James and Münsterberg. In chemistry a great figure was passing in Cooke, in geology the astounding Shaler had a wide influence. In mathematics three men had borne the heat and burden of the day since the death of Benjamin Peirce, namely his son, James Mills Peirce (A.B.1853), William Elwood Byerly (A.B.1871) and Benjamin Osgood Peirce (A.B.1876), a distant relative of the other Peirces." [**15**] Of those men, Benjamin O. Peirce became a good friend.

Mathematics had recently been strengthened by the appointment of two younger men, William Fogg Osgood of the class of 1886 and Maxime Bôcher, class of 1888. Both had studied at Göttingen under Felix Klein, both had written dissertations which still are read, both set their permanent imprint on the Harvard mathematics division as scientists as well as teachers. However, in Coolidge's undergraduate days Harvard had not yet advanced to a school where students worked for a higher degree in mathematics, and so Coolidge, intent on making teaching his career, set out for Oxford, where at Balliol he finished with a B.Sc., the first degree in natural science awarded by the hitherto severely humanistic university. After his return he settled as a schoolteacher, and from 1897–99 taught at the school in Groton, Massachusetts, at that time under the leadership of its founder, the Reverend Endicott Peabody. Among his pupils was Franklin D. Roosevelt, with whom Coolidge always maintained a friendly relationship.

Archibald Cary Coolidge, Julian's elder brother, later founder and head of the Widener Library, had at that time entered the Harvard history department after a diplomatic career abroad. Under his influence Julian accepted a position as instructor in the mathematics department—initial salary $250 a year. From this period date his first publications, which already show the trend of his interests: one on a geometric representation of imaginary elements in the plane, the other on the construction of the intersection of two conics with a common focus with the aid of compass and straight edge alone, if at least three points or tangents of each conic are given [**27**]. This construction has a neat application

to the problem of Apollonius, to construct a circle tangent to three given circles. There exists another paper of these years concerning the classification of quadric surfaces in hyperbolic three space [**3**]. The classification is based on the intersection of the surfaces with the absolute, and Coolidge improves on Clebsch by sharply discriminating between real and imaginary elements, and between the regions inside and outside the absolute.

In 1902 Coolidge became a member of the Faculty and was granted a two year leave of absence to study abroad. He was a married man now, and accompanied by Mrs. Coolidge, he first went to Paris, and then to Greifswald. Here he studied under Kowalewski and especially under Study. Afterwards he travelled to Turin, where he heard D'Ovidio and in particular Corrado Segre; then returned to continue his studies with Study, who had moved from Greifswald to Bonn. There, in 1904, he received the doctor's degree with a thesis on non-euclidean line geometry, a field in which both Segre and Study had done outstanding work [**5**].

These years, 1902 to 1904, were decisive for Coolidge's mathematical career. With both Segre and Study he formed strong scientific and personal bonds, to which he repeatedly refers in later work. The thesis was written in almost daily contact with Study, who had just written his *Geometrie der Dynamen*—essentially a highly original line geometry in euclidean space—and whose methods Coolidge learned to apply. The thesis is still an interesting bit of research; those who like to study its ideas can also find them in Coolidge's first book, that on Non-euclidean Geometry, in a section that has always struck us as one of the most interesting parts of this book. We also find here the essentials of another paper by Coolidge dating from the time of his study with Study and Segre. This paper, placed by Segre in the Atti of the Turin Academy, dealt with isotropic congruences of lines in elliptic space—congruences of which the focal surface is a developable circumscribed to the absolute [**6**]. Here again we meet a subject which always fascinated Coolidge: the geometrical interpretation of complex numbers and their functions.

Back home, Coolidge continued teaching at Harvard, where he became an assistant professor in 1908. In those days he began to work on the first of his books and approached the Clarendon Press in Oxford to find out whether it was interested. The first result of the agreement between author and publishing house was the *Non-euclidean Geometry* of 1909. This was the beginning of an association which lasted for fifty years: all eight of Coolidge's books have been published by the Clarendon Press. These books have become well known in professional circles for their clear exposition, the excellence of the material, the facility of their style and their personal touch. They also brought the author's own contributions, organized in the framework of the general text.

The papers of this first period after the return from Bonn belong in the main to the Segre-Study school, with its generous touch of Lie. In 1909 appears Coolidge's first article on the theory of probability, in 1915 his first on the algebraic theory of curves [**11, 17**]. Life at Harvard continued at an even pace in a

growing department; in 1912 he attended the International Mathematical Congress at Cambridge, England. In 1916 he published his second book, the *Treatise on the Circle and the Sphere.* In 1918 he became a full professor.

The world war interrupted for a while the work at Cambridge. Coolidge served as a major in the United States army, and from 1918–19 was an A.E.F. liaison officer with the French general staff, a function in which his fluent command of French was of great help. In 1919 courses were organized at the Sorbonne for American officers and enlisted men who still had to stay in France; Coolidge was placed in command of the 1800 men involved and spent much time and effort in making the best of an unusual and arduous assignment in which he had, it has been said, the duties of a dean without the sanctions usually at the disposal of such a dignitary. However, he seemed to have found the teaching and administrating assignment quite rewarding. It also brought him the *Lêgion d'honneur*, as *Chevalier* in 1919, as *Officer* in 1936.

After the war Coolidge resumed his duties on the Harvard faculty. In 1924 he published the *Geometry of the Complex Domain*, still a tribute to his Bonn and Turin days. He had lectured on this subject at the Sorbonne in 1919, he again took it up in a lecture given at Rome in 1926 on a visit from Paris, where he stayed during 1927 as an exchange professor at the Sorbonne [**42**]. At that time his *Introduction to Mathematical Probability* (1925) had already appeared. Of all Coolidge's books this one is probably the best known; Coolidge found this out not only because it received a German translation in 1927, but also through word from his publishers that it had a vogue at Monte Carlo. In 1927 he became chairman of the mathematics department, where he organized the main features of that department's tutorial system. In 1929, after President Lowell had inaugurated his so-called House Plan for the reorganization of the college into smaller residential units, Coolidge accepted the position of Master of Lowell House. It was a delicate assignment in a new university venture of considerable consequence [**59**], which gave Coolidge the opportunity to build an atmosphere of friendly social life combined with serious scholarship. The task was congenial to him and he stayed at Lowell House until 1940, when he retired both as professor and as master. During the period of Lowell House Coolidge finished his *Algebraic Curves* (1931) and his *History of Geometrical Methods* (1940). The first was a token of his indebtedness to the geometers of the Italian school—it is dedicated "*Ai geometri italiani, morti, viventi*"—the latter opens the series of his books on the history of mathematics, a field in which we see his interest growing after 1924, the year of first paper on the subject. The other two historical books were published when Coolidge was already retired; the *Conic Sections and Quadric Surfaces* dates from 1943, the *Mathematics of Great Amateurs* from 1949. Retirement, however, did not prevent him from offering his services to the university during the war emergency, and so we see him, during 1942–43, return to his old love, the classroom, where he taught the calculus to enlisted men. He was productive well-nigh till the end, his last paper, on the length of curves

[71], appeared in 1953, when he was in his eightieth year. He died March 5, 1954. He is survived by his wife, Mrs. Theresa Reynolds Coolidge, five daughters and two sons.

Coolidge was deeply interested in both the Mathematical Association of America and the American Mathematical Society. He was vice-president of the Association in 1924 and president in 1925, member of the Society's Council during 1911–13 and 1924–25, vice-president in 1918 and a trustee in 1929–30. During his year of service as President of the Association Coolidge suggested the establishment of the Chauvenet Prize for noteworthy expository writing in mathematics. He himself donated $100 to the Association to be used for the first award of the Chauvenet Prize. His most important contribution to the Society was his chairmanship of the Endowment Fund drive. The campaign to place the finances of the Society on a firm foundation was inaugurated in 1922 and lasted till 1926; it was not only projected as a fund-raising device, but also as an educational venture, to place before the public the basic character of mathematics and mathematical research in our civilization. There were newspaper and magazine articles and the inauguration of the Josiah Willard Gibbs Lectureship. By 1926 the chairman could feel satisfaction: the financial condition of the Society was improved by about $70,000 in gifts, sustaining memberships and a subvention from the National Academy of Sciences. The main burden of the drive, which often consisted in day by day soliciting, was carried by Coolidge, assisted by Professors Veblen and Dresden.

Coolidge's wit was proverbial, it leavened his teaching, his conversation and, we are glad to say, his written work, so that readers can still enjoy it though the writer has passed. It is strong in his book reviews, where the personal touch gives the feeling of direct contact between reader and author, but also tickles us in his textbooks, as in his comment on the unsatisfactory character of Bayes' theorem in probability: "Steyning tuk him for the reason the theif tuk the hot stove—bekaze there was nothing else that season." Suspecting (and by the way, rightly so), that his collection of "mathematical amateurs" might be judged rather arbitrary, he announced: "If consistency is a vice of small minds, it is a vice I have successfully avoided." Even when engaged in a stern mathematical reasoning he liked to jump off the high horse and allowed his readers or his audience to relax through a colloquialism or a personal remark.

Teaching was Coolidge's first professional love and he continued to cherish it throughout his life, building up a reputation as an excellent instructor with a plastic way of reasoning. He liked to do strenuous things, as a student at Harvard he gained a gold medal for his record-breaking mile, 4 min. and 30 4/5 sec.; at Balliol he was on the crew. He indulged in hiking and swimming, and loved to sail off the Maine coast, where he had a summer home. Interested in the conservation of our national resources, he forested himself a large tract of woodland. He was also an amateur astronomer of considerable vigor; later in life he presented the 6-inch telescope at his house in Cambridge to the American

Association of Variable Star Observers, who have put it to new duty in Minneapolis.

We have already mentioned his relation to the Clarendon Press. An extract from a letter by Mr. A. M. Wood of this house may properly finish this section of our paper:

"He first got in touch with the Press in 1904, at a time when we did very little mathematical publishing. He continued to publish with us throughout his working life and our relations with him were uniformly happy. He used to tell us of his plans for writing and, looking through past correspondence, I am impressed by the way in which he saw each book whole before he started. He would tell us that his next book would be on a certain topic; he would outline its scope and its relationship with other books in the field and would tell us how many years it would take him to write it. His estimates of this time were very accurate."

Coolidge was lucky with his publishers: few mathematicians can boast of so many beautifully printed books.

II

The Bonn thesis [5] is a study of lines in an elliptic three-space S_3. Such lines have a polar with respect to the absolute, line and polar together form what Study has called a *cross*. The figure is equivalent to that of two totally perpendicular planes at a point in euclidean four-space. It is also equivalent to a pair of points in an elliptic plane, or to its dual, a pair of lines in such a plane. Two real lines in S_3 which meet the same pair of left (right) generators of the absolute are called left (right) *paratactic*, a term introduced by Study for what often is called *Clifford parallel*. Such left (right) paratactic lines have an infinite number of common perpendiculars and are at constant distance. Through a point of the S_3 one right and one left paratactic can be drawn to the lines of a given cross. A cross can therefore be represented by two lines through (1, 0, 0, 0), the left paratactic has now coordinates ${}_lX_i$, $i=1, 2, 3$, different from zero and proportional to the coordinates of intersection with the plane $X_3=0$. There are also 3 coordinates ${}_rX_i$ for the right paractactic. The planes ${}_lX$ and ${}_rX$ are the representative planes. There is a dual representation ${}_lU$, ${}_rU$. Coolidge now introduces for the coordinates of the cross not the X themselves, but the "dual" combinations

$${}_lX_i\epsilon_1 + {}_rX_i\epsilon_2 = X_i, \qquad {}_lU_i\epsilon_1 + {}_rU_i\epsilon_2 = U_i,$$

where

$$\epsilon_1^2 = \epsilon_1, \qquad \epsilon_2^2 = \epsilon_2 > \epsilon_1\epsilon_2 = \epsilon_2\epsilon_1 = 0.$$

The properties of crosses can be very elegantly expressed in terms of these dual coordinates; *e.g.*, the scalar product $U\cdot X=0$ means orthogonality. The group of this dual projective geometry G_{16} is the group of linear transformation of the X_i with dual coefficients.

The object of the dissertation is to study certain systems of crosses. Those which are selected are mapped on the representative planes by two projective point ranges (a *chain*), by a point in one plane and a line in the other (a *strip*) and a collinear relation between the planes (*homographic congruence*). In the case of a chain the *chain surface* in S_3, belonging to the general chain in the form

$$ {}_rX_1 = a_1\, {}_lX_1, \qquad {}_rX_2 = a_2\, {}_lX_2, \qquad {}_rX_3 = {}_lX_3 = 0, \qquad a_1^2 = a_2, $$

is of fourth order and class and its own polar, with two mutually absolute polar double lines and two absolute generators of each set and given by

$$ (a_1 - a_2)(x_0^2 + x_3^2)x_1x_2 + (a_1 + a_2)(x_1^2 + x_2^2)x_0x_3 = 0 $$

in projective coordinates.

The strip surface is the well-known Clifford surface of zero curvature. The general homographic congruence is of the third order and class, is its own absolute polar, contains all generators of the absolute and may be constructed in 15 ways as partial intersection of two of six tetrahedral complexes.

This dissertation has many traits which are constantly returning, in some form or another, throughout Coolidge's later work; especially his interest in non-euclidean geometry, in line geometry and his preoccupation with the geometry of complex numbers, as well as his solid technical skill in handling the analytical apparatus of a particular geometry. Most of his later results he embodied in his books, and we can therefore do fair justice to them by following the trend of these books.

The *Non-euclidean Geometry* [**10**] was one of the first books on this subject written in English. The author had several foreign examples to draw from, especially the works by Killing and Liebmann, but he preferred to follow his own approach. He chose an axiomatic introduction, selecting his axioms so that at one point he could make his choice of three avenues: one leading to euclidean, and the two others to the two non-euclidean geometries. After the synthetic buildup he develops the trigonometry and the analytic formulae with the extension to space and congruent transformations. The author's love, as he tells us, is in fruits rather than in roots, and so we get a full treatment of conics and quadrics, where he can utilize his own work [3]. Areas and volumes follow. Of permanent value is the algebraic and differential geometry of line and line crosses where we find the subject of the thesis anew. Here we also meet Study's representation of all real rays of elliptic space on pairs of real points of two euclidean spheres, which establishes a relation between analytic functions of a complex variable and the isotropic congruences in elliptic space [**6**]. The importance of this representation has since been brought out again by Blaschke (Annali di Matem., vol. 28, 1940, p. 205), now with the use of quaternions.

Interest in line geometry easily leads to interest in sphere geometry by means of the transition from Plücker coordinates into pentaspherical coordinates. They play an important role in Coolidge's *Treatise on the Circle and Sphere* [**19**].

However, circles and spheres have an importance beyond that indicated by Klein and Lie transformations, as already Euclid has shown. Coolidge tried to bring some organization into the colossal mass of literature dealing with circle and sphere, and so we have a book in which we can find the circles of Apollonius, Feuerbach, Lemoine and Brocard as well as the chains of DeLongchamps (these and other sets of circles were favorites with Coolidge, see [12] and [18]), the cyclids of Darboux and the results of Laguerre and Lie. Among Coolidge's own contributions we find his proof that equilong transformations of the plane depend on an analytic function of one complex variable (or the conjugate complex) [7], his study of metrical relations in Lie transformations by comparing distances on lines with angles of spheres, keeping in mind that the contact transformations which carry lines into lines form a G_6, while those that leave the angles of spheres intact form a G_{10} [13]. Restriction of this group leads to a close analogy between non-euclidean line geometry and the euclidean of the so-called circle cross. Such a circle cross, corresponding to Study's line cross, is composed of a pair of circles so related that every sphere through one is orthogonal to the other. There are also *paratactic* circles, which are cut twice orthogonally by at least one third circle, and hence cut orthogonally twice by ∞^1 circles. This theory leads to the so-called pseudo-cylindroid, which is a one-parameter family of circle crosses dependent linearly upon those of two fixed crosses and is a surface of the eighth order [14]. In this book we find also Coolidge's theory of congruences, as well as of complexes of circles in space [16]. It is impossible here to do full justice to this highly original treatise so full of pleasant theorems; it has had considerable influence on the later development of conformal algebraic and differential geometry. In later life Coolidge also tried his hand at congruences of conics in space [54], where the methods used for circles must be changed; he succeeded by applying Darboux' method of the trihedron moving with the conic with respect to a fixed trihedron.

In the *Geometry of the Complex Domain* [35] Coolidge returns, again in a different way, to the ideas of Segre and Study. The central thought is the geometrical representation of points with complex coordinates in the binary and ternary domain and the investigation of their configurations in their own right, and not, as is customary, as direct generalizations of configurations in real space. The central concept is the chain, which is defined in the complex binary domain as collection of points for which a) the cross ratios of any four are real, and b) for which there exists one point having with any three points of the system any given real cross ratio different from 0, 1, ∞. A point is defined as an object in one-to-one correspondence with a pair of homogeneous complex coordinate values (x_1, x_2) not both zero. In the Gaussian plane the chain appears as a circle or line. Any collineation and anti-collineation carries a chain into a a chain—we see how close we are again to non-euclidean and to circle geometry. Involutory anti-collineations have fixed points defined by an equation of the form $(ax)(\bar{a}\bar{x})=0$ ($\bar{x}$ conjugate complex to x) and thus we arrive at Hermitian forms, and later to forms $(ax)^n(\bar{a}\bar{x})^n=0$. These Hermitian forms and their geom-

etry were subject of special papers by Coolidge [**23, 26**]. Much of the text is taken up by the theory of chain congruences and their relation to Hermitian metrics. Differential properties are included as well. The book ends with complex space of three dimensions and an axiomatic treatment of complex geometry based on Von Staudt. It was in some respects a forerunner of modern Hermitian geometry, just as the previous book anticipated modern conformal geometry.

Where the three previous books form a kind of loose trilogy, the fourth geometrical treatise has a position of its own. Coolidge's first paper on algebraic curves [**17**] dealt with his attempt to attach a genuinely geometrical meaning to the order n of a plane algebraic curve $a^n x = 0$ with at least one real continuous branch. He found it by defining the order as the number greater by unity than the number of successive polar systems necessary to determine the curve, and on this based definitions of class and deficiency. Somewhat later he proved that the only algebraic identities involving any combination of real or total singularities, which are valid for all real algebraic plane curves, are those deducible from the Plücker and Klein equations [**20**]. There are 16 characteristics denoting their irregularities, of which seven are "total," that is, defined without taking the distinction between real and imaginary into account. The theory of algebraic curves continued to interest him [**31, 34, 41, 44**] and eventually brought him to write his book on plane algebraic curves [**46**]. Here, as in the case of his first book, he entered a field in which there was a need for a modern English book, though several good books in other languages existed.

The method of the book is algebraic, though no reference is made to algebraic number theory. In the exposition stress is laid on the concept of correspondence. There is much on real circuits of curves and on the reduction of singularities, and on systems of points on a curve. Linear and non linear systems of curves are studied, the latter in the form indicated in [**44**]; and a good deal is said about Cremona transformations. The emphasis is on the general curve rather than on curves of the third and fourth degree, though they are not neglected. The whole treatment is strongly influenced by the Italian school of Bertini, Segre, Castelnuovo, Enriques and Severi, all inspired by the genius of Max Noether.

At present, with so many books on probability in the English language, it is a little strange to realize that Coolidge's work on this subject [**37**] appeared at a time that there existed hardly any modern English treatise in this field. Coolidge here followed the example he set in two of his previous books, so that his text helped to open a new period of interest in probability, at any rate in the English speaking countries, an interest which did no longer center on the classical theory of games, but also allowed ample room for error and correlation theory and statistics in general. However, Coolidge's first paper on probability [**11**] was still on the theory of games; it elaborated on variation of the problem of the ruin of the player. It contains a moral lesson:

"The average gambler will say 'the player who stakes his whole fortune on

a single play is a fool, and the science of mathematics can not prove him to be otherwise.' The reply is obvious: 'The science of mathematics never attempts the impossible, it merely shows that other players are greater fools'."

Some papers of a more statistical nature followed [**24, 27, 32**], which eventually found their niche in the book; however, it was not only the attention paid to statistics which added interest to the book. One of its features was its break with the classical definitions of probability as a measure of ignorance, and its introduction of the so-called frequency definition. This definition took probability out of the field of abstract mathematics and played it into that of the natural sciences. We shall not evaluate its merits and demerits here, suffice to say that this approach to probability had also strongly been argued in Germany by Von Mises and others of his school. The favorable review of the book in the *Jahresbericht der Deutschen Mathematiker Vereinigung* stressed this aspect; and shortly afterwards a German edition appeared [**37a**]. The book makes entertaining reading and has a particular Coolidge touch, with its already quoted crack from Kipling and the following formulation of a famous 18th century problem:

"If all the inhabitants of Chicago should meet together in one place and get extremely drunk, and then try to go home by guess-work, the chances that at least one would get back to his own bed are almost two out of three."

Coolidge would not for the world have substituted Boston for Chicago, or Cambridge for Boston.

The three books on the history of mathematics are full of thorough information, the result of the mature thinking of a scientist about the way his own field of learning came into being. The *History of Geometrical Methods* [**55**], written in independent emulation of Chasles' century old *Aperçu Historique*, is a broad survey of almost all fields of geometrical endeavor from the Greeks to the present. In his mastery of so many different branches, his understanding of the relations between them and the directness of his approach, Coolidge endowed his text with something of the spirit of Felix Klein. The book will remain an excellent introduction to all these fields of geometry which were predominant in the nineteenth century, while keeping a window open to the present. The *History of Conic Sections and Quadric Surfaces* [**61**] is not only a report on the many authors who, from Menaechmos on, built up our present theory of conics and quadrics; it is a critical appraisal of their work as well, and a good and quite original introduction into the theory. The *Mathematics of Great Amateurs* [**64**] is a collection of essays on several men of the past who contributed to mathematics but were essentially prominent in other fields, such as Plato, Omar Khayyam and Diderot. For some reason Napier and L'Hospital are also included. The merit of such a book lies in the fact that here we find a modern professional opinion on the mathematical works of these authors of a by-gone age by one who has taken the trouble to read their works. Particularly valuable to those who are interested in geometry and art are the chapters on Pietro dei Franceschi,

Leonardo Da Vinci and Albrecht Dürer. Some more on this subject can be found in Coolidge's review of Ivins' book [**78**].

The many smaller papers on the history of mathematics cover different territory. Some deal with the growth of American mathematics [**36, 38, 45, 52, 56**] some with persons [**29, 38, 40, 48, 50, 58, 60, 68**], some with whole fields [**43, 49, 51, 63**] and several, especially those of his later years, with special topics on which he was able to throw some historic light [**53, 65, 66, 67, 69, 70, 71**]. The latter are particularly useful to instructors and college textbook writers who like to obtain a better grasp of the historical foundation of their topics.

More than a passing word should be said about Coolidge's book reviews. Let us not forget, there are reviews and reviews. Many do not figure in the creative output of a person, they are written for little else than duty, the desire to own a free copy, or to see one's name in print. But there are reviews that must count as creative writing. Reviews are a common form of literary criticism, one of our most discussed philosophical treatises is in the form of a book review, Leibniz first announced the integral calculus in a book review, and Asa Gray's review of Darwin's *Origins of Species* made biological history. When Coolidge reviewed a book which caught his fancy, he gave it his whole attention, matched his wits against those of the author, spared no praise nor spared the rod, and by the time he was through the review had become a veritable essay. Several of these reviews are even now instructive and can be evaluated as regular papers comparable to [**33**] or [**62**]; they contain some of that science-between-the-lines which is usually reserved for the classroom or for private conversation, and is responsible for half the fun of teaching and study. Our bibliography contains some of these valuable book reviews.

It is a curious fact that with all his vigorous scientific activity Coolidge always seems to have labored under the thought that his type of mathematics was somehow passé, that he was in a certain sense an epigone. With him it was not a reaction due to old age, a phenomenon common enough among scientists. We find the thought expressed in the very first sentence of his very first book, written when Coolidge was still fresh from Bonn: "The heroic age of non-euclidean geometry has passed." Later we see him writing about the "rise and fall" of projective geometry [**49**]. There are more such places in his writings. It is true that certain methods and certain aspects of geometry with which Coolidge was particularly familiar had passed their prime. But Coolidge, in his diffidence, may not have seen as clearly as others how his own work contributed to that rejuvenation through which all science must pass in order to remain alive. Coolidge, thinking in 1909, that the hey-day of non-euclidean geometry was over, did not foresee the great flowering of non-euclidean geometries which came in the wake of Einstein's theory of relativity. And we have already seen that some of his books can even now be considered as pioneer ventures in fields that are very much alive. Significant mathematics is never antiquated, even if there are periods in which certain fields lie dormant.

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NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

BELL TELEPHONE LABORATORIES FELLOWSHIPS

The Bell Telephone Laboratories has announced the establishment of a fellowship program through which it will grant funds for students doing graduate study in electrical communications.

To be known as the Bell Telephone Laboratories Fellowships, the awards are for study of one or two years leading to a doctorate. Each fellowship carries a grant of $2,000 to the fellow and an additional $2,000 to cover tuition, fees, and other costs to the institution at which he chooses to study.

Recipients of the fellowships will not be required to limit their study to

electrical engineering, although the field of study and research must have a bearing on electrical communications. They may, for example, pursue various branches of mathematics, physics, chemistry, engineering mechanics, and mechanical engineering. Fellows may make their own choice of an academic institution within the United States.

EDUCATIONAL TESTING SERVICE PSYCHOMETRIC FELLOWSHIPS

The Educational Testing Service is offering for 1956–1957 its ninth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,500 a year and are normally renewable. Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School.

Suitable undergraduate preparation may consist either of a major in psychology with supporting work in mathematics, or a major in mathematics together with some work in psychology. However, in choosing fellows, primary emphasis is given to superior scholastic attainment and demonstrated research ability rather than to specific course preparation.

The closing date for completing applications is January 12, 1956. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

PERSONAL ITEMS

Professor J. R. Mayor of the University of Wisconsin has been appointed Director of the AAAS Science Teaching Improvement Program.

Georgia Institute of Technology announces the following: Associate Professor M. B. Sledd has been promoted to a professorship; Assistant Professor J. A. Nohel has been promoted to an associate professorship; Dr. W. R. Smythe, Jr., of Duke University has been appointed to an assistant professorship.

Michigan State University reports: Professor J. S. Frame, head of the Department of Mathematics, is temporarily serving as Acting Head of a new Department of Statistics, recently formed at the University; Professor Joseph Meixner of the Technische Hochschule, Aachen, Germany, has been appointed Visiting Professor for the year 1955–1956; Dr. M. L. Tomber of Amherst College and Assistant Professor R. L. Blair of the University of California at Davis have been appointed to assistant professorships; Mr. S. K. Berberian, Mr. C. C. Faith, and Mrs. Delia Koo have been appointed to instructorships.

United States Naval Postgraduate School anounces: Assistant Professor E. J. Stewart of California State Polytechnic College has been appointed to an associate professorship; Associate Professor B. J. Lockhart has been promoted to a professorship; Professor C. H. Rawlins has retired with the title of Professor Emeritus.

University of Connecticut reports the following: Dr. Felix Haas of Princeton University and Dr. R. P. Gosselin of Youngstown College have been appointed to assistant professorships; Dr. J. A. Schatz of Lehigh University and Dr. Violet B. Haas of Immaculata College have been appointed to instructorships; Dr. Lida K. Barrett has been appointed to an instructorship at the Waterbury Branch.

At the University of Nebraska: Assistant Professors Edwin Halfar and L. K. Jackson have been promoted to associate professorships; Dr. Arne Magnus has been promoted to an assistant professorship; Mr. D. W. Miller and Dr. E. J. Schweppe have been appointed to instructorships.

University of New Hampshire announces the following: Mr. R. W. Sloan of the University of Illinois has been appointed to an assistant professorship; Dr. S. L. Ross of Boston University has been appointed to an instructorship; Assistant Professors J. B. Crabtree and H. G. Rice have been promoted to associate professorships.

University of Washington reports: Dr. Anne C. Davis of the University of California, Dr. Joel Franklin of New York University, and Dr. R. S. Pierce of Harvard University have been appointed to assistant professorships; Mr. Alfred Descloux of the University of North Carolina has been appointed to an instructorship; Dr. H. C. Wang of the Institute for Advanced Study has been appointed Visiting Lecturer; Dr. A. E. Livingston and Dr. R. F. Tate have been promoted to assistant professorships; Professor Edwin Hewitt is on leave of absence for the academic year 1955–1956 on a Guggenheim Fellowship at the Institute for Advanced Study; Associate Professor D. G. Chapman has returned from his leave of absence at Oxford University; Professor R. A. Beaumont, who was a Ford Fellow at the Institute for Advanced Study, has returned from his leave of absence.

Dr. S. S. Abhyankar, previously a teaching fellow at Harvard University, is now an associate in mathematics at Columbia University.

Dr. C. M. Ablow, formerly a research engineer for Boeing Airplane Company, Seattle, Washington, has a position as a senior research mathematician for Stanford Research Institute, Menlo Park, California.

Mr. Walter Abramowitz, previously a physicist for Solid State Research Institute, New York City, now has a position as an analytical engineer for Pratt and Whitney Aircraft, East Hartford, Connecticut.

Mr. W. R. Allen, formerly an associate at Forrestal Research Center, Princeton University, is now a member of the technical staff of the David Sarnoff Research Center, Radio Corporation of America Laboratories, Princeton, New Jersey.

Associate Professor A. G. Anderson, chairman of the Department of Mathematics of Duquesne University, has accepted a position as a mathematician at the Research Center of Jones and Laughlin Steel Corporation, Pittsburgh, Pennsylvania.

Mr. W. P. Anderson, formerly an instructor at the University of Minnesota,

has a position as a mathematician at the Rand Corporation, Santa Monica, California.

Mr. R. R. Archer, previously a student at Massachusetts Institute of Technology, has been appointed to an instructorship at the Institute.

Dr. W. G. Bade of Yale University has been appointed to an assistant professorship at the University of California.

Professor L. C. Bagby of Lawrence Institute of Technology has accepted a position as staff industrial engineer with Burns and Roe, Detroit, Michigan.

Assistant Professor R. W. Bagley of the University of Kentucky has a position as a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Associate Professor G. A. Baker of the University of California at Davis has been promoted to the position of Professor of Mathematics and Statistician in the Experiment Station.

Mr. Sheldon Balk, formerly a graduate student at Arizona State College, has a position as an equipment engineer for the Douglas Aircraft Company, Tucson, Arizona.

Mr. C. W. Barnett, previously a mathematician in the Mathematical Analysis Department of Lockheed Aircraft Corporation, Marietta, Georgia, is now a research programmer at Louisiana State University.

Dr. R. G. Bartle of Yale University has been appointed to an assistant professorship at the University of Illinois.

Dr. J. D. Baum of Oberlin College has been promoted to an assistant professorship.

Mr. T. H. Bedwell, previously an instructor at Southern State Teachers College, Springfield, South Dakota, has been appointed to an instructorship at California State Polytechnic College.

Mr. R. J. Beeber, formerly an instructor at St. Peter's College, Jersey City, New Jersey, is now a mathematician with the I. B. M. Corporation, New York City.

Dr. P. R. Beesack, previously a research assistant at Washington University, has been appointed to an assistant professorship at McMaster University.

Dr. Donald C. Benson of Princeton University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Mr. W. H. Benson, retired captain of the United States Navy, has been appointed to an assistant professorship at Dickinson College.

Mr. H. S. Berg, previously an instructor at State Teachers College, St. Cloud, Minnesota, has taken a position as a mathematical analyst at Lockheed Aircraft Corporation, Burbank, California.

Mr. H. H. Berry, formerly a numerical analyst at General Electric Company, Cincinnati, Ohio, has a position as a mathematician at the Avco Manufacturing Company, Crosley Division, Cincinnati.

Mr. D. W. Blakeslee of San Francisco State College has been promoted to an assistant professorship.

Associate Professor W. J. Blundon of Memorial University of Newfoundland is on sabbatical leave at University College, London, England.

Dr. C. C. Braunschweiger, recently a teaching assistant at the University of Wisconsin, has been appointed to an instructorship at Purdue University.

Mr. E. B. Bridgforth, previously a statistician in the Nutrition Division of Vanderbilt Medical School, has been appointed Assistant Professor of Biostatistics, School of Medicine, Vanderbilt University.

Dr. W. E. Briggs, recently a research assistant at the University of Colorado, has been appointed to an instructorship at the University.

Professor S. K. Bright of Austin Peay State College has been appointed Research Scientist at the University of Texas.

Mr. R. C. Brown, Jr., of Glenville State College, West Virginia, has a position as a senior aerophysics engineer at Consolidated-Vultee Aircraft Corporation, Fort Worth, Texas.

Assistant Professor W. P. Brown of Michigan State University has been appointed to an assistant professorship at the University of Toronto.

Mr. R. G. Buschman, formerly a fellow at the University of Colorado, has been appointed to an instructorship at the University.

Mr. C. M. Callahan of Florida State University is now a mathematician at the U. S. Navy Mine Defense Laboratory, Panama City, Florida.

Mr. L. N. Caplan, formerly a student at Carnegie Institute of Technology, has a position as a numerical analyst for the General Electric Company, Evendale, Ohio.

Mr. J. A. Carpenter, previously a mathematician with the Ultrasonic Corporation, Cambridge, Massachusetts, has a position as an engineer for Melpar, Inc., Cambridge.

Miss Margaret J. Cotter, recently a teacher at West Orange High School, New Jersey, is now a mathematics teacher at Dwight Morrow High School, Englewood, New Jersey.

Mr. D. C. Davis, formerly a graduate assistant at the University of Oklahoma, has a position as a development engineer at McDonnell Aircraft Corporation, St. Louis, Missouri.

Dr. Ruth M. Davis of the University of Maryland has accepted a position as a mathematician with the Applied Mathematics Laboratory, David Taylor Model Basin, Carderock, Maryland.

Dr. R. A. Dean of California Institute of Technology has been promoted to an assistant professorship.

Mr. E. J. Delate, previously a group leader in the Yerkes Film Plant, E. I. DuPont de Nemours, Buffalo, New York, has been promoted to the position of Statistical Control Supervisor.

Mr. A. W. Dickinson, formerly a student at the University of Massachusetts, is now a National Science Foundation Fellow at the University of North Carolina.

Mr. J. O. Distad, recently a graduate student at Montana State College,

has been appointed to an instructorship at the University of Alaska.

Assistant Professor L. J. Dixon of Arkansas State College has been promoted to an associate professorship.

Mr. D. L. Dunning, previously an assistant actuary at Warner-Watson Inc., Chicago, Illinois, is now teaching at Elmhurst Junior High School, Illinois.

Professor L. A. Dye of the Citadel has been appointed Head of the Department of Mathematics.

Professor Emeritus W. E. Edington of DePauw University has been appointed *Margaret Pilcher Visiting Professor of Mathematics at Coe College for* the year 1955–1956.

Professor D. O. Ellis of the University of Florida has accepted a position as associate mathematician with the Rand Corporation, Santa Monica, California.

Mr. R. C. Foster of Tri-State College has a position as an associate engineer at Lockheed Aircraft Corporation, Burbank, California.

Assistant Professor Cleota G. Fry of Purdue University has been promoted to an associate professorship.

Dr. W. M. Gilbert of the State College of Washington has been appointed to an assistant professorship at Iowa State College.

Mr. R. L. Greene, formerly a design engineer with Westinghouse Electric Corporation, Sharon, Pennsylvania, has a position as a design engineer at Chicago Standard Transformer, Chicago, Illinois.

Dr. H. C. Griffith of the University of Connecticut has been appointed to an assistant professorship at Florida State University.

Dr. J. E. Hafstrom of the University of Minnesota, Duluth Branch, has been promoted to an assistant professorship.

Professor N. A. Hall of the University of Minnesota has been appointed Assistant Dean of the Graduate Division of the College of Engineering, New York University.

Mr. S. S. Holland, Jr., previously a mathematician with Army Chemical Corps, Army Chemical Center, Maryland, has a position as a mathematician at Technical Operations Inc., Arlington, Massachusetts.

Mr. R. H. Hoskins, recently actuarial associate with John Hancock Mutual Life Insurance Company, Boston, Massachusetts, has been appointed Assistant Group Actuary of the Company.

Assistant Professor A. V. Houghton, III, of Bradley University has been promoted to an associate professorship.

Dr. J. L. Howell of the University of Delaware has been appointed to an assistant professorship at the University of Alabama.

Dr. P. F. Hultquist of the University of Colorado has been promoted to an assistant professorship.

Assistant Professor T. A. Jeeves of the University of California has a position as a research mathematician for Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania.

Professor Mark Kac of Cornell University is on leave during the current academic year on an Air Force Contract in Geneva, Switzerland.

Associate Professor J. W. Kaiser of Kent State University has been promoted to a professorship.

Assistant Professor Karlis Kaufmanis of Gustavus Adolphus College has been promoted to an associate professorship.

Mr. R. B. Kelman, formerly a student at the University of California, is now a computing analyst for North American Aviation, Downey, California.

Mr. R. R. Kemp, previously a teaching assistant at Massachusetts Institute of Technology, has been promoted to an instructorship at the Institute.

Mr. Hewitt Kenyon of the University of Rochester has been promoted to an assistant professorship.

Mr. E. G. Kimme, formerly a teaching assistant at the University of Minnesota, has been appointed to an instructorship at Oregon State College.

Associate Professor V. L. Klee, Jr., of the University of Washington has been appointed Visiting Associate Professor at the University of California at Los Angeles for the academic year 1955–1956.

Assistant Professor L. A. Kokoris of the University of Washington has been appointed to an assistant professorship at Washington University.

Mr. H. P. Kuang, formerly a research fellow and senior statistician at the University of Minnesota, has been appointed to a professorship at Agricultural and Technical College, Greensboro, North Carolina.

Mr. Y. L. Luke, research mathematician at Midwest Research Institute, Kansas City, Missouri, has been promoted to the position of Head of the Mathematical Analysis Section.

Dr. M. D. Marcus of the University of British Columbia has been promoted to an assistant professorship.

Mr. P. L. Marshall has a position as a project aerodynamicist with North American Aviation Corporation, Columbus, Ohio.

Mr. H. J. McBlaine, Jr., formerly a training instructor in meteorology, Department of Weather School, Chanute Air Force Base, Illinois, is now a mathematician at the U. S. Naval Ordnance Laboratory, Corona, California.

Associate Professor S. S. McNeary of Drexel Institute of Technology has been promoted to a professorship.

Assistant Professor D. M. Merriell of Robert College, Istanbul, Turkey, has been promoted to an associate professorship.

Professor W. E. Milne of Oregon State College has retired.

Associate Professor T. W. Moore of the United States Naval Academy has been promoted to a professorship.

Mr. T. P. Mulhern, previously a graduate student at Brown University, has been appointed to a research assistantship at the University.

Dr. D. J. Myatt, a research mechanical engineer with the Atlantic Research Corporation, Alexandria, Virginia, has been appointed to an associate professorship at Antioch College.

Associate Professor Henry Parkus of Michigan State University has been appointed to a professorship at the University of Technology in Vienna.

Mr. K. C. Peng, previously a statistical quality control engineer at Parke, Davis and Company, Detroit, Michigan, has a position as a mathematical statistician with the Chrysler Corporation, Detroit.

Associate Professor Ruth M. Peters of St. Lawrence University has been promoted to a professorship.

Mr. C. W. Pflaum, formerly an instructor at John Marshall High School, Rochester, New York, has been appointed Head of the Department of Mathematics of the School.

Professor Wladimir Seidel of the University of Rochester has been appointed to a professorship at the University of Notre Dame.

Dr. Harry Siller, previously a statistician for the Jewish Education Committee, New York City, has been appointed to an assistant professorship at Hofstra College.

Professor W. S. Snyder of the University of Tennessee has been appointed to the staff of the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Mr. W. K. Spears has accepted a position as engineer with the Electronics Corporation of America, Cambridge, Massachusetts.

Mr. V. E. Thomas, formerly an instructor at West Virginia University, has a position as a performance analyst for Pratt and Whitney Aircraft, East Hartford, Connecticut.

Assistant Dean C. W. Trigg of Los Angeles City College has been promoted to the position of Dean of Instruction; Mr. Trigg also served as Dean of the Summer Session of the College.

Assistant Professor J. M. Wolfe of Brooklyn College has been promoted to an associate professorship.

Mr. T. J. Wolinski, previously a student at St. John's College, Brooklyn, New York, has a position as a mathematician at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

Professor R. C. Yates of Virginia Polytechnic Institute has been appointed Professor and Head of the Department of Mathematics of the College of William and Mary.

Mr. F. R. Yett of Iowa State College has been appointed to an assistant professorship at Long Beach State College.

Mr. J. A. Zilber of Johns Hopkins University has been appointed to an instructorship at the University of Illinois.

Professor M. A. Girshick of Stanford University died on March 2, 1955.

Mr. George Hartnell, formerly a research geologist with the U. S. Coast and Geodetic Survey, died on March 20, 1955. He was a member of the Association for thirty years.

Professor H. C. Shaub, head of the Department of Mathematics of Washington and Jefferson College, died on July 20, 1955. He was a member of the As-

sociation for thirty-four years.

Professor Augustus Sisk of Maryville College died on February 1, 1955. He was a member of the Association for twenty years.

Professor Emeritus E. J. Townsend, formerly dean of the College of Science, University of Illinois, died on July 8, 1955.

Mr. E. L. Vanderburgh of Pueblo Junior College, Colorado, died on October 27, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SIXTH SUMMER MEETING OF THE ASSOCIATION

The thirty-sixth summer meeting of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Monday and Tuesday, August 29 and 30, 1955, in conjunction with the summer meetings of the American Mathematical Society, the Association for Symbolic Logic, the Econometric Society, the Industrial Mathematics Society, the Institute of Mathematical Statistics, the Pi Mu Epsilon Fraternity, and the Society for Industrial and Applied Mathematics. Eleven hundred and seventy persons were registered, including the following four hundred and eighty-nine members of the Association:

Smbat Abian, R. P. Agnew, M. I. Aissen, A. A. Albert, H. W. Alexander, Bess E. Allen, W. R. Allen, C. B. Allendoerfer, A. G. Anderson, R. V. Andree, H. A. Antosiewicz, K. J. Arnold, R. A. Askey, W. L. Ayres, J. L. Bagg, R. W. Bagley, R. P. Bailey, Frances E. Baker, J. M. Barbour, R. H. Bardell, Joshua Barlaz, R. W. Barnard, W. E. Barnes, I. A. Barnett, D. Y. Barrer, J. H Barrett, R. C. F. Bartels, M. A. Basoco, P. T. Bateman, M. T. Battles, Jr., J. D. Baum, H. M. Beatty, W. A. Beck, E. G. Begle, Louise G. Belai, Richard Bellman, J. S. Bendat, H. A. Bender, Dean C. Benson, Dorothy L. Bernstein, H. H. Berry, H. R. Beveridge, W. S. Bicknell, Kurt Bing, R. H. Bing, C. J. Blackall, D. W. Blackett, Shirley A. Blackett, David Blackwell, R. L. Blair, J. H. Blau, H. D. Block, G. M. Bloom, M. Isobel Blyth, R. D. Boswell, Jr., S. G. Bourne, Julia W. Bower, C. B. Boyer, J. W. Brace, Fred Brafman, G. U. Brauer, C. C. Braunschweiger, C. S. Brewster, C. F. Briggs, H. W. Brinkmann, Paul Brock, Arlen Brown, Marjorie L. Browne, R. H. Bruck, R. C. Buck, G. C. Byers, J. M. Calloway, E. A. Cameron, E. J. Camp, H. H. Campaigne, H. E. Campbell, C. E. Capel, Dorothy I. Carpenter, R. E. Carr, W. B. Carver, Jeremiah Certaine, Lamberto Cesari, C. Y. Chao, S. S. Chern, P. L. Chessin, R. V. Churchill, A. G. Clark, E. H. Clarke, Nathaniel Coburn, C. J. Coe, L. M. Coffin, L. W. Cohen, A. J. Coleman, E. P. Coleman, N. B. Conkwright, C. H. Cook, T. F. Cope, A. H. Copeland, Sr., R. R. Coveyou, W. H. H. Cowles, V. F. Cowling, H. S. M. Coxeter, C. C. Craig, E. H. Crisler, J. W. Crispin, Jr., Helen F. Cullen, A. B. Cunningham, H. B. Curry, J. H. Curtiss, John F. Daly, D. A. Darling, Robert Davies, R. L. Davis, Ruth M. Davis, Violet B. Davis, D. E. Deal, E. R. Deal, R. B. Deal, Jr., R. A. Dean, R. Y. Dean, M. L. DeMoss, E. K. Dorff, Jim Douglas, Jr., H. H. Downing, W. L. Duren, Jr., P. S. Dwyer, J. C. Eaves, W. F. Eberlein, Albert Edrei, Carolyn Eisele, J. D. Elder, J. G. Elliott,

D. O. Ellis, Wade Ellis, M. P. Emerson, M. P. Epstein, H. P. Evans, H. S. Everett, W. H. Fagerstrom, Catherine S. Feeley, W. J. Feeney, H. F. Fehr, F. A. Ficken, J. V. Finch, N. J. Fine, D. T. Finkbeiner, II, C. H. Fischer, N. C. Fisk, G. E. Forsythe, M. K. Fort, Jr., Tomlinson Fort, J. S. Frame, Evelyn Frank, C. H. Frick, C. V. Fronabarger, T. C. Fry, R. E. Fullerton, J. W. Gaddum, T. M. Gallie, Jr., R. A. Gambill, G. N. Garrison, H. M. Gehman, K. G. Getman, B. P. Gill, Leonard Gillman, Wallace Givens, A. M. Gleason, Casper Goffman, Michael Goldberg, Samuel Goldberg, Michael Golomb, W. A. Golomski, D. B. Goodner, R. M. Gordon, S. H. Gould, Arthur Grad, L. M. Graves, R. L. Graves, J. W. Green, F. L. Griffin, J. S. Griffin, Jr., G. W. Grotts, V. G. Grove, W. E. Grove, P. E. Guenther, H. M. Gurk, Theodore Hailperin, Franklin Haimo, Marshall Hall, Jr., P. R. Halmos, J. W. Hamblen, P. C. Hammer, H. W. Handsfield, Frank Harary, W. J. Hardell, W. L. Hart, H. L. Harter, M. C. Hartley, T. W. Hatcher, G. E. Hay, Katharine E. Hazard, T. J. Head, H. S. Heaps, G. A. Hedlund, E. R. Heineman, R. G. Helsel, Melvin Henriksen, Edwin Hewitt, T. H. Hildebrandt, T. W. Hildebrandt, J. T. Hinely, Jr., J. J. L. Hinrichsen, A. J. Hoffman, Roslyn B. Hoffman, S. P. Hoffman, Jr., Walter Hoffman, R. V. Hogg, F. E. Hohn, L. Aileen Hostinsky, D. B. Houghton, A. S. Householder, C. C. Hsiung, R. C. Huffer, M. Gweneth Humphreys, W. R. Hutcherson, Jane C. Ingersoll, H. G. Jacob, Jr., Bernard Jacobson, L. A. Jehn, F. I. John, L. W. Johnson, R. E. Johnson, A. W. Jones, F. B. Jones, H. T. Jones, L. O. Jones, P. S. Jones, Mark Kac, L. H. Kanter, Wilfred Kaplan, Irving Kaplansky, L. C. Karpinski, Chosaburo Kato, Leo Katz, M. W. Keller, C. E. Kelley, J. E. Kelley, L. M. Kells, J. G. Kemeny, J. H. B. Kemperman, D. E. Kibbey, E. C. Kiefer, S. H. Kimball, M. S. Klamkin, L. A. Knowler, R. J. Koch, F. W. Kokomoor, L. A. Kokoris, Jacob Korevaar, R. R. Korfhage, D. M. Krabill, Max Kramer, G. R. Kuhn, Harold W. Kuhn, Solomon Kullback, L. C. Labowitz, R. J. Lambert, R. E. Langer, E. H. Larguier, J. P. LaSalle, C. G. Latimer, E. B. Leach, J. R. Lee, Marguerite Lehr, R. A. Leibler, Walter Leighton, K. B. Leisenring, F. C. Leone, W. J. Leveque, Norma L. Lindemann, B. W. Lindgren, A. E. Livingston, B. J. Lockhart, K. L. Loewen, F. W. Lott, Jr., D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, Y. L. Luke, R. W. MacDowell, W. G. Madow, C. G. Maple, Morris Marden, A. M. Mark, A. D. Martin, N. M. Martin, Ceslovas Masaitis, J. C. Mathews, K. O. May, B. H. McCandless, P. J. McCarthy, N. H. McCoy, S. W. McCuskey, W. C. McDaniel, Edith A. McDougle, W. R. McEwen, A. W. McGaughey, J. H. McKay, J. E. McLaughlin, R. M. McLeod, A. E. Meder, Jr., L. E. Mehlenbacher, Paul Meier, D. F. Mela, E. P. Merkes, B. E. Meserve, D. M. Mesner, Fred Meyer, D. D. Miller, E. E. Moise, Harriet F. Montague, Mabel D. Montgomery, J. T. Moore, W. K. Moore, Vera T. Morris, F. C. Mosteller, L. T. Moston, E. J. Moulton, H. T. Muhly, S. B. Myers, W. H. Myers, H. W. Nace, H. E. Nelson, E. D. Nering, Morris Newman, E. A. Nordhaus, E. S. Northam, E. P. Northrop, C. O. Oakley, E. N. Oberg, R. H. Oehmke, Theresa M. C. Oehmke, Rufus Oldenburger, E. G. Olds, L. F. Ollman, C. E. Olsen, Joseph Oppenheim, G. M. Ortner, W. R. Orton, Jr., Morris Ostrofsky, F. F. Otis, R. R. Otter, J. C. Oxtoby, L. J. Paige, O. O. Pardee, G. P. Paternoster, G. A. Paxson, W. H. Pell, F. W. Perkins, H. P. Pettit, C. W. Pflaum, C. R. Phelps, George Piranian, Everett Pitcher, J. C. Polley, G. B. Price, K. S. Purdie, A. L. Putnam, J. W. Querry, Gustave Rabson, Tibor Rado, Henry Rainbow, G. Y. Rainich, E. D. Rainville, Gordon Raisbeck, G. E. Raynor, C. B. Read, M. O. Reade, W. T. Reid, Irving Reiner, Haim Reingold, C. E. Rhodes, D. E. Richmond, P. R. Rider, J. D. Riley, R. F. Rinehart, J. M. Robb, G. deB. Robinson, Robin Robinson, Selby Robinson, F. Virginia Rohde, M. I. Rose, Saul Rosen, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, L. J. Rouse, Mary E. Rudin, H. J. Ryser, Hans Samelson, W. C. Sangren, A. C. Schaeffer, Alice T. Schafer, R. D. Schafer, Robert Schatten, Henry Scheffe, E. D. Schell, Edith R. Schneckenburger, J. J. Schoderbek, Pincus Schub, B. L. Schwartz, W. T. Scott, J. G. Sharp, A. L. Shields, Annette Sinclair, David Singer, James Singer, Sister Mary Carmel, M. F. Smiley, H. W. Smith, Ernst Snapper, E. J. Specht, G. L. Spencer, II, Abraham Spitzbart, J. R. Stallings, Jr., Maria W. Steinberg, H. E. Stelson, C. F. Stephens, Rothwell Stephens, A. D. Stewart, B. M. Stewart, Ruth W. Stokes, R. R. Stoll, Irwin Stoner, W. L. Strother, Anna K. Suter, R. L. Swain, W. C. Swift, T. T. Tanimoto, O. E. Taulbee, William Clare Taylor, H. P. Thielman, G. B. Thomas, Jr., G. L. Thompson, D. E. Thoro, R. M. Thrall, W. J. Thron, H. S. Thurston, John Todd, M. L. Tomber, C. B. Tompkins, Eugenia I. Trapp, C. K. Tsao, A. W. Tucker, J. W. Tukey, J. B. Tysver,

F. E. Ulrich, W. R. Utz, Jr., Henry Van Engen, Helen E. Van Sant, T. L. Wade, Jr., R. W. Wagner, E. A. Walker, R. J. Walker, A. D. Wallace, S. E. Warschawski, G. C. Webber, J. V. Wehausen, W. G. Weideman, E. T. Welmers, F. J. Weyl, Frederick R. White, P. M. Whitman, L. R. Wilcox, Marie S. Wilcox, R. L. Wilder, S. S. Wilks, R. L. Wilson, G. N. Wollan, F. L. Wren, F. B. Wright, Jr., C. T. Yang, J. E. Yarnelle, J. W. T. Youngs, J. L. Zemmer, Jr., J. A. Zilber, R. E. Zindler.

Sessions of the Association were held on Monday morning and afternoon in the Natural Science Auditorium of the University of Michigan with President W. L. Duren and Vice-President H. S. M. Coxeter presiding. The Session on Tuesday morning was held in the Auditorium of Rackham Hall with Vice-President G. B. Price and President Duren presiding. The fourth series of Earle Raymond Hedrick Lectures was delivered by Professor Mark Kac of Cornell University. The Program Committee for the meeting consisted of J. W. T. Youngs, Chairman; R. H. Bing, and V. L. Klee, Jr.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture I, "From Vieta to Foundations of Modern Probability Theory," by Professor Mark Kac, Cornell University.

Mathematics for Social Scientists, II (Joint Session with the Econometric Society). Panel Chairman: Professor R. M. Thrall, University of Michigan. Panel Members: Professor J. G. Kemeny, Dartmouth College, Professor W. G. Madow, University of Illinois and Stanford University, Professor Frederick Mosteller, Harvard University, Professor C. O. Oakley, Haverford College.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture II, "From Vieta to Foundations of Modern Probability Theory," by Professor Mark Kac.

"Mathematics as a Profession Today": "In the university," by Professor G. B. Price, University of Kansas; "In government," by Dr. R. A. Leibler, Department of Defense; "In industry," by Dr. T. C. Fry, Bell Telephone Laboratories.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Familiar Things from an Unfamiliar Point of View"; Lecture III, "From Kinetic Theory to Continued Fractions," by Professor Mark Kac.

Business Meeting of the Association.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Michigan Union. Twenty-five of the forty-one members of the Board were present. Among the more important items of business transacted were the following:

President Duren announced the appointment of a Committee on Mathematical Personnel and Education consisting of Tomlinson Fort, Chairman, E. A. Cameron, H. P. Evans, M. Gweneth Humphreys, Ivan Niven, E. P. Northrop, and E. T. Welmers.

It was also announced that G. B. Price had agreed to serve as chairman of the Committee to Study the Activities of the Association.

It was voted to authorize the publication of "Irrational Numbers" by Ivan Niven as Carus Monograph No. 11.

The Board voted to approve the action of the Putnam Trustees in increasing from $1500 to $2500 the value of the scholarship at Harvard University or at Radcliffe College which is awarded annually to one of the five highest individual contestants in the Putnam Mathematical Competition.

BUSINESS MEETING OF THE ASSOCIATION

At a business meeting of the Association held on Tuesday morning, announcement was made of a grant of $25,000 received by the Association from the Ford Foundation for the work of the Committee on the Undergraduate Program and a grant of $20,500 from the National Science Foundation for the continued support of the Program of Visiting Lecturers.

Professor D. E. Richmond reported for the Committee on Visiting Lecturers; Professor J. S. Frame reported on the Employment Register; Professor W. L. Duren reported for the Committee on the Undergraduate Program; and Professor A. W. Tucker told of the newly appointed Commission on Mathematics of the College Entrance Examination Board.

The Secretary announced that the membership of the Association was 5999 on August 25.

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Michigan Union. Thirty-six persons were present representing twenty-one Sections and the New England Region.

Among the topics discussed were contests among high school students sponsored by sections, programs of section meetings, and the financial support of the sections. It was agreed that these meetings of section officers serve a useful purpose and it was voted to continue to hold such meetings in connection with the summer meetings of the Association.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday morning through Friday afternoon. The colloquium lectures on "Jordan Algebras" were delivered by Professor Nathan Jacobson of Yale University. Invited addresses were given by Professors Raoul Bott and Edwin Hewitt. The Econometric Society held its meetings from Monday through Thursday. The Pi Mu Epsilon Fraternity met on Tuesday. The Society for Industrial and Applied Mathematics

met on Tuesday and Wednesday. The Institute of Mathematical Statistics met from Tuesday through Friday. The Association for Symbolic Logic met on Thursday and the Industrial Mathematics Society met on Friday.

The Committee on the Mathematical Training of Social Scientists held a conference on Saturday and Sunday, August 27 and 28. The Commission on Mathematics of the College Entrance Examination Board met on these same dates.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Wilfred Kaplan, Chairman; P. S. Dwyer, H. M. Gehman, P. S. Jones, W. J. LeVeque, R. M. Thrall, R. L. Wilder, J. W. T. Youngs.

Registration headquarters was in the lobby of the West Quadrangle Dormitories of the University of Michigan. Dormitory accommodations and meals were available in West Quadrangle.

A tea in Inglis House for the women attending the meeting was held on Tuesday afternoon, and a square dance on Tuesday evening. A guided tour of the campus was scheduled for Wednesday morning, a picnic at Kensington Metropolitan Park for Wednesday afternoon and evening, a coffee hour for the ladies on Thursday morning and a concert for Thursday evening.

A supervised play program for children was provided throughout the week at the University Elementary School and free baby-sitting service was provided from Monday through Thursday evenings.

The Employment Register, listing positions available for mathematicians, was open for inspection during the week in Mason Hall.

On Tuesday morning, Vice-President M. L. Niehuss of the University of Michigan greeted the assembled mathematicians on behalf of the University. Later, Professor Harold W. Kuhn presented a motion on behalf of the eight organizations meeting at the University of Michigan expressing "thanks and appreciation, especially to the local members of the Committee on Arrangements who, in spite of the unusually large attendance, have provided exceptionally complete accommodations and services to make the week in Ann Arbor a memorable one for us and for our families."

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 54 persons have been elected to membership by the Board of Governors on applications duly certified.

ARLYNE G. ADELSTEIN, M.A. (Western Reserve) University Heights, Ohio.

J. H. BAILEY, Student, University of Rhode Island.

MAJ. J. F. BLACKBURN, Ph.D. (North Carolina) Asst. Professor, U. S. Air Force Academy, Denver, Colo.

C. K. BRADSHAW, M.S. (Nevada) Instr., University of Nevada.

J. P. BRANNEN, B.S. (A.&M.C. of Texas) Instr., Sweeny High School, Texas.

F. C. BREISCH, A.B. in Ed. (Michigan) Whitmore Lake, Michigan.

R. K. BUTNER, Ph.D. (S.U. of Iowa) Asst. Professor, Ohio University.

H. L. CARLSON, M.S. (Wisconsin) Electronic Applications Specialist, National Cash Register Co., Dayton, O.

M. M. CHIRICO, M.A. (Syracuse) Mathematician, Aberdeen Proving Ground, Md.

L. C. DEAN, JR., M.S. (North Texas S.C.) Grad. Asst., Iowa State College.

M. R. DEMERS, M.A. (Buffalo) Asst. Professor, University of Nevada.

C. W. DYCHE, B.A. (Linfield) Grad. Asst., Oregon State College.

MRS. HAZEL E. EVANS, B.A. (Allegheny) Lecturer, University of Pittsburgh.

MILTON FELSTEIN, B.A. (Texas) Fire Control Design Engr., U. S. Naval Gun Factory, Washington, D. C.

D. JOAN GALLAGHER, B.A. (Colorado) Kimball, South Dakota.

G. H. F. GARDNER, Ph.D. (Princeton) Instr., Cornell University.

C. D. GORMAN, A.B. (Missouri) Asst. Instr., University of Missouri.

E. H. GROSSMAN, Vancouver, B. C., Canada.

F. L. HARDY, A.B. (Oglethorpe) Graduate School, Emory University.

ENID B. JACOBSON, M.A. (Temple) Instr., Temple University.

C. E. JAEGER, M.A. (Claremont Grad. School) Instr., Chaffey High School, Ontario, Calif.

F. I. JOHN, M.A. (Boston U.) Pfc., United States Army.

J. A. KALMAN, Ph.D. (Harvard) University College, Auckland, New Zealand.

MR. MYUNGHWAN KIM, Student, University of Alabama.

H. G. LANGE, Chicago, Illinois.

R. P. LANGLANDS, Student, University of British Columbia.

R. A. LUEBBE, M.A. (Cincinnati) Grad. Student, University of Cincinnati.

Y. L. LUKE, M.S. (Illinois) Head, Mathematical Analysis Section, Midwest Research Institute, Kansas City, Mo.

JEANNETTE S. MAGASINY, Student, Temple University.

PAUL MANOS, Student, Western Reserve University.

W. C. MASON, Student, Massachusetts Institute of Technology.

J. C. MATHEWS, Student, Iowa State College.

MRS. DOROTHY L. B. MCCULLOUGH, Student, Baylor University.

R. A. MCHAFFEY, B. S. (Brooklyn P.I.) Grad. Asst., Iowa State College.

G. E. NEU, B.A. (Houghton) Instr., University of Buffalo.

M. G. OSSESIA, B.S. (Pittsburgh) Grad. Student, University of Pittsburgh.

R. V. PARKER, Teacher, Scole School, Diss, Norfolk, England.

F. R. POOLE, B.S. (Nebraska) Instr., Ohio University.

R. M. RILEY, M.S. (Minnesota) Member of the Technical Staff, Bell Telephone Labs., Murray Hill, N.J.

SOLOMON RUBINSTEIN, Student, City College of the City of New York.

F. D. RUDOLPH, JR., B.A. (Williams) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

W. H. SAYRE, Student, Columbia University.

R. M. SHANK, B.A. (Marshall) Grad. Student, University of Kentucky.

BROCK SHUGART, B.S. (Texas) Division Engr., Pacific Valves, Houston, Texas.

J. J. STIPANOWICH, M.S. (Illinois) Asst. Professor, Western Illinois State College.

JANE M. SUNDGAARD, M.A. (Minnesota) Grad. Student, University of Minnesota.

G. M. TRUE, B.A. (Williams) Field Representative, Addison-Wesley Publishing Co., Cambridge, Mass.

J. F. ULRICH, M.A. (Chicago) Teacher, Arlington Heights Township High School, Ill.

S. P. VESELKA, M.S. (N.Y.S.C. for Teachers, Albany) United States Army.

JANET M. WEBER, B.S. (Illinois Wesleyan) Instr., Township High School, Sterling, Ill.

ELIZABETH A. WELLS, B.A. (Phillips) Grad. Asst., University of Oklahoma.

DAVID WERNER, M.S. (Michigan) Mathematician, Naval Proving Ground, Dahlgren, Va.

J. J. WESOLOWSKI, B.S. (St. Mary's) Grad. Student, St. Mary's University.

T. M. WILM, Student, St. John's College, Brooklyn, N. Y.

THE MARCH MEETING OF THE KANSAS SECTION

The fortieth annual meeting of the Kansas Section of the Mathematical Association of America was held at Fort Hays State College, Hays, Kansas, on March 26, 1955. Professor E. C. Stopher, Chairman of the Section, presided.

There were 65 in attendance, including the following 13 members of the Association:

Paul Eberhart, W. C. Foreman, R. W. Gibson, Laura Z. Greene, V. D. Nyhoff, G. B. Price, C. B. Read, L. M. Reagan, Sister M. Nicholas, E. C. Stopher, Robert H. Thompson, Wilmont Toalson, Ferna E. Wrestler.

The following officers were elected for the year 1955–1956: Chairman, Professor Ferna E. Wrestler, University of Wichita; Vice-Chairman, Professor W. R. Scott, University of Kansas; Secretary-Treasurer, Professor Laura Z. Greene, Washburn University of Topeka.

The following papers were presented at the morning and afternoon sessions:

1. *Empirical formula for survival*, by Mr. L. W. Whiteside, Boeing Airplane Company, Wichita, introduced by the Chairman.

2. *Analysis of certain special power forms*, by Mr. R. D. Sinkhorn, University of Wichita.

The paper consisted of an analysis of infinite continued powers with special emphasis on forms in which the base and each of the individual exponents were equal. It was shown that such a power exists for a positive base if and only if this base is less than or at most equal to e; in the case of the zero base, the power oscillates between zero and unity. The cases where the exponents were unlike, negative, or complex were not discussed.

3. *Magic circles*, by Sister M. Nicholas, Marymount College.

There are magic circles as well as magic squares. Magic circles are n concentric circles cut by n diameters with numbers arranged on their intersections in such a way that the sums on all the circles and diameters are equal.

We find mention of magic circles and magic squares in the early Chinese and Japanese books on arithmetic. They placed the 1 in the center of the circles and did not use it in their sums. We find that the sums for this arrangement are equal to $2n^3+3n$, where n is the number of circles and diameters. A second arrangement can be made using 1. The sums for this second arrangement are equal to $2n^3+n$.

Another type of magic circles is a set of interesting circles with numbers placed on the points of intersection of the circles. The numbers on each circle give the same sum.

4. *Some remarks on distance geometries*, by Professor W. L. Stamey, Kansas State College, by title.

5. *A modified course in college geometry*, by Professor J. D. Haggard, Kansas State Teachers College, by title.

6. *A problem of an artist*, by Professor G. W. Smith, University of Kansas, by title.

LAURA Z. GREENE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11-12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20—21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17-18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA, University of Nebraska, Linclon, April 21, 1956.

NEW ENGLAND, Organizational Meeting, University of New Hampshire, Durham, November 26, 1955.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 26, 1955.

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SYMMETRIC AND SELF-DISTRIBUTIVE SYSTEMS

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1. Introduction. A binary operation $a+b$ will be called *symmetric* if it obeys the law:

$$(a + b) + (c + d) = (a + c) + (b + d). \tag{1}$$

This identity will be called the *symmetric law*. The associative and commutative laws together imply the symmetric law, but it neither implies nor is implied by the associative or commutative laws individually. Most of the cases of symmetric operations we shall consider are neither associative nor commutative.

Some of the more interesting cases of symmetric operations are means or averages, which obey also the *idempotent law*:

$$a + a = a. \tag{2}$$

If an operation is both symmetric and idempotent it also obeys the left and right *self-distributive laws*:

$$a + (b + c) = (a + b) + (a + c) \tag{3}$$

$$(a + b) + c = (a + c) + (b + c). \tag{4}$$

The self-distributive laws result from the ordinary distributive laws when the two operations appearing in them are identified. A self-distributive operation is distributive over itself. An operation which is both left and right self-distributive is not automatically also symmetric, although this is often the case. Likewise in the absence of the idempotent law, the symmetric law does not imply the self-distributive laws. But these various laws are related, and may be studied together.

In this paper we shall first study systems of elements with a single binary operation obeying the symmetric or self-distributive laws. These systems will be called symmetric or self-distributive systems. Examples will be given showing how these systems arise in a natural way. We shall then study systems called *pseudo-rings* with two binary operations called addition and multiplication. A pseudo-ring is a symmetric system with respect to its addition operation, a semi-group with respect to its multiplication operation, and the two operations are related by the distributive laws. Generalizing known results on rings, it will be shown that the set of all endomorphisms of a symmetric additive system is a pseudo-ring with the natural definitions of sum and product of endomorphisms, and that any pseudo-ring with right unity element is a sub-pseudo-ring of the pseudo-ring of all endomorphisms of its additive system. We study also the symmetric system of all subsystems of a symmetric system.

Simple systems with one or two operations obeying familiar laws have been much studied in abstract algebra. Examples of these are groups, semi-groups, quasi-groups, loops, semi-lattices, fields, skew-fields, rings, and lattices. The formal laws assumed as postulates in such systems include the following:

A. $a(bc)=(ab)c$. (*associative law*)

B. $aa=a$. (*idempotent law*)

C. $ab=ba$. (*commutative law*)

D. For every pair of elements a and b, distinct or not, there exists a unique element x such that $xa=b$, and a unique element y such that $ay=b$ (*unique inverse law*).

E. There exists an identity element I such that for every element a we have $aI=Ia=a$.

For example, a system with a single operation ab is called a *semi-group* if it obeys A, a *quasi-group* if it obeys D, a *loop* if it obeys D and E, a *semi-lattice* if it obeys A, B, and C, a *commutative semi-group* if it obeys A and C, a *group* if it obeys A and D (and hence also E). A system is called an *abelian group* if A, C, D (and hence also E) hold. If postulates A and B are assumed, we have an *idempotent semi-group*, which may also be called a *skew semi-lattice*. Such systems have been studied by David McLean. Some references to work on various of these abstract systems will be found in the bibliography. In this paper we hope to show that other formal laws, in particular the symmetric and self-distributive laws, either by themselves or in combination with the more familiar laws, may lead to neat theories having useful interpretations and applications.

2. Additive and multiplicative systems. A set of elements will be called a *system* if there is a binary operation defined in it assigning to each ordered pair of elements of the set, distinct or not, a unique element of the set. If the operation is denoted by $a+b$, the set will be called an *additive system*; if it is denoted by ab, the set will be called a *multiplicative system*. There is no essential difference between the two notions, the only difference being that of notation. A system in which the binary operation obeys the symmetric law (1) will be called a *symmetric system*; a system in which the self-distributive laws (3) and (4) hold will be called a *self-distributive system*. We proceed to show the relation of these laws to the idempotent law and to each other, to give some examples of such systems, and to study the set of endomorphisms of these systems.

THEOREM 1. *A symmetric system in which every element is idempotent is also a self-distributive system.*

Proof. By (1) and (2) we have:

$$a+(b+c)=(a+a)+(b+c)=(a+b)+(a+c),\ \text{and} \tag{5}$$

$$(a+b)+c=(a+b)+(c+c)=(a+c)+(b+c). \tag{6}$$

THEOREM 2. *The set of all idempotent elements of a symmetric system forms a self-distributive system.*

Proof. Because of Theorem 1, it is sufficient to verify that in a symmetric system, the sum of idempotent elements is idempotent, namely:

$$a+b=(a+a)+(b+b)=(a+b)+(a+b).$$

THEOREM 3. *In a self-distributive multiplicative system, any multiple of an idempotent element is idempotent.*

Proof. If the element a is idempotent, then by the distributive laws:

$$ab = (aa)b = (ab)(ab), \quad \text{and} \quad ba = b(aa) = (ba)(ba).$$

3. Examples of symmetric systems. Any set of elements forms a symmetric system if addition is defined by $a+b=a$, or $a+b=b$. These operations are also associative and idempotent, and hence by Theorem 1 self-distributive, but not commutative. Such systems are non-commutative self-distributive semi-groups.

Since the symmetric law holds if the associative and commutative laws are both present, any system with an operation that is both associative and commutative is a symmetric system. Hence every abelian group and every commutative semi-group is a symmetric system. Every semi-lattice is both a symmetric and a self-distributive system, since it is idempotent. Every lattice is both a symmetric system and a self-distributive system with respect to both lattice operations.

Most means or averages are symmetric and self-distributive, such as the arithmetic, geometric, and harmonic means. More generally, the elements of any field or ring form a symmetric system with respect to an operation $x \circ y$ defined by

$$x \circ y = ax + by, \tag{7}$$

where a and b are fixed ring elements. If $a+b=1$, where 1 is the unity element of the ring or field, then the operation $x \circ y$ of (7) is also idempotent, and hence self-distributive. In this case the operation is a kind of mean or average.

This procedure for forming means may be generalized as follows. Let $x+y$ be the addition operation of any symmetric additive system S, and let $\alpha(x)$ and $\beta(x)$ be any two endomorphisms of S which commute under multiplication, that is, such that $\alpha(\beta(x))=\beta(\alpha(x))$. Then it will be shown below that with respect to the operation $x \circ y$ defined by

$$x \circ y = \alpha(x) + \beta(y), \tag{8}$$

the elements of S form also a symmetric system. The operation $x \circ y$ defined by (8) is idempotent if $\alpha(x)+\beta(x)=x$, that is, if the sum of the endomorphisms α and β is the identity automorphism. In this case the operation is a kind of mean or average, and the resulting symmetric system is also self-distributive. This gives a method, starting with a symmetric system such as an abelian group, of forming other symmetric systems having the same elements. As a special case of (7), note that in an abelian group the elements form a symmetric system with respect to the difference operation $x-y$.

With respect to the operation $x \circ y=ax+by$ of (7), the elements of any field form a symmetric system which is also a quasi-group, provided a and b are

not 0. In this case unique inverses with respect to the operation exist. This quasi-group is usually not a loop, since in general there is no identity element.

Interesting examples of symmetric and self-distributive systems with a finite number of elements may be formed in this way from the elements of finite fields. Tables I and II below exhibit such systems formed from the fields of integers reduced modulo 3 and 5 respectively, the operations being $2x+2y$ and $2x+4y$ in these cases.

TABLE I

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

TABLE II

	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

The operation of Table I is commutative, idempotent, symmetric, and self-distributive, but not associative. That of Table II is idempotent, symmetric, and self-distributive, but neither commutative nor associative. Both systems are also quasi-groups, but neither is a loop, since there is no identity element.

As an example of an operation that has some of the properties of a mean or average, but is neither symmetric nor self-distributive, consider the operation $x \circ y$ defined for the non-negative real numbers by

$$x \circ y = (x^2 - xy + y^2)^{1/2}. \tag{9}$$

Geometrically, this represents the third side of a triangle with sides x and y and included angle 60°. This operation is idempotent. Unlike other means, it has an identity element, namely 0. With respect to this operation, inverse elements, when they exist, are usually not unique. The corresponding system is neither a loop nor a quasi-group. Besides being neither symmetric nor self-distributive, then, this operation has so many peculiar properties that it should probably not be considered as a mean.

4. Endomorphisms of symmetric systems. Although one can study the set of all endomorphisms of a general additive system, in particular in connection with the notions of direct and subdirect reducibility, it is only when the symmetric law is assumed that the set of all endomorphisms acquires a neat structure. It will be seen that the symmetric law is just what is required in order that the sum of two endomorphisms shall be an endomorphism. It is for this reason that the set of all endomorphisms of an abelian group is an important notion, since it is a ring, and in a sense the model for all rings, while the set of all endomorphisms of a non-abelian group has no such structure. This is because an abelian group is a symmetric system, while a non-abelian group is not.

By an *endomorphism* of an additive system we mean a mapping $\alpha(x)$ of the system into itself which preserves the addition operation, that is, identically in x and y,

$$\alpha(x + y) = \alpha(x) + \alpha(y). \tag{10}$$

By the product $\alpha\beta(x)$ of two mappings $\alpha(x)$ and $\beta(x)$, and in particular of two endomorphisms, we mean as usual the mapping $\gamma(x)$ such that

$$\gamma(x) = \alpha\beta(x) = \alpha(\beta(x)). \tag{11}$$

The following theorem is well known and is stated without proof for convenience of reference.

THEOREM 4. *The set of all endomorphisms of an additive system is closed and associative under multiplication, and forms with respect to multiplication a semi-group.*

By a semi-group is meant as usual an associative system. By the *sum* of two mappings $\alpha(x)$ and $\beta(x)$, and in particular of two endomorphisms of an additive system, is meant the mapping $\gamma(x)$ such that:

$$\gamma(x) = (\alpha + \beta)(x) = \alpha(x) + \beta(x). \tag{12}$$

THEOREM 5. *The sum of two endomorphisms of a symmetric additive system is an endomorphism.*

Proof. Let $\alpha(x)$ and $\beta(x)$ be two endomorphisms of a symmetric system. By the symmetric law (1) we have:

$$\begin{aligned}(\alpha + \beta)(x + y) &= \alpha(x + y) + \beta(x + y) = (\alpha(x) + \alpha(y)) + (\beta(x) + \beta(y)) \\ &= (\alpha(x) + \beta(x)) + (\alpha(y) + \beta(y)) = (\alpha + \beta)(x) + (\alpha + \beta)(y).\end{aligned} \tag{13}$$

As can be seen from the proof, the symmetric law is exactly what is needed to insure that the sum of endomorphisms is an endomorphism. It is to this fact that the symmetric law owes its chief importance. As will be shown, the set of all endomorphisms of a symmetric additive system has a neat structure, that of a pseudo-ring, in consequence of being closed under two operations.

THEOREM 6. *The set of all endomorphisms of a symmetric additive system is a symmetric additive system under addition.*

Proof. Theorem 5 shows that the set of endomorphisms of a symmetric system is closed under addition. It remains to verify that this addition operation of endomorphisms is itself symmetric. The proof is immediate.

5. Pseudo-rings. By a *pseudo-ring* we mean a system R of elements which is a symmetric system under an addition operation $a+b$, a semi-group under a multiplication operation ab, and obeys both distributive laws. The formal postulates for a pseudo-ring are:

$$\text{I.}\quad (a+b)+(c+d) = (a+c)+(b+d).$$
$$\text{II.}\quad a(bc) = (ab)c.$$
$$\text{III.}\quad a(b+c) = ab+ac.$$
$$\text{IV.}\quad (a+b)c = ac+bc.$$

This is clearly a generalization of the notion of a ring.

THEOREM 7. *The set of all endomorphisms of a symmetric additive system is a pseudo-ring with identity of multiplication.*

Proof. Theorems 4 and 6 show that the set of endomorphisms is a semi-group under multiplication, and a symmetric system under addition. It remains to verify the distributive laws. Let α, β, and γ be three endomorphisms of a symmetric system. By definition of sum and product of endomorphisms we have:

$$(14)\quad \alpha(\beta+\gamma)(x) = \alpha(\beta(x)+\gamma(x)) = \alpha(\beta(x))+\alpha(\gamma(x)) = \alpha\beta(x)+\alpha\gamma(x).$$

The identity of multiplication is the identity automorphism $\alpha(x)=x$. The other distributive law follows similarly. Theorem 7 is a generalization of the result that the set of all endomorphisms of an abelian group is a ring with unity element of multiplication. Since semi-lattices and commutative semi-groups are symmetric systems, we have the following corollary:

COROLLARY. *The set of all endomorphisms of a semi-lattice or of a commutative semi-group is a pseudo-ring with identity of multiplication.*

THEOREM 8. *Every pseudo-ring with a right identity of multiplication is isomorphic to a sub-pseudo-ring of the pseudo-ring of all endomorphisms of its additive system.*

Proof. The proof is like that for the corresponding theorem for rings. To the elements a, b, and c of the pseudo-ring P we assign the mappings $\alpha(x)=ax$, $\beta(x)=bx$, and $\gamma(x)=cx$. These mappings are endomorphisms of the additive system of P, since

$$(15)\quad \alpha(x+y) = a(x+y) = ax+ay = \alpha(x)+\alpha(y).$$

The correspondence between elements and mappings also preserves addition and multiplication, since if $a+b=c$,

$$(16)\quad (\alpha+\beta)(x) = \alpha(x)+\beta(x) = ax+bx = (a+b)x = cx = \gamma(x),$$

and if $ab=c$, then

$$(17)\quad \alpha\beta(x) = \alpha(\beta(x)) = a(bx) = (ab)x = cx = \gamma(x).$$

The correspondence is also one-to-one, since if a and b are distinct elements of P, and I is a right identity of multiplication, then

$$(18)\quad \alpha(I) = aI = a, \quad\text{and}\quad \beta(I) = bI = b.$$

Hence $\alpha(x)$ and $\beta(x)$ are distinct endomorphisms. This completes the proof, since these endomorphisms, which are left multiplications, form a sub-pseudo-ring of the pseudo-ring of all endomorphisms.

Theorems 7 and 8 show the significance of the distributive laws in algebra; multiplication of endomorphisms of a system with one binary operation is always both left and right distributive over addition of endomorphisms when the latter is defined, that is, for symmetric systems. Conversely, when both distributive laws hold, we have

$$(ac + ad) + (bc + bd) = (ac + bc) + (ad + bd), \tag{19}$$

which is a particular case of the symmetric law. The close relation between the rather weak symmetric law and the rather strong distributive laws is thus seen.

6. Construction of symmetric systems. Given a symmetric system S (such as an abelian group), one may define in terms of it other symmetric systems having the same elements as S, but a different binary operation.

THEOREM 9. *If $\alpha(x)$ and $\beta(x)$ are commuting endomorphisms of a symmetric additive system S (that is, if $\alpha\beta=\beta\alpha$), then the elements of S form also a symmetric system with respect to the operation $x \circ y=\alpha(x)+\beta(y)$. If in addition $\alpha+\beta$ is the identity automorphism, then the resulting symmetric system is also idempotent and self-distributive.*

Proof. To verify that the operation $x \circ y$ obeys the symmetric law, we note that

$$\begin{aligned}(x \circ y) \circ (z \circ w) &= \alpha(\alpha(x) + \beta(y)) + \beta(\alpha(z) + \beta(w)) \\ &= (\alpha\alpha(x) + \alpha\beta(y)) + (\beta\alpha(z) + \beta\beta(w)) \\ &= (\alpha\alpha(x) + \alpha\beta(z)) + (\beta\alpha(y) + \beta\beta(w)) = (x \circ z) \circ (y \circ w).\end{aligned} \tag{20}$$

Since left multiplication in a ring is an endomorphism of the additive group, it follows that the operation $x \circ y$ given by

$$x \circ y = ax + by, \tag{7}$$

where a and b are elements of the ring, is a symmetric operation, which may also be verified directly. If the ring elements a and b have multiplicative inverses, then the elements of the ring form also a quasi-group with respect to the operation $x \circ y$. If in addition $a+b=1$, where the element 1 is the identity of multiplication of the ring, then with respect to the operation $x \circ y$, the elements of the ring form a symmetric quasi-group which is also idempotent and self-distributive. The operation $x \circ y$ in this case has most of the properties one would expect of a mean or average; nevertheless in general it satisfies neither the associative nor commutative laws.

7. Abstract means and mid-points. We have seen that in rings and fields, operations analogous to familiar means or averages may be defined which obey

the symmetric, idempotent, and self-distributive laws. This suggests the question: what are the weakest and strongest formal laws that a binary operation $x \circ y$ should obey in order that it should be considered an abstract mean? If an order relation is present, it might be reasonable to require that if $x \leqq y$, then $x \leqq x \circ y \leqq y$. Likewise in a topological space, it would seem reasonable to require that the operation $x \circ y$ be continuous in the topology.

Aside from this, the weakest natural requirement for a mean is that it be idempotent, that is, that $x \circ x = x$. In addition, it would be reasonable to require that $x \circ y = x$ only if $y = x$, and $x \circ y = y$ only if $x = y$. This would follow automatically if the system of elements forms a quasi-group with respect to the operation. The strongest set of formal laws would seem to be the following:

DEFINITION. A system of elements closed under a binary operation $x \circ y$ is called a *mean system* if it is a quasi-group relative to the operation, and if the operation obeys the symmetric and idempotent laws, and hence also the self-distributive law. The operation is then called a *mean*. If in addition the commutative law holds, the operation $x \circ y$ may be called a *mid-point*.

The term *hoop* has also been suggested for what is here called a mean system. Interesting examples of commutative hoops or mid-point systems with a finite number of elements may be constructed as follows:

(a) Let the elements of M be the vertices of a regular polygon with an odd number of sides. The vertex $x \circ y$ which is the "mid-point" of the vertices x and y is then defined in an obvious way. It may be verified that the operation $x \circ y$ is commutative, symmetric, idempotent, and self-distributive, but not associative, and that the system is a quasi-group.

(b) Let M be a finite geometry, with the same odd number of points on each line. If x and y are distinct points, let $x \circ y$ be a point on the line xy determined in a manner somewhat like that in (a).

8. Subsystems of symmetric systems. By a subsystem A of an additive system S is meant a subset of S which is closed under the defining binary operation. The set of all subsystems of an additive system always forms a complete lattice relative to the subset relation as order relation. The lattice *meet* AB of two subsystems is their intersection. The lattice *join* $A \cup B$ in general has a more complicated determination. If the system S is symmetric, a third operation, the *sum* $A+B$ of two subsystems may be defined. This consists of all elements of the form $a+b$, where a is in A, and b is in B. This is an additional reason for the importance of the symmetric law.

DEFINITION. By the *sum* $A+B$ of two subsystems A and B of a symmetric additive system S is meant the set of all elements of S of the form $a+b$, where a is in A, and b is in B.

THEOREM 10. *The sum $A+B$ of two subsystems A and B of a symmetric additive system S is a subsystem of S.*

Proof. If c and c' are both in $A+B$, it must be verified that $c+c'$ is also in

$A+B$. But by definition of sum and by the symmetric law, we have

$$c + c' = (a + b) + (a' + b') = (a + a') + (b + b'), \tag{21}$$

where a, a', and hence $a+a'$ are in A, and b, b', and hence $b+b'$ are in B, since A and B are subsystems. It follows that $c+c'$ is in $A+B$.

THEOREM 11. *The set Σ of all subsystems of a symmetric additive system S is a symmetric additive system relative to addition. If S obeys the idempotent law, so does Σ, and in this case Σ contains a subsystem isomorphic to S.*

Proof. The symmetric law

$$(A + B) + (C + D) = (A + C) + (B + D) \tag{22}$$

for Σ follows from the definition of addition and the symmetric law for S. Note that $A+A$ is always a subset of A, since A is a subsystem. If S is idempotent, and a is in A, then $a=a+a$ is also in $A+A$. Hence A and $A+A$ are identical, and Σ is idempotent. In this case, the set of one-element sets of S is a set of subsystems of S, and is a subsystem of Σ isomorphic to S. This completes the proof.

Note that the sum $A+B$ of two subsystems of a symmetric additive system S is always a subset of the lattice join $A\cup B$. In certain cases, for example whenever S is an abelian group or a semi-lattice, the operations $A+B$ and $A\cup B$ are identical. In this case the symmetric system Σ is associative, commutative, and idempotent, since the lattice join operation $A\cup B$ has these properties. In general, however, the addition operation $A+B$ is neither associative, commutative, nor idempotent.

9. The modular law. In modular lattices, the two lattice operations obey the modular law. This law may be written as a weakened distributive law, namely

$$AB + AC = A(B + AC). \tag{23}$$

In a lattice, the operations AB and $A+B$ in (23) would be the lattice operations meet and join. The subsystems of a symmetric system sometimes obey the modular law (23) when $A+B$ means the sum of subsystems, rather than lattice join.

THEOREM 12. *The system of all sub-quasi-groups of a symmetric quasi-group obeys the modular law* (23).

Proof. It may be verified immediately that the sum and intersection of sub-quasi-groups are quasi-groups. It must be shown that $AB+AC=A(B+AC)$, where A, B, and C are sub-quasi-groups of a symmetric quasi-group S. The element x of S is in $AB+AC$ if and only if $x=y+z$, where y is in A and B, and z is in A and C. Likewise x is in $A(B+AC)$ if and only if x is in A, and $x=y+z$, where y is in B, and z is in A and C. Under the given hypotheses, these conditions are equivalent. For if y and z are both in A, so is $x=y+z$, since A is a subsystem. Likewise, if x and z are both in A and $x=y+z$, then y is in A also since A is a

quasi-group. This shows that the two conditions are equivalent, and completes the proof.

A corollary of Theorem 12 is the well known result that the lattice of all subgroups of an abelian group is modular, since in this case the operations sum $A+B$ and join $A\cup B$ are identical.

10. Identity elements. The examples of new types of symmetric and self-distributive systems we have given, such as those where the operation is a mean or average, had no identity element. This is related to the fact that the existence of an identity element in a symmetric or self-distributive system implies other formal laws.

THEOREM 13. *In the presence of an identity element, the symmetric law implies both the commutative and associative laws.*

Proof. Let 0 be an identity element for the symmetric operation $a+b$, that is, assume that $a+0=a=0+a$ for all a. Then by the symmetric law we have:

$$(24) \qquad a + b = (0 + a) + (b + 0) = (0 + b) + (a + 0) = b + a.$$

Hence the commutative law holds. Likewise by the symmetric law we have:

$$(25) \quad a + (b + c) = (a + 0) + (b + c) = (a + b) + (0 + c) = (a + b) + c.$$

Therefore the system is also associative.

One can take advantage of Theorem 13 in formulating postulate systems for abelian groups, fields, and similar systems. If the existence of identity elements has been postulated, then the associative and commutative laws may be replaced by the single symmetric law.

COROLLARY. *A loop is a symmetric system if and only if it is an abelian group.*

Recall that a loop is a quasi-group with identity element. It should be noted that a quasi-group can be a symmetric system without being either associative or commutative. This is in fact the case with the mean systems discussed in section 7.

THEOREM 14. *Every self-distributive system with identity element is idempotent. More generally, a left self-distributive system with a right identity element is idempotent.*

Proof. Assume that $a+0=a$ for all a. Then by the left self-distributive law (3)we have:

$$(27) \qquad a = a + 0 = a + (0 + 0) = (a + 0) + (a + 0) = a + a.$$

11. Conclusion. In this paper the more immediate and obvious consequences of the symmetric and self-distributive laws have been investigated. Such results form the necessary groundwork for further studies. It has been shown that formal laws differing slightly from the usual ones have interesting properties and

interpretations, and that the study of simple systems of abstract algebra has not yet been exhaustive.

Certain topics suggest themselves for further study. One of these is the application of the pseudo-ring of endomorphisms of a symmetric system to the question of direct and subdirect reducibility of systems. Another is a further investigation of abstract means and averages. Since the set of all subsystems of a symmetric and idempotent, and therefore self-distributive system is a complete atomic lattice, there are possible geometric applications and interpretations of this notion.

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CHARACTERIZATION OF THE SINE AND COSINE

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1. In studying trigonometry one is likely to be impressed by the fundamental character of the formula $\cos(x-y)=\cos x \cos y+\sin x \sin y$. A little experimentation may suggest that this in itself must very nearly characterize the sine and cosine functions. It is the purpose of the present note to investigate the extent to which this suggestion is justified by examining in some detail the solutions of the functional equation

$$g(x-y) = g(x)g(y) + f(x)f(y) \tag{A}$$

and of the more special one

$$g(x - y) = g(x)g(y) + g(x - s)g(y - s), \tag{A'}$$

in which s is a constant. My interest in this subject was reawakened by the consideration of some solutions offered to a problem suggested by Professor V. L. Klee. These appeared as E1079, on page 197 in volume 61 of this MONTHLY.

The main results are summarized in the next three paragraphs:

If g, f is a solution of (A) then $0 \leqq g(0) \leqq 1$. If $g(0)=0$ there is just one solution. For each c such that $0<c<1$ there are precisely two solutions with $g(0)=c$. If $g(0)=1$ then $g(x+y)=g(x)g(y)-f(x)f(y)$ and $f(x \pm y)=f(x)g(y) \pm g(x)f(y)$. If, in this case, either g or f is continuous at any point then there is a constant a such that, for all x, $g(x)=\cos ax$ and $f(x)=\sin ax$. However, totally discontinuous solutions do exist (and a method will be indicated for constructing some).

The hypothesis $g(0)=1$ can be deduced from: for some s, $g(s)=0$ and $f(s) \neq 0$. This hypothesis is stronger than the former but is equivalent to: g has a zero but is not identically zero. From this it follows that the constant a referred to above is not zero, and that there exists a constant t such that, for all x, $g(x-t) =f(x)$.

If g is a solution of (A′) then $g(0)=0$, $\frac{1}{2}$ or 1. In each of the first two cases there is a single solution, while in the third $g(s)=0$. In this last case, g is a solution of (A′) if and only if $g(s)=0$ and there is an f such that g, f is a solution of (A).

The methods used are, with two possible exceptions, quite elementary. Both these have to do with properties of the additive group R of the real numbers. In establishing that for $0<c<1$ there are just two solutions of (A) with $g(0)=c$ it seems necessary to refer to the fact that this group has no subgroup of index 2; and the method used to show the existence of discontinuous solutions depends on the fact that R can be represented as the sum of two of its non-trivial subgroups. These two results are perhaps of subordinate interest and none of the others depends on either of them. Both properties of R follow from the fact that it is the direct sum of subgroups each isomorphic to the additive group of rational numbers, *i.e.*, from the existence of a Hamel basis for R.

2. Setting $y=x$ in (A) we get

$$g(0) = g(x)^2 + f(x)^2, \tag{1}$$

and from this

$$|g(x)| \leqq g(0), \qquad |f(x)| \leqq g(0). \tag{2}$$

Setting $x=0$ in (1),

$$f(0)^2 = g(0) - g(0)^2, \tag{3}$$

whence

$$0 \leqq g(0) \leqq 1. \tag{4}$$

Setting $y=0$ in (A) we find

$$[1 - g(0)]g(x) = f(0)f(x), \tag{5}$$

and using the results so far obtained we can prove that, *for every* x,

$$\text{if } g(x) = g(0) \quad \text{then } f(x) = f(0), \quad \text{and} \tag{6a}$$

$$\text{if } g(x) = -g(0) \quad \text{then } f(x) = -f(0). \tag{6b}$$

For from (2), if $g(0)=0$ then $f=0$ (*i.e.*, $f(x)=0$ for all x). From (3), if $0<g(0)<1$ then $f(0)\neq 0$ while from (3) and (5) if $g(x)=\pm g(0)$ then $\pm f(0)^2=f(0)f(x)$. From (1), if $g(0)=1$ and $g(x)=\pm g(0)$ then $f(x)=0$ while, from (3), if $g(0)=1$ then $f(0)=0$. According to (4) this exhausts all cases.

Using (A), (6) and (3) it is now clear that, *for every* x *and* y,

$$\text{if } g(x) = g(0) = g(y) \quad \text{then } g(x-y) = g(0), \quad \text{and} \tag{7a}$$

$$\text{if } g(x) = -g(0) = g(y) \quad \text{then } g(x-y) = g(0). \tag{7b}$$

We next note that, for every x,

$$[1 - g(0)][g(x)^2 - g(0)^2] = 0. \tag{8}$$

For this results from squaring both sides of (5), using (3), transposing and factoring out $1-g(0)$, and using (1). From this we learn that

$$\text{if } g(0) \neq 1 \quad \text{then, for every } x, \quad g(x) = g(0) \quad \text{and} \quad f(x) = f(0). \tag{9}$$

For, by (8), if $g(0)\neq 1$ then $\{x \mid g(x)=\pm g(0)\}$ is the additive group R of all real numbers. By (7a), $\{x \mid g(x)=g(0)\}$ is a subgroup which, by (7b), is either all of R or is of index 2 in R. Since R has no subgroup of index 2, $g(x)=g(0)$ for all x. Then, by (6a), $f(x)=f(0)$ for all x. Hence,

(10) if $g(0)=0$ then $g=0$ and $f=0$; if $0<g(0)=c<1$ and $b=\sqrt{c-c^2}$ then, for all x, $g(x)=c$ and either, for all x, $f(x)=b$ or, for all x, $f(x)=-b$.

Since the functions referred to are obviously solutions of (A), this completes the discussion of the cases for which $g(0)\neq 1$.

3. We now assume

$$g(0) = 1. \tag{B}$$

Setting $x=0$ in (A) and using (B) (or (5)) we find that, for all y,

$$g(-y) = g(y), \tag{11}$$

and from this, together with (1), that, for all x,

$$f(x)^2 = f(-x)^2. \tag{12}$$

An immediate consequence of (12) is that, *for all* x,

$$\textit{if } f(x) = 0 \quad \textit{then} \quad f(-x) = 0. \tag{13}$$

Setting $y=-x$ in (A) and using (11) yields, *for all* x,

$$g(2x) = g(x)^2 + f(x)f(-x). \tag{14}$$

For all x *and* y,

$$\text{if } f(x) = 0 = f(y) \quad \text{then} \quad f(x-y) = 0. \tag{15}$$

For, by (1) and (B), if $f(x)=0=f(y)$ then $g(x)g(y)=\pm 1$. Using this and (A), if $f(x)=0=f(y)$ then $g(x-y)=\pm 1$. Now (15) is a consequence of this, (1) and (B).

It can now be shown that, *for all* x,

$$f(2x) = g(x)[f(x) - f(-x)]. \tag{16}$$

To do so, set $y=2x$ in (A) and use (11) to obtain $g(x)=g(x)g(2x)+f(x)f(2x)$. Use (14) to eliminate $g(2x)$. By transposing, factoring out $g(x)$, and applying (1) and (B) one obtains $f(x)g(x)[f(x)-f(-x)]=f(x)f(2x)$ which yields (16) under the assumption that $f(x)\neq 0$. But if $f(x)=0$ then (16) is a consequence of (13) and (15).

Now it is possible to sharpen (12) by showing that, *for all* y,

$$f(-y) = -f(y). \tag{17}$$

For, by (12), for each y, either $f(-y/2)=f(y/2)$ or $f(-y/2)=-f(y/2)$. In the first case, by (16), $f(y)=0$ and, by (13), $f(-y)=-f(y)$. In the second case, by (16), $f(y)=2g(y/2)f(y/2)$ and $f(-y)=-2g(-y/2)f(y/2)$. By (11), again $f(-y) = -f(y)$.

That, *for all* x *and* y,

$$g(x+y) = g(x)g(y) - f(x)f(y) \tag{18}$$

is an immediate consequence of (A), (11) and (17). To establish

$$f(x+y) = f(x)g(y) + g(x)f(y) \tag{19}$$

replace y in (A) by $x+y$ and use (11) to obtain $g(y)=g(x)g(x+y)+f(x)f(x+y)$. Use (18) to eliminate $g(x+y)$, transpose and use (1) and (B) to obtain $f(x)f(x+y) =f(x)[f(x)g(y)+g(x)f(y)]$. This establishes (19) if $f(x)\neq 0$ and, by symmetry, if $f(y)\neq 0$. If $f(x)=0=f(y)$ then (19) follows from (13) and (15). The companion formula

$$f(x-y) = f(x)g(y) - g(x)f(y) \tag{20}$$

is now a consequence of (19), (11) and (17).

For future use it is desirable to note that

$$g(2x) = g(x)^2 - f(x)^2 = 2g(x)^2 - 1 \tag{21a}$$

$$f(2x) = 2f(x)g(x) \tag{21b}$$

$$(3x) = f(x)[4g(x)^2 - 1] \tag{21c}$$

(22a) $$g(x) - g(y) = -2f((x+y)/2)f((x-y)/2)$$

(22b) $$g(x) + g(y) = 2g((x+y)/2)g((x-y)/2)$$

(23a) $$f(x) - f(y) = 2g((x+y)/2)f((x-y)/2)$$

(23b) $$f(x) + f(y) = 2f((x+y)/2)g((x-y)/2)$$

can now be established in the usual manner.

4. The methods and results up to this point have been purely algebraic, but in this section continuity considerations will be paramount. First,

(24a) *if* $\lim_{x\to 0^+} g(x)$ *exists then* $\lim_{x\to 0} f(x)=0$, *and*

(24b) *if* $\lim_{x\to 0^+} f(x)^2$ *exists then* $\lim_{x\to 0} g(x)=1$.

(By (11) and (12), 0^+ could be replaced by 0^-, and, of course, by 0 and, in (24b), $f(x)^2$ could be replaced by $f(x)$.) For, from (21a), if $\lim_{x\to 0^+} g(x)=L$, then $L = 2L^2-1$ and $L=1$ or $-\frac{1}{2}$. If $L=-\frac{1}{2}$ then, (21c) and (2), $\lim_{x\to 0^+} f(3x)=0$. But from (1) and (B), if $L=-\frac{1}{2}$, $\lim_{x\to 0^+} f(x)^2=\frac{3}{4}$. Hence $L=1$, and from (1) and (B) $\lim_{x\to 0^+} f(x)=0$. By (17), $\lim_{x\to 0} f(x)=0$. This establishes (24a). By (1), if $\lim_{x\to 0^+} f(x)^2$ exists then so does $\lim_{x\to 0^+} g(x)^2$ and, by (21a), so does $\lim_{x\to 0^+} g(2x)$. Hence, as shown above, $\lim_{x\to 0^+} g(x)=1$. By (11) $\lim_{x\to 0} g(x)=1$.

Next,

(25) *if f or g is continuous from the right at some point then both are continuous at* 0.

For note that, by (24), if either f or g is continuous at 0 then both are, and that, by (1) and (B), f and g have no common zero. Suppose now that f is continuous from the right at x. If $f(x)\neq 0$ then, from (23b), $\lim_{y\to x^+} g((x-y)/2)=1$ so $\lim_{x\to 0^-} g(x)=1$ and, by (11) and (B), $\lim_{x\to 0} g(x)=g(0)$. If $f(x)=0$ (so $g(x)\neq 0$) then, from (19), $\lim_{y\to 0^+} f(y)=0$. Hence, by (24b) and (B), $\lim_{x\to 0} g(x)=g(0)$. On the other hand, if g is continuous from the right at x it follows in a similar manner, using (22b), if $g(x)\neq 0$ and (18) if $f(x)\neq 0$, that f is continuous at 0.

Now,

(26) *if f is continuous at* 0 *then both g and f are continuous everywhere,*

as is easily seen from (22a), (23a) and (1). Hence the members g, f of a solution of (A) and (B) are either continuous everywhere or are not continuous (from either side) anywhere. An interesting side result, not needed in establishing the results described in §1, is that

(27) *f is continuous if and only if there is an interval* $(0, b)$ *in which f does not change sign.*

For, on the one hand, if $0<y<x<b$ then $0<(x-y)/2<(x+y)/2<b$ so, by (22a), if f does not change sign in $(0, b)$ then g is monotone non-increasing in this interval. By (2) it follows that $\lim_{x\to 0^+} g(x)$ exists so, by (24a), f is continuous at zero and, by (26), is continuous everywhere. On the other hand, if f were

continuous but were not of constant sign in any interval $(0, b)$ then each such interval would contain a zero of f. By (15), the zeros of f form a group which would, then, be dense in R. Since f is assumed to be continuous it would have to be identically zero, contradicting the assumption that it changes sign in each interval $(0, b)$.

To wind up the continuous case we prove that

(28a) *if g is continuous everywhere then f is differentiable everywhere and* $f'(x) = f'(0)g(x)$, *and*

(28b) *if f is continuous everywhere then g is differentiable everywhere and* $g'(x) = -f'(0)f(x)$.

The proofs of these depend on the fact that each continuous function has an indefinite integral of which it is the derivative. Suppose then that g is continuous everywhere. Then f is continuous and, by the argument at the end of the preceding paragraph, either $f=0$ (and (28a) holds) or there is an interval $(0, b)$ which contains no zeros of f. In the latter case, if $0<x=nh$, where $0<h<b$,

$$\begin{aligned} f(h/2)\sum_{k=1}^{n} g(kh) &= \sum_{k=1}^{n} g((2k-1)h/2 + h/2)\, f(h/2) \\ &= \frac{1}{2}\sum_{k=1}^{n} [f((2k+1)h/2) - f((2k-1)h/2)] \\ &= \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) - f\left(\frac{h}{2}\right)\right], \quad \text{and} \end{aligned}$$

$$\sum_{k=1}^{n} g(kh)h = \left[f\left(x+\frac{h}{2}\right) - f\left(\frac{h}{2}\right)\right]\frac{h/2}{f(h/2)}\cdot$$

Since g and f are continuous and $f(0)=0$ but $f\neq 0$,

$$\int_0^x g(t)dt = f(x)\lim_{h\to 0}\frac{h/2}{f(h/2)}\cdot$$

Since $g(x)>0$ for sufficiently small x, the limit on the right is not zero, so $f'(0)$ exists and $f'(0)\int_0^x g(x)dx = f(x)$. On differentiating we obtain (28a) for $x>0$. By (17) and (11) this extends to $x\neq 0$. That $f'(0)=f'(0)g(0)$ follows from (B). The result (28b) can be established in a similar manner.

It follows that g and f are solutions of $y''+a^2y=0$, with the initial conditions $g(0)=1$, $g'(0)=0$, $f(0)=0$ and $f'(0)=a$. Hence, by (2), they are represented by the usual power series for $\cos ax$ and $\sin ax$.

5. So far we have found only differentiable solutions of (A). Unless (B) holds these are the only solutions. We can however obtain other, necessarily everywhere discontinuous solutions, of (A) and (B) as follows. First express R as the

direct sum of two of its proper subgroups, say R_1 and R_2, and choose two solutions, say g_1, f_1 and g_2, f_2. Each x in R is uniquely representable as $x = x_1 + x_2$ where x_i is in R_i and each solution satisfies (20) as well as (A). Define g and f by

$$g(x) = g_1(x_1)g_2(x_2) - f_1(x_1)f_2(x_2) \quad \text{and}$$
$$f(x) = f_1(x_1)g_2(x_2) + g_1(x_1)f_2(x_2).$$

Straightforward computations show that g, f also satisfies (A) and (20) whence, unless $g = 0 = f$, it also satisfies (B).

This procedure can be generalized by replacing g_1, f_1 and g_2, f_2 by the function pairs which they induce in R_1 and R_2 through arbitrary isomorphisms of the latter in R.

6. If $g(s) = 0$ and $f(s) \neq 0$ then, by (1), $g(0) \neq 0$, so $g \neq 0$. On the other hand, if $g \neq 0$ but $g(s) = 0$ then $f(s) \neq 0$. For if $g(s) = 0 = f(s)$ then, by (1), $g(0) = 0$ and $g = 0$. Hence the hypotheses: $g(s) = 0 \neq f(s)$, for some s, and: g has a zero but is not identically zero, are equivalent. Taken together with (A) they imply (B). For setting $x = s$, $y = 0$, in (A) the first form of the hypothesis gives at once $f(0) = 0$ whence, by (3), $g(0) = 0$ or 1. But, since $g \neq 0$, $g(0) \neq 0$ (by (1)).

That this hypothesis is stronger than (B) now follows from the fact that $g = 1, f = 0$ satisfies the latter but not the former. Furthermore, the former implies

$$\text{(29)} \qquad \textit{for some } t,\ g(x - t) = f(x).$$

To prove this we note that since $g(s) = 0$ and $g(0) = 1$, $f(s)^2 = 1$. Hence, by (17), if we choose t properly as s or $-s$ then $f(t) = 1$, $f(-t) = -1$, and, by (11), $g(t) = 0$. Property (29) follows by setting $y = t$ in (A).

As to (A′), it is clear that this is a special case of (A) and that we need merely sort out from the solutions g, f of (A) those g's which satisfy (A′). If $g = c$ then evidently $c = 0$ or $\frac{1}{2}$. In particular, $g = 1$ is obviously not a solution. If $g(0) = 1$ then, setting $x = s = y$, $g(s) = 0$ and, from the preceding discussion, there is a t (either s or $-s$) for which $g(x - t) = f(x)$. The only continuous solutions of (A′) are then $g = 0$, $g = \frac{1}{2}$, and $g(x) = \cos ax$, $a > 0$. There are discontinuous solutions some of which can be obtained as in the preceding section by choosing, if s is in R_1, $g_1(s) = 0$. Automatically then $g(s) = 0$ and (possibly after a permissible change in the sign of f) $f(x) = g(x - s)$.

7. From the material developed above (including (27)) it is easy to deduce:

THEOREM. *If g, f satisfy* (A) *and $\lim_{h \to 0^+} f(h)/h = 1$ then $g(x) = \cos x$ and $f(x) = \sin x$.*

SOME BALANCED HOWELL ROTATIONS FOR DUPLICATE BRIDGE SESSIONS

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1. Introduction. Duplicate bridge rotation systems are tactical configurations in the sense of Carmichael [2] and they are also experimental designs [3] of a specialized but particularly exacting kind in that there are several factors involved and time for only one or two replications. The special results presented in this paper will probably find their widest application in bridge circles and we use that language.

For two to seven or eight tables the Howell rotations are the most satisfactory for an evening of duplicate bridge with fixed partnerships. For larger numbers of players the evening becomes too long and other less satisfactory rotations must be used. The Howell rotations have the following properties:

(a) Every partnership plays once at a table with every other partnership.

(b) Every partnership plays every board once.

A *balanced* rotation has the additional property:

(c) Every partnership competes equally often with every other partnership.

The rotations commonly used are presented by Beynon [1]; those for two and four tables are balanced while those for other numbers of tables are not balanced. Much effort has gone into searching for balanced rotations with no further success. However, by doubling the minimum number of rounds (which is $2T-1$ for T tables) Gold [1, p. 17] constructed a balanced rotation for the case of three tables.

We shall show here that a balanced rotation is impossible in $2T-1$ rounds when T is odd and we shall exhibit balanced rotations for 3, 5, and 7 tables in double the minimum number of rounds. Also we present balanced rotations for 6 and 8 tables in the minimum number of rounds.

So that this paper will contain a complete set of balanced rotations for 2 to 8 tables, we reproduce here a four table rotation by General Gruenther which will aid in introducing notation and terms. The partnerships (here called teams for

FOUR TABLES

Round	Table 1		Table 2		Table 3		Table 4	
	Teams	Board	Teams	Board	Teams	Board	Teams	Board
1	8, 1	1	6, 3	4	2, 7	6	4, 5	7
2	8, 2	2	7, 4	5	3, 1	7	5, 6	1
3	8, 3	3	1, 5	6	4, 2	1	6, 7	2
4	8, 4	4	2, 6	7	5, 3	2	7, 1	3
5	8, 5	5	3, 7	1	6, 4	3	1, 2	4
6	8, 6	6	4, 1	2	7, 5	4	2, 3	5
7	8, 7	7	5, 2	3	1, 6	5	3, 4	6

short) are labeled 1 to 8 and since every one is to play every other there must be seven rounds. The boards are numbered from 1 to 7 and, of course, each board may represent a stack of several boards. The team to the left of the comma plays the north-south (NS) hand. (Incidentally, all Howell rotations may be used when there is one missing team by omitting the first table and the highest numbered team; each team then sits out one round).

This four table rotation is balanced. Thus on board 1 teams 3, 4, 5, 8 play NS and others EW. The score of team 8 on this board is determined by the scores of 3, 4, and 5; 8 competes directly with 3, 4, 5 on it. On the other boards 8 competes with other triples. The balanced rotation insures that team 8 competes in this manner equally often with every other, and the same is true for all teams.

2. Requirement of even T. The following argument shows that a balanced Howell rotation for T tables in $2T-1$ rounds with $2T-1$ boards is impossible when T is odd. Each board partitions the $2T$ teams into two equal groups—those that play the NS hand in one and those that play the EW hand in the other. Team p, for example, competes with $T-1$ others on a given board and, there being $2T-1$ boards, p competes $(T-1)(2T-1)$ times in all. In a balanced rotation p competes equally with all its $(2T-1)$ opponents; hence it competes $T-1$ times with each opponent.

The $4T-2$ parts of the partitions mentioned above are described as P parts or $\overline{P}$ parts according as they do or do not contain p. Another team q appears in $T-1$ of the P parts and T of the $\overline{P}$ parts. Similarly a third team r appears in $T-1$ of the P parts, T of the $\overline{P}$ parts, $T-1$ of the Q parts, and T of the $\overline{Q}$ parts. Let x be the number of parts that contain all three of p, q, and r. Then there must be

$$\begin{array}{rl} x & PQ \text{ parts containing } r \\ T-1-x & P\overline{Q} \text{ parts containing } r \\ T-1-x & \overline{P}Q \text{ parts containing } r \\ x+1 & \overline{PQ} \text{ parts containing } r, \end{array}$$

the last because there are $2T-1$ R parts in all. Now the $T-1$ PQ parts have $T-1$ corresponding $\overline{PQ}$ parts and there must be an r in each partition; hence $x+x+1=T-1$ and T must be even if x is to be integral.

3. Balanced rotations. Unfortunately we have no general method for finding these rotations. All were obtained by trial and error and it was just luck or perseverance that the natural first choice of cyclic permutations worked out before we gave up. This three table rotation appears to be a five round solution but is not because 10 boards (or stacks of boards) are necessary. At table 3 the boards are turned 90° halfway through the round and we indicate this by using a dash instead of a comma between teams. Thus in round one team 3 plays the NS hand of board 2 and the EW hand of board 5.

THREE TABLES

Round	Table 1		Table 2		Table 3	
	Teams	Boards	Teams	Boards	Teams	Boards
1	6, 1	1, 6	5, 2	8, 9	3−4	2, 5
2	6, 2	2, 7	1, 3	9, 10	4−5	3, 1
3	6, 3	3, 8	2, 4	10, 6	5−1	4, 2
4	6, 4	4, 9	3, 5	6, 7	1−2	5, 3
5	6, 5	5, 10	4, 1	7, 8	2−3	1, 4

Since we have nothing but cyclic permutations in the columns (excepting the fixed highest numbered team) it is necessary to write down only the first round. The five tables use 18 boards; 1 follows 9, and 10 follows 18 in writing down the permutations; similarly 1 follows 13, and 14 follows 26 in the seven table case. It is impossible to do two tables in three rounds; the rotation given here for two tables uses six boards and is balanced; actually it requires only two stacks of boards as 1 and 4 are completed after the first round and may be reshuffled for the second.

FIRST ROUNDS

No. of Tables	Table 1		Table 2		Table 3		Table 4		Table 5		Table 6		Table 7		Table 8	
	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds	Tms	Bds
5	10, 1	1, 10	9, 2	12, 17	3−8	4, 7	4, 7	18, 11	6−5	8, 3						
6	12, 1	1	2, 11	6	10, 3	2	4, 9	5	5, 8	10	6, 7	4				
7	14, 1	1, 14	2−13	11, 24	12, 3	9, 22	4−11	5, 18	10−5	13, 26	9, 6	8, 21	8, 7	12, 25		
8	16, 1	1	15, 2	9	14, 3	5	4, 13	14	5, 12	2	6, 11	4	7, 10	6	8, 9	12
2	4, 1	1, 4	2−3	4, 1												

From the point of view of experimental design these rotations are similar to graeco-latin squares in the sense that main effects of several factors may be estimated with relatively few observations. Thus, regarding NS teams as one factor, EW teams as a second, boards as a third, rounds as a fourth, and tables as a fifth, one could estimate 44 main effects from 55 observations using the six table rotation. The highest numbered team and first table would ordinarily be omitted in using the rotation as a design; tables would then be at five levels and the other four factors at eleven levels each. (A factor at n levels has $n-1$ main effects or independent differences in statistical terminology.)

References

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2. Carmichael, R. D., Groups of Finite Order, Ginn and Co., Boston, 1937.
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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

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A GENERALIZED THEOREM OF CENTER OF GEODESIC CURVATURE

T. K. PAN, University of Oklahoma

Let S: $x^i = x^i(u^1, u^2)$ $(i=1, 2, 3)$ denote a surface of class C^2 in three dimensional euclidean space referred to a rectangular cartesian coordinate system. Let C: $u^\alpha = u^\alpha(s)$ $(\alpha=1, 2)$ denote a curve of class C^2 on S, s being its arc length. Let R denote the ruled surface formed by the tangents to S which are perpendicular to C at its points $P(x^i)$. Let A denote the point of the generator PG of R at which the plane tangent to R is perpendicular to the plane tangent to S at P. It is shown in classical differential geometry that A is the center of geodesic curvature of C at P, and that A is the point of contact of PG with the edge of regression when R is developable. We attempt to give a generalization of this theorem.

Replace the unit tangent vector field t along C by an arbitrary but fixed unit vector field v of class C^1 with components $v^i = x^i{}_{,\alpha}\, p^\alpha$; thus $g_{\alpha\beta}p^\alpha p^\beta = 1$, where $g_{\alpha\beta}$ is the first fundamental tensor of S. It is assumed in this note that all Greek indices run from 1 to 2 and that repeated indices mean summation. The derived vector of v in S with respect to C is called the angular spread vector of v with respect to C, and its magnitude ${}_v\kappa_g$ is known as the angular spread of v with respect to C. The reciprocal ${}_vr_g = ({}_v\kappa_g)^{-1}$ is called the radius of angular spread of v with respect to C. Let θ denote the angle between C and v at P, and let Q denote the point on the angular spread vector of v at a distance ${}_vr_g \cos\theta$ from P; we call Q the *center of associate curvature for P of v with respect to C*. It is evident that if $v=t$ then Q coincides with the center of geodesic curvature of C at P. Our generalized theorem is as follows.

THEOREM. *Let C be a curve and v a vector field on a surface S. Denote by R the ruled surface formed by the tangents to S which are perpendicular to v at each point P of C. If A is the point on the generator PG of R at which the plane tangent to R is perpendicular to the plane tangent to S at P, then A is the center of associate curvature for P of v with respect to C. When R is developable, A is the point of contact of PG with the edge of regression.*

Proof. Let q^α denote the unit angular spread vector of p^α with respect to C. Then we have

$$(1) \qquad p^\alpha{}_{,\gamma}\frac{du^\gamma}{ds} = {}_v\kappa_g q^\alpha$$

where $p^\alpha{}_{,\gamma}$ is the covariant derivative of the contravariant vector p^α based upon

$g_{\alpha\beta}$. It can be shown readily that equation (1) implies

$$q^{\alpha}_{,\gamma}\frac{du^{\gamma}}{ds} = - {}_{v}\kappa_{g}p^{\alpha}. \tag{2}$$

Denote the coordinates of a generic point on R by

$$y^{i} = x^{i}(s) + \lambda q^{\alpha} x^{i}_{,\alpha} \tag{3}$$

where s, λ are parameters. It is obvious that λ is geometrically the distance from x^i to y^i. By the help of equation (2) and the equations of Gauss, we obtain from equation (3)

$$\frac{\partial y^{i}}{\partial s} = x^{i}_{,\alpha}\left(\frac{du^{\alpha}}{ds} - \lambda_{v}\kappa_{g}p^{\alpha}\right) + \lambda d_{\alpha\delta}q^{\alpha}\frac{du^{\delta}}{ds}X^{i}$$

where X^i are the components of the unit normal to S at P and where $d_{\alpha\beta}$ is the second fundamental tensor of S. The plane tangent to R and through the generator PG is perpendicular to the plane tangent to S at P if and only if $\partial y^i/\partial s$ is orthogonal to v^i, that is

$$g_{\alpha\rho}p^{\rho}\frac{du^{\alpha}}{ds} - {}_{v}\kappa_{g}\lambda = 0 \tag{4}$$

or

$$\lambda = {}_{v}r_{g} \cos \theta. \tag{5}$$

This proves the first part of the theorem.

It is shown [1, p. 962] that R is developable but not a cylinder if and only if either the generators of R are tangent to C or, with respect to C, the unit angular spread vector of v has its derived vector in S equal to its derived vector in the enveloping space. In the first case, the edge of regression is the curve C itself defined by $\lambda = 0$ in equation (3). This value $\lambda = 0$ is evidently given by equation (5) if ${}_{v}\kappa_{g} \neq 0$. If ${}_{v}\kappa_{g} = 0$, ${}_{v}r_{g}$ in equation (5) is then undefined. But from equation (4) we have $\cos\theta = 0$, which implies that R is developable with C as its edge of regression. In the second case, the edge of regression is found [2] to be the locus of orthocenter of associate curvature of the unit angular spread vector of v with respect to C. By the relation between equations (1) and (2), this orthocenter can be shown to be identical with the center of associate curvature of v with respect to C. Hence the second part of the theorem is proved.

References

1. T. K. Pan, Normal curvature of a vector field, Amer. J. Math., vol. 75, 1952, pp. 955–966.

2. T. K. Pan, Complementary surfaces for a vector field, Proc. Amer. Math. Soc., vol. 6, 1955, pp. 151–158.

$$Y_{ijklm} = x_i x_j x_k x_l x_m - \frac{10}{9} x^2 Y_{(ijk}\delta_{lm)} - \frac{3}{7} x^4 Y_{(i}\delta_{jk}\delta_{lm)},$$

and for basic h.s.h.'s of the sixth degree,

$$Y_{ijklmn} = x_i x_j x_k x_l x_m x_n - \frac{15}{11} x^2 Y_{(ijkl}\delta_{mn)} - \frac{5}{7} x^4 Y_{(ij}\delta_{kl}\delta_{mn)} - \frac{1}{7} x^6 \delta_{(ij}\delta_{kl}\delta_{mn)}.$$

The trace, obtained by contracting over any pair of indices, is zero: $Y_{ii}=0$, $Y_{iij}=0$, $Y_{iijk}=0$, $Y_{iijkl}=0$, $Y_{iijklm}=0$, *etc.* A further contraction property may also be noted: $x_iY_i=x^2$, $x_iY_{ij}=\frac{2}{3}x^2Y_j$, $x_iY_{ijk}=\frac{3}{5}x^2Y_{jk}$, $x_iY_{ijkl}=\frac{4}{7}x^2Y_{jkl}$, $x_iY_{ijklm}=\frac{5}{9}x^2Y_{jklm}$, $x_iY_{ijklmn}=\frac{6}{11}x^2Y_{jklmn}$, *etc.*

The method herein used gives a complete but not a minimal set of h.s.h.'s of any stated degree. It is possible, in various ways, to construct from the set obtained a minimal set which is also a conjugate set, that is, orthogonal over the surface of a sphere.

A NOTE ON VECTOR SPACES

R. H. Ackerson, Alabama Polytechnic Institute

Let V be a vector space of dimension n over a field F, and A a linear transformation of V into itself. Let $R(A)$ denote the range of A, and $N(A)$ denote the null space of A. $R(A)$ and $N(A)$ are subspaces of V. Let the dimension of $R(A)=r(A)$, the dimension of $N(A)=n(A)$. It is well known that $r(A)+n(A)=n$. However, it is not true in general that V is expressible as the direct sum of $R(A)$ and $N(A)$, for $R(A)$ and $N(A)$ may have non-zero vectors in common. Define $I(A)$ to be $R(A)\cap N(A)$. Since the intersection of any number of subspaces is a subspace, $I(A)$ is a subspace.

Theorem. *Let $I(A)=R(A)\cap N(A)$. The dimension of $I(A)=r(A)-r(A^2)$.*

Proof. Let $z_1, z_2, \cdots, z_k$ be a basis for $I(A)$. (The case $k=0$ is not excluded.) Since $z_i \epsilon R(A)$, there exists x_i such that $z_i=Ax_i$ ($i=1, 2, \cdots, k$). The vectors Ax_i are linearly independent vectors of $R(A)$ and may be extended to form a basis for $R(A)$, say $Ax_1, Ax_2, \cdots, Ax_k, Ax_{k+1}, \cdots, Ax_{r_1}$, where $r_1=r(A)$.

Now let x be arbitrary in V. Since Ax belongs to $R(A)$, there exist scalars $a_1, a_2, \cdots, a_{r_1}$ such that

$$Ax = a_1Ax_1 + a_2Ax_2 + \cdots + a_kAx_k + a_{k+1}Ax_{k+1} + \cdots + a_{r_1}Ax_{r_1}.$$

Any vector in $R(A^2)$ is therefore expressible as

$$A^2x = a_1A^2x_1 + a_2A^2x_2 + \cdots + a_kA^2x_k + a_{k+1}A^2x_{k+1} + \cdots + a_{r_1}A^2x_{r_1}.$$

But $Ax_1, Ax_2, \cdots, Ax_k$ belong to $N(A)$, and thus $A^2x_i=0$, $i=1, 2, \cdots, k$. Hence the vectors $A^2x_{k+1}, \cdots, A^2x_{r_1}$ span $R(A^2)$.

Suppose there exist scalars $b_{k+1}, \cdots, b_{r_1}$ such that

$$b_{k+1}A^2x_{k+1} + \cdots + b_{r_1}A^2x_{r_1} = 0.$$

The linearity of A gives

$$A[b_{k+1}Ax_{k+1} + \cdots + b_{r_1}Ax_{r_1}] = 0.$$

Hence $y = b_{k+1}Ax_{k+1} + \cdots + b_{r_1}Ax_{r_1}$ belongs to $N(A)$. Clearly y also belongs to $R(A)$. Hence y belongs to $I(A)$, and is expressible as a linear combination of $Ax_1, Ax_2, \cdots Ax_k$. The linear independence of the set $Ax_1, \cdots, Ax_k, Ax_{k+1}, \cdots, Ax_{r_1}$ implies $b_j = 0, j = k+1, k+2, \cdots, r_1$.

It follows that the set $A^2x_{k+1}, A^2x_{k+2}, \cdots, A^2x_{r_1}$ is a linearly independent set and is therefore a basis for $R(A^2)$. Hence $r(A^2) = r_1 - k = r(A) -$ dimension of $I(A)$. This concludes the proof.

Several corollaries follow immediately from the theorem.

COROLLARY 1. *$N(A) \subset R(A)$ if and only if $n(A) = r(A) - r(A^2)$.*

COROLLARY 2. *$R(A) \subset N(A)$ if and only if $A^2 = 0$.*

COROLLARY 3. *$R(A) = N(A)$ if and only if $A^2 = 0$, and $n(A) = r(A)$.*

COROLLARY 4. *$I(A) = 0$ if and only if $r(A) = r(A^2)$.*

COROLLARY 5. *$V = R(A) \oplus N(A)$ if and only if $r(A) = r(A^2)$.*

Toward a proof of Corollary 5, we note that if $V = R(A) \oplus N(A)$, then $I(A) = 0$, which implies $r(A) = r(A^2)$ by Corollary 4.

Suppose $r(A) = r(A^2)$. Consider the subspace $M(A) = R(A) + N(A)$. From the relation dim $(S+T) =$ dim $S +$ dim $T -$ dim $(S \cap T)$, where S, T are subspaces of a space V, we obtain dim $M(A) = r(A) + n(A)$, since dim $R(A) \cap N(A) = 0$ by Corollary 4. Therefore, dim $M(A) = n$, *i.e.* $M(A) = V$. We have $V = R(A) + N(A)$, where $R(A) \cap N(A) = 0$, and by the theory of direct sums $V = R(A) \oplus N(A)$.

We note that Corollary 2 is true for any vector space, finite dimensional or not.

CLASSROOM NOTES

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ONE MORE CORRECTION FORMULA

RALPH W. SNYDER, Indianapolis, Indiana

From Taylor's expansion

$$f(x) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{\text{iv}}(x_0)}{4!}h^4 + \cdots,$$

in which x_0 is an approximation to a root of $f(x)=0$ in the proper vicinity of the root, with continuity assumed, and $h=x-x_0$, we shall derive a new and closer correction formula—call it a "double-improved" Newton's formula.

Using Stewart's notation [1], $a_n=f^{(n)}(x_0)/n!$, we write Taylor's series thus:

$$f(x) = a_0 + a_1h + a_2h^2 + a_3h^3 + a_4h^4 + \cdots. \tag{1}$$

Since $f(x)=0$, we have

$$\frac{-a_0}{a_1h} = 1 + \frac{a_2}{a_1}h + \frac{a_3}{a_1}h^2 + \frac{a_4}{a_1}h^3 + \cdots.$$

We now invert both members of this equation, and obtain

$$\begin{aligned} -a_0 = a_1h &- \frac{a_0a_2}{a_1}h + a_0\left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right]h^2 \\ &- a_0\left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right]h^3 + \cdots, \end{aligned} \tag{2}$$

which converges more rapidly than (1). From (2) we get the familiar Newton-Raphson correction formula by retaining the first term on the right, and the less known and recently rediscovered Halley's formula [1] [2] by using two terms:

$$h_1 = \frac{-a_0}{a_1}\ \text{(Newton)}, \tag{3}$$

$$h_2 = \frac{-a_0}{a_1 - \dfrac{a_0a_2}{a_1}}\ \text{(Halley)}. \tag{4}$$

Both of these may of course be found from (1). But to get Halley's formula from (1) involves the replacing of h in the denominator by its approximate value h_1.

By retaining the third term of (2), and adopting the replacement technique, we get the following formula:

$$h_3 = \frac{-a_0}{a_1 - \dfrac{a_0a_2}{a_1} + a_0\left[\left(\dfrac{a_2}{a_1}\right)^2 - \dfrac{a_3}{a_1}\right]h_2} \tag{5}$$

where h_2 is Halley's result. This is our "double-improved" formula.

The subscripts 1, 2, 3, with h, do not signify an iterative sequence; they merely denote an h found by the use of one, two, or three terms of series (2).

By equating (2) successively with (3), (4), and (5), through the common term a_1/a_0, we obtain error terms:

$$(6)\qquad \epsilon_1 = h - h_1 \cong -\frac{a_2}{a_1} h_1^2, \qquad (7)\ \epsilon_2 = h - h_2 \cong \left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right] h_2^3,$$

$$(8)^*\qquad \epsilon_3 = h - h_3 \cong -\left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right] h_3^4.$$

It may be noted that the error term (6) for h_1 is similar to that shown by Scarborough in his *Numerical Mathematical Analysis*, 1930, page 183: $-Mh_1^2/2f'(x_0)$, except that his M denotes the maximum value of $f''(x)$ in the neighborhood of x_0+h_1.

Trying a new correction formula on the square root of 2 seems to be a recognized test: $f(x)=x^2-2$. Let $x_0=1$; then $a_0=-1$, $a_1=2$, $a_2=1$, $a_3=a_4=\cdots=0$. The computation by formula (5) is as follows:

Numerator: $-a_0$	=	1	
Denominator:			
1st term: a_1	=	2	
2nd term: $-a_0a_2/a_1$	=	.5	
Denominator for h_2	=	2.5	$h_2=.4$
3rd term:			
$(a_2/a_1)^2$	=	.25	
$-a_3/a_1$	=	0	
		.25	
$.25a_0h_2$	=	$-$ .1	
Denominator for h_3	=	2.4	$h_3=.41\dot{6}$.

Then, $x_1=x_0+h_3=1.416$. The first iteration, using $x_1=1.416$, gives $h_3=-.002,453,104,291,977$, whence $x_2=1.414,213,562,374,689$, which is in error by $-.0^{11}1595$. By (8), we find the computed error to be:

$$\epsilon_3 \cong -\left[\left(\frac{12}{34}\right)^3 - 0 + 0\right].002,453,1^4 = -\ .04396 \times .0^{10}3623 = -\ .0^{11}1592.$$

Thus we would have fourteen places correct by applying the error term.

By (3) and (4) respectively, from the same starting point of $x_0=1$, and with one iteration each, we would have x_2's in which the actual error is $-.00245$ and $+.000,000,364$, and the computed errors are: by (6), $\epsilon_1=-.00231$, and by (7), $\epsilon_2=.0^6366$.

* A more complete statement of the error for h_3 is obtained by including a preceding term, thus:

$$(8a)\qquad \epsilon_3 = h - h_3 \cong \left[\left(\frac{a_2}{a_1}\right)^2 - \frac{a_3}{a_1}\right][h_3 - h_2]h_3^2 - \left[\left(\frac{a_2}{a_1}\right)^3 - \frac{2a_2a_3}{a_1^2} + \frac{a_4}{a_1}\right]h_3^4.$$

The value of the first term is usually small enough to discard, but in some cases its effect cannot be ignored.

As another example, we take the equation $x-\cos x=0$, letting $x_0=\pi/4=.785{,}398{,}163{,}4$. Then $\cos x_0=\sin x_0=\sqrt{2}/2=.707{,}106{,}781{,}2$. Here $f(x)=x-\cos x$, $f'(x)=1+\sin x$, $f''(x)=\cos x$, $f'''(x)=-\sin x$, $f^{\text{iv}}(x)=-\cos x$; and $a_0=\pi/4-\sqrt{2}/2$, $a_1=1+\sqrt{2}/2$, $a_2=\sqrt{2}/4$, $a_3=-a_2/3$, $a_4=a_3/4$.

$$
\begin{array}{lll}
\text{Numerator: } -a_0 & =- & \underline{.078{,}291{,}382{,}2} \\
\text{Denominator:} & & \\
\quad \text{1st term: } a_1 & = & 1.707{,}106{,}781 \\
\quad \text{2nd term: } -a_0a_2/a_1-a_0\left(\dfrac{\sqrt{2}}{2}-\dfrac{1}{2}\right) & =- & \underline{.016{,}214{,}676} \\
\qquad \text{Denominator for } h_2 & = & \underline{1.690{,}892{,}105} \quad h_2=-.046{,}301{,}82 \\
\quad \text{3rd term:} & & \\
\qquad (a_2/a_1)^2=\dfrac{3}{4}-\dfrac{\sqrt{2}}{2} & = & .042{,}893{,}2 \\
\qquad -a_3/a_1=\dfrac{1}{3}\left(\dfrac{\sqrt{2}}{2}-\dfrac{1}{2}\right) & = & \underline{.069{,}035{,}6} \\
 & & \underline{.111{,}928{,}8} \\
\qquad .111{,}928{,}8a_0h_2 & =- & \underline{.000{,}405{,}746} \\
\qquad \text{Denominator for } h_3 & = & \underline{\underline{1.690{,}486{,}359}} \quad h_3=-.046{,}312{,}933{,}4.
\end{array}
$$

By (8) the error is

$$
\begin{aligned}
\epsilon_3 &\cong -\left[\left(\frac{5\sqrt{2}}{8}-\frac{7}{8}\right)-(-)\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\right)+\left(\frac{1}{4}\,\frac{a_3}{a_1}\right)\right]h_3^4 \\
&= -\ .020220\times .0^5 4600 = -\ .0^7 930.
\end{aligned}
$$

The approximate value of x is $x_1=x_0+h_3=.739{,}085{,}230{,}0$, true to six places. Applying the error term, we have $x_{1c}=x_1+\epsilon_3=.739{,}085{,}137{,}0$, which we might expect to be accurate to nine places. However, in this case the effect of the additional term in ϵ_3 is $+.0^8 27$, and x_{1c} is about three units low in the ninth place.

If we had stopped with h_2 (Halley), we would have $x_1=x_0+h_2=.739{,}096{,}34$, with $\epsilon_2\cong .111{,}93h_2^3=-.0^4 1111$, so that Halley corrected would give $x_{1c}=x_1+\epsilon_2=.739{,}085{,}23$.

References

1. J. K. Stewart, Another variation of Newton's method, this MONTHLY, vol. 58, 1951, pp. 331–334.

2. J. S. Frame, The solution of equations by continued fractions, this MONTHLY, vol. 60, 1953, pp. 293–305.

Both these papers contain references to several other papers.

A SIMPLE DERIVATION OF THE LEIBNITZ-GREGORY SERIES FOR $\pi/4$

D. K. KAZARINOFF, University of Michigan

The Leibnitz-Gregory formula,

$$\frac{\pi}{4} = \sum_{0}^{\infty} \frac{(-1)^k}{2k+1},$$

is usually obtained from the series for $\tan^{-1}x$ by applying Abel's theorem of limits. In this note we give a simpler and more elegant derivation of this result and the series

$$\ln 2 = \sum_{1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

We consider the integral

$$I_n = \int_0^{\pi/4} \tan^n x dx, \qquad n \geqq 2,$$

and note that it is monotone decreasing with n. It is obvious that

$$I_n + I_{n-2} = \frac{1}{n-1},$$

and

$$I_n + I_{n+2} = \frac{1}{n+1}.$$

Since I_n is monotone decreasing,

$$\frac{1}{2(n-1)} > I_n > \frac{1}{2(n+1)}.$$

On the other hand, for n a positive integer, by repeated application of the reduction formula

$$\int_0^{\pi/4} \tan^n x dx = \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2} x dx,$$

we know that

$$I_{2n} = \frac{1}{2n-1} - \frac{1}{2n-3} + \cdots \pm 1 \mp \frac{\pi}{4},$$

and

$$I_{2n+1} = \frac{1}{2n} - \frac{1}{2n-2} + \cdots \pm \frac{1}{2} \mp \frac{1}{2} \ln 2.$$

Thus

$$\frac{1}{2(2n+1)} < \left| \sum_{0}^{n-1} \frac{(-1)^k}{2k+1} - \frac{\pi}{4} \right| < \frac{1}{2(2n-1)}, \qquad n = 1, 2, \cdots,$$

and

$$\frac{1}{4(n+1)} < \left| \sum_{1}^{n} \frac{(-1)^{k-1}}{2k} - \frac{1}{2} \ln 2 \right| < \frac{1}{4(n-1)}, \qquad n = 1, 2, \cdots.$$

Letting n become infinite, we obtain the series for $\pi/4$ and $\ln 2$.

We remark that the inequalities above provide a sharper estimate of the error made by using n terms of these series than the usual estimate of "less than the first neglected term."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1191. *Proposed by C. W. Trigg, Los Angeles City College*

Identify the mathematical terms corresponding to the following unorthodox definitions: (1) an ice-cream container, (2) dull, stupid, (3) the forefinger, (4) a carpenter's tool, (5) a commission merchant, (6) having a nice discernment, (7) a writing instrument, (8) to yoke, (9) to turn up the earth with a snout, (10) one who does not conform to the standards of the gang, (11) to court in Colonial times, (12) a small stream, (13) a mercenary in the American Revolution, (14) to increase in strength and influence, (15) a prescribed movement of a fleet, (16) a somite or metamere, (17) the first entrance of any house in the calculation of nativities, (18) an attribute attached to the predicate narrowing its extent, (19) a genus of pulmonate land snails, (20) a group of two coupled feet, (21) a sandpiper which breeds in Arctic regions.

E 1192. *Proposed by Edgar Karst, Independence, Missouri*

Let AB, BC be two adjacent sides of a regular nonagon inscribed in a circle of center O. Let M be the midpoint of AB and N the midpoint of the radius perpendicular to BC. Show that angle $OMN=30°$. (Attention Professor Umbugio.)

E 1193. *Proposed by L. C. Barrett, University of Utah, and Herbert Knothe, Holloman Air Development Center*

(1) An ellipse has one focus at the center of two concentric circles, is tangent internally to the larger circle, and has at least one point in common with the smaller circle. Find the length of the maximum major axis of all such ellipses.

(2) Let the inner of two concentric circles represent a homogeneous spherical mass and the outer a prescribed orbit. Determine the range of specific energy values a particle may be given if it is to traverse a plane elliptical path from the surface of the sphere and just reach, but not cross, the circular orbit.

E 1194. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a triangle ABC given A, $ma+b$, $na+c$, where m and n are given positive integers. (Dedicated to Professor W. B. Carver, the geometer of precision.)

E 1195. *Proposed by G. E. Bardwell, University of Denver*

If $n=2, 3, 4, \cdots$ and m is fixed and positive, for what positive values of p less than 1 is $\ln n<mn^p$?

SOLUTIONS

Professor Umbugio's Fourth Degree Equation

E 1161 [1955, 254]. *Proposed by C. S. Ogilvy, Hamilton College*

Professor E. P. B. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}.$$

Deflate the professor by solving this equation using nothing higher than quadratic equations.

Solution by P. M. Anselone and Sam Cook, Johns Hopkins University Radiation Laboratory. Set $b=(x-1/x)^{1/2}$, $a=(1-1/x)^{1/2}$. Then

$$b + a = x \tag{1}$$

and, since $x\neq 0$,

$$b - a = (b^2 - a^2)/(b + a) = (x - 1)/x = 1 - 1/x. \tag{2}$$

Adding (1) and (2) we find

$$2b = x - 1/x + 1 = b^2 + 1,$$

or $b = 1$, whence

$$x^2 - x - 1 = 0,$$

or $x = (1 \pm \sqrt{5})/2$. Checking these results in the original equation we find that the only solution is $x = (1 + \sqrt{5})/2$.

Also solved by A. N. Aheart, Leon Bankoff, H. F. Bennett, B. J. Boyer, Julian Braun, Louis Brickman, R. G. Buschman, Leonard Carlitz, G. B. Charlesworth, P. L. Chessin, Hüseyin Demir, I. A. Dodes, R. P. Eisinger, A. J. Goldman, W. D. Googe, J. E. Householder, A. R. Hyde, E. S. Keeping, M. S. Klamkin, Sam Kravitz, B. R. Leeds, Jerome Manheim, E. W. Marchand, D. C. B. Marsh, R. E. Messick, Walter Penney, J. V. Pennington, C. F. Pinzka, J. A. Prieto and Max Rosenberg (jointly), James Robertson and Dale Woods (jointly), Azriel Rosenfeld, F. W. Saunders, R. E. Shafer, F. W. Sharbrough, III, Michael Skalskyj, M. R. Spiegel, O. E. Stanaitis, J. D. Thomas, C. J. Vanderlin, W. W. Varner, G. H. Vosper, G. W. Walker, Chih-yi Wang, Harry Weingarten, Maud Willey, Hazel Schoonmaker Wilson, David Zeitlin, and the proposer.

The Proposer pointed out that this is Problem No. 1 of a little book called *Mathematical Recreations*, by Horatio N. Robinson, of Albany, published in 1851. The solution there given is similar to that given above, except that the author, like many of the above solvers, failed to note that the root $(1 - \sqrt{5})/2$ is extraneous. Many solvers also employed in their solutions equations of degree higher than two. Umbugio's fourth degree equation actually factors into $(x^2 - x - 1)^2 = 0$.

Bankoff pointed out that Umbugio's deep-rooted preference for fourth degree equations stems from his long established reputation as the square of squares (see E 961 [1951, 700] and E 1111 [1954, 712]). Shafer relayed the information that Umbugio has stated that the occasion of this problem is an auspicious one, since the number of the problem (1161) has the property that $11^{6n+3} + 61^{6n+3}$ is divisible by 7, a lucky number.

More-than-Similar Triangles

E 1162 [1955, 255]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find two non-congruent similar triangles having two sides of one equal to two sides of the other.

Solution by C. F. Pinzka, Princeton, N. J. Denoting the lengths of the sides by a, b, c and a', b', c', where $a < a'$, $a:a' = b:b' = c:c'$, and $a \leqq b \leqq c$, then we must have $b = a'$ and $c = b'$. This leads to $a = a'^2/b'$. The sequence a, $b = a'$, $c = b'$, c' must therefore be a geometric progression. The further restriction $b^2/a - b = c - b < a < b$ gives $1 < b/a < (1 + \sqrt{5})/2$. The triangles with sides 8, 12, 18 and

12, 18, 27 fulfill the conditions.

Also solved by M. J. Aissen and P. M. Anselone (jointly), Leon Bankoff, B. J. Boyer, Julian Braun, Louis Brickman, W. E. Briggs, R. G. Buschman, G. B. Charlesworth, P. L. Chessin, J. E. D'Atri, Hüseyin Demir, Monte Dernham, I. A. Dodes, A. J. Goldman, J. E. Householder, A. R. Hyde, J. F. Kennison, M. S. Klamkin, Jerome Manheim, D. C. B. Marsh, G. E. Martin, Leo Moser, Herbert Nadler, C. S. Ogilvy, M. J. Pascual, Walter Penney, L. A. Ringenberg, Avinina Schub, R. E. Shafer, A. H. Simon, D. D. Strebe, G. W. Walker, Chih-yi Wang, Alan Wayne, Maud Willey, Hazel Schoonmaker Wilson, the proposer, and an anonymous solver. Late solutions by Natan Jacobson, L. V. Mead, and Ruben Perelis.

Moser and Schub located the problem in W. Leitzmann, *Lustiges und Merkwürdiges von Zahlen und Formen*, 7th ed., p. 202, and Wayne pointed out that it occurs in *The Mathematics Teacher*, vol. 47, Dec. 1954, pp. 561–2. Triangles in which $b^2=ca$ are known as *geometric mean* triangles, and are easily constructed. Aissen and Anselone generalized the problem to that of finding two non-congruent similar n-gons having $n-1$ sides of one equal to $n-1$ sides of the other.

Distributing m Similar Objects Among n Persons

E 1163 [1955, 255]. *Proposed by H. Gupta, Hoshiarpur, India*

Find the number of ways in which m similar objects can be distributed among n persons where there is no restriction as to the number of objects that any of them may receive.

I. *Solution by J. V. Pennington, Houston, Texas.* Let each of the n persons and m objects be designated by p and o, respectively. Let a permutation of all the p's and o's indicate, by the number of o's immediately following each p, the number of o's distributed to that p. Thus *pooppop* is the distribution 2, 0, 1, 0. The first symbol will therefore always be a p and we seek the number of permutations of the remaining p's and o's, $m+n-1$ in number, of which m are alike and $n-1$ are alike, that is,

$$\binom{m+n-1}{m}.$$

II. *Solution by Julian Braun, U. S. Army, White Sands Proving Ground, N. M.*

Let $f(m, n)$ be the required number of ways. We have $f(m, 1)=1$ and $f(m, n+1)=\sum_{i=0}^{m} f(i, n)$, from which it can be shown, by induction, that

$$f(m, n)=\binom{m+n-1}{n-1}=\binom{m+n-1}{m}.$$

Also solved by P. M. Anselone, H. W. Becker, D. G. Brennan, A. D. Freedman, A. R. Hyde, D. C. B. Marsh, Leo Moser, Azriel Rosenfeld, R. E. Shafer,

R. R. Sharma (3 ways), Michael Skalskyj, G. W. Walker, and Jagan Nath Mital and the proposer (jointly).

Several solvers pointed out that the problem occurs in W. Feller, *Probability Theory and Its Applications*, vol. 1, p. 52, with a solution like solution I above. Klamkin remarked that the problem appears as a theorem in Whitworth, *Choice and Chance*. Hyde and Sharma formulated solutions based on the fact that the number of distributions is the coefficient of x^m in the expansion of

$$(1 + x + x^2 + \cdots + x^m)^n.$$

A Sequence of Rational Numbers

E 1164 [1955, 255]. *Proposed by M. J. Mansfield, Purdue University*

Every rational number in the closed interval [0, 1] appears at least once in the sequence

$$0, 1, 1/2, 1/3, 2/3, 1/4, 2/4, 3/4, 1/5, 2/5, 3/5, 4/5, \cdots.$$

Find the nth, or general, term of this sequence.

Solution by E. S. Keeping, University of Alberta. Ignoring the first two terms of the sequence, the nth term is p/q where $q \geqq 2$, $1 \leqq p \leqq q-1$, and

$$q^2 - 3q + 2 = 2(n - p).$$

Therefore, if $[x]$ represents the integer equal to or next below x,

$$q = [(3 + \sqrt{8n - 7})/2], \qquad p = n - (q^2 - 3q + 2)/2.$$

If we replace n by $n-2$, the nth term, for $n > 2$, of the original sequence is p/q, where

$$q = [(3 + \sqrt{8n - 23})/2], \qquad p = n - (q^2 - 3q + 6)/2.$$

Also solved by P. M. Anselone, Julian Braun, R. G. Buschman, Leonard Carlitz, G. B. Charlesworth, P. L. Chessin, Monte Dernham, J. E. Householder, A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, Herbert Nadler, Walter Penney, Azriel Rosenfeld, F. W. Saunders, R. E. Shafer, Michael Skalskyj, C. J. Vanderlin, Jr., Vladeta Vučković, G. W. Walker, Chih-yi Wang, K. B. Williams, and the proposer.

This problem, with a solution, can be found in Walmsley, *Mathematical Analysis*, Cambridge, 1926, pp. 57–8. Klamkin remarked that a more interesting problem is to determine the nth term of the given sequence after we have cast out all repeat terms. Buschman showed interest in the series $\sum n^{-2}/|x-a_n|$ and $\sum f(n)/|x-a_n|$, where $0 \leqq x \leqq 1$ and a_n is the nth term of the given sequence. Is there, for example, an $f(n)$ such that the second series converges for all irrational values of x in the interval?

Convex Polygons and Regular Vertices

E 1165 [1955, 255]. *Proposed by Andrew Sobczyk, Los Alamos Scientific Laboratory*

A vertex V of a closed polygon C having an odd (even) number of sides is *regular* in case a triangle formed by extending the sides incident on V and having for base a line segment containing the opposite side (vertex) to V circumscribes C. Show that every convex C has at least one regular vertex. (A convex pentagon may have non-regular vertices.)

Solution by G. W. Walker, Buffalo, N. Y. Let us be given a convex polygon of n sides. A point moving around the perimeter will travel in succession in n different directions, changing directions as it passes each vertex by an amount equal to the exterior angle at that vertex.

Draw a set of n rays through any point O, each ray parallel with one of the n directions taken by the moving point. Each ray, in circuit, will correspond to one of the sides of the polygon, and each sector between rays will correspond to the vertex between the sides.

Draw a line through O and let it swing around O as a pivot.

Case 1. When n is even.

Consider any vertex of the polygon, V. In the figure around O place the swinging line so that one end is within the sector corresponding to V and the other end of the line is within some other sector. Count the rays between the two ends on the two sides of the line. If the numbers are the same, the far end of the line is in the sector corresponding to the vertex numerically opposite to the vertex V, and V will necessarily be "regular" in the sense defined; since the extension of the two sides of the vertex V and a line through the opposite vertex parallel to the given position of the swinging line will circumscribe the polygon.

If more rays are on one side of the line than on the other, call that side positive and the other negative. Swing the line slowly around O. Every time either end of the line crosses a ray, the number of rays on the positive side will be increased or decreased by one (and the number on the negative side decreased or increased correspondingly) unless it just happens that both ends of the swinging line cross rays simultaneously, in which case the score remains unchanged. By the time the line has swung around through 180°, the side originally positive will have become negative. Since every change involved a shift of a single unit, and the sum of the two sides was always even, there must have been some position of the swinging line in between when there were exactly as many rays on one side as on the other. The vertex corresponding to either sector marked out by this position of the line must be a regular vertex in the sense defined.

Case 2. When n is odd.

Choose a ray in the figure around O which does not have a ray directly opposite, and place the swinging line so that one end coincides with that ray, the

other end falling in a sector. Count the rays on the two sides of the line. If the numbers are equal, the vertex corresponding to the indicated sector will be "regular" in the sense defined, since the extension of the two sides of the vertex and the extension of the side opposite will circumscribe the polygon.

If the numbers are not equal, label the two sides of the line positive and negative, and swing the line slowly around O. Every time one end of the line leaves a ray, a score of one is added to either the positive or the negative side, making the total of the two sides odd. If either end later comes into coincidence with a new ray which has no ray exactly opposite, a score of one is subtracted from either the positive or the negative side, making the total even again. Cases where the line comes into coincidence with or separates from two opposite rays simultaneously will keep the score uneven and can be ignored. If the line swings through 180°, the positive side will have become negative, by steps involving single units, and there must have been some position in between when the scores on the two sides of the line were exactly equal. If the line is placed in that position, the vertex corresponding to the sector within which one end of the line is found must be regular in the sense defined.

Therefore every polygon, whether the number of sides is even or odd, must have at least one vertex "regular" in the sense defined in the problem.

Also solved by P. M. Anselone, Enrique Rubin, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4663. *Proposed by H. S. Shapiro, New York University*

Let $P(z)$ and $Q(z)$ be any two polynomials and let $F(n)$ denote the number of zeros of the polynomial $P(z)+z^nQ(z)$ which lie in $|z|<1$. Prove that

$$\lim_{n\to\infty}\frac{F(n)}{n}=\frac{\mu}{2\pi}$$

where μ is the measure of the set of θ for which $|P(e^{i\theta})|<|Q(e^{i\theta})|$.

4664. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Are there any other laws of attraction beside the inverse square law such that the time of descent (from rest) through any straight tunnel through a uniform spherical planet is independent of the path?

4665. *Proposed by W. K. Ergen and W. C. Sangren, Oak Ridge National Laboratory*

Prove, or give a counter example of, the inequality

$$\int_0^a ds \int_0^b dt \int_0^p dx \frac{y(x-s)\cdot y(x-t)}{y(x)} \geqq ab \int_0^p y(x)dx,$$

where a and b are positive constants and where $y(x)$ is a positive, continuous and periodic function with period p.

4666. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p = p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x} + \frac{1}{T_y} = \frac{1}{T_z}\cdot$$

4667. *Proposed by J. L. Ullman, University of Michigan*

Let λ_i be an infinite sequence of positive numbers such that $\sum \lambda_i = 1$. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{\infty} ((N\lambda_i)) = 0,$$

where $((x))$ means the fractional part of x.

SOLUTIONS

An Infinite Sequence of Pythagorean Triangles

4600 [1954, 476]. *Proposed by Leon Bankoff, Los Angeles, California*

Vertices A–C and B–D of square $ABCD$ are joined by quadrants of circles (B) and (C). A semi-circle (O_1) is described internally on the diameter BC and a circle (O_2) is drawn tangent to the three arcs. Another circle (O_3) is drawn tangent to circle (O_2) and to arcs AC and BC, and a right triangle is formed by joining O_3 to O_1 and dropping a perpendicular from O_3 upon BC. Successively tangent circles are drawn in the same manner (with (O_n) tangent to (O_{n-1}) and to arcs AC and BC) and right triangles are formed (with O_1O_n for hypotenuse).

Show that the infinitude of triangles so constructed are Pythagorean.

Solution by W. J. Blundon, Memorial University of Newfoundland. Let the radius of (O_n) be r_n, let P_n be the projection of O_n on BC, and let $AB=2$. Then $O_nB=2-r_n$, $O_1O_n=1+r_n$, $O_1P_n=1-3r_n$, $O_nP_n=(8r_n-8r_n^2)^{1/2}$, and O_nO_{n+1}

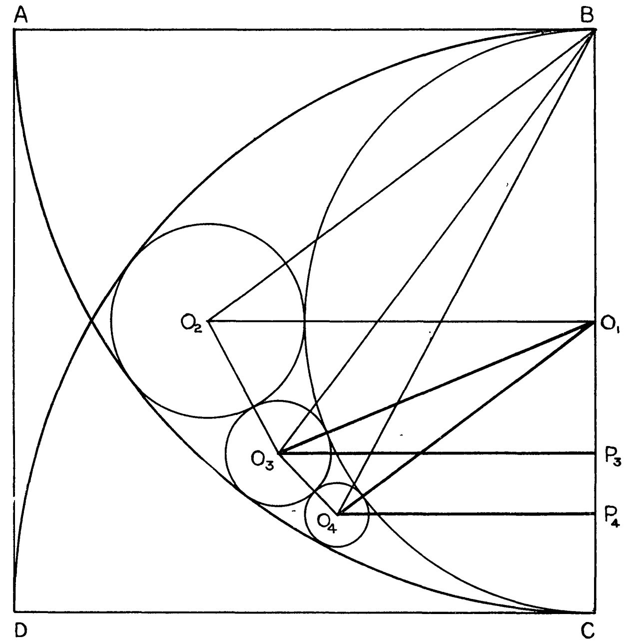
$=r_n+r_{n+1}$.

The relationship $O_nO_{n+1}^2=(O_1P_{n+1}-O_1P_n)^2+(O_nP_n-O_{n+1}P_{n+1})^2$ gives

$$(1)\qquad 9r_{n+1}^2r_n^2 - 4r_{n+1}^2r_n - 4r_{n+1}r_n^2 + 4r_n^2 - 8r_nr_{n+1} + 4r_{n+1}^2 = 0.$$

Solving this quadratic in $1/r_{n+1}$ and selecting the root that gives $r_{n+1}<r_n$, we have

$$(2)\qquad 1/r_{n+1} = 1/r_n + 1/2 + (2/r_n - 2)^{1/2}.$$



Now from triangle O_1O_2B, $r_2=1/3$ and (2) gives $r_3=2/11$, $r_4=2/18$, $r_5=2/27$, $\cdots$. The conjecture that $r_n=2/(n^2+2)$ is easily proved by induction. Hence the sides of triangle $O_1O_nP_n$ are found to be

$$\frac{n^2-4}{n^2+2},\qquad \frac{4n}{n^2+2},\qquad \frac{n^2+4}{n^2+2}.$$

Thus the triangles are all Pythagorean.

It is interesting to note that the only duplicated ratios of sides are 5:12:13 corresponding to $n=3$ and $n=10$, and 3:4:5 corresponding to $n=4$ and $n=6$. Also the centers of the circles lie on the arc of an ellipse with foci B and O_1.

Also solved by Hüseyin Demir, A. R. Hyde, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy, Roscoe Woods, and the Proposer.

Editorial Note. The Proposer remarks that Pythagorean triangles are also

generated when O_nB are taken as hypotenuse, also when lines joining the centers of consecutive circles are taken as hypotenuse with legs perpendicular and parallel to BC. See also the Proposer's paper, The Golden Arbelos, *Scripta Mathematica*, v. 21 (1955), pp. 70–76.

A Quadratic Recurrence

4601 [1954, 477]. *Proposed by Harry Goheen, Iowa State College*

What is the necessary and sufficient condition that as $t\to\infty$ the limit of n_i exists, if n_i is defined by the recurrence relation

$$n_{i+1} = bn_i^2 + c,$$

n_0 being given?

Editorial Note. The problem has been considered previously. C. D. Olds calls attention to a paper, The Convergence of Sequences Defined by Quadratic Recurrence Formulae, by T. W. Chaundy and Eric Phillips in the *Quarterly Journal of Mathematics* (Oxford series), vol. 7 (1936), pp. 74–80. The answer to the present question may be summarized as follows:

Simplify the given recurrence relation by putting $u_i = bn_i$ to get

$$u_{n+1} = u_n^2 + bc, \tag{1}$$

and let $k \geqq \frac{1}{2}$, $1-k$ be the real roots of $x^2 - x + bc = 0$. Take also $b \neq 0$ (for $b=0$ the problem is trivial).

The limit as $n\to\infty$ of u_n, if it exists, must be either k or $1-k$. Further, if any u_n is $\pm k$ or $\pm(1-k)$ all subsequent u_{n+i} are stationary at k or $1-k$, respectively. Corresponding values of u_0 (other than k, $1-k$) are easily obtained from (1) by setting $u_n = k-1$ or $-k$ and solving for $u_{n-1}, u_{n-2}, \cdots$. To obtain real values it is necessary that $k \geqq 1$ and $k \geqq 2$, respectively. If $k=2$ we can put $u_0 = 2\cos(r\pi/2^s)$ where r and s are positive integers.

In all cases where the sequence does not become at length stationary, the necessary and sufficient condition for the existence of the limit of u_n is given by $1/2 \leqq k \leqq 3/2$ and $|u_0| < k$. Then $\lim u_n = 1-k$.

Also solved by W. S. Lawton, and C. A. Rogers.

A Trigonometric Sum

4602 [1954, 477]. *Proposed by C. M. Ablow and D. L. Johnson, Seattle, Washington*

Show that

$$\sum_{i=1}^{n} A_i \cos(B_i t + C_i)$$

$$\bar{t}(v) = \sum_{n=0}^{\infty} (T + nr)e^{-pT} \cdot \frac{(pT)^n}{n!} = T(1 + pr) = \frac{x}{v}(1 + p(v)r).$$

The velocity v_m for minimum $\bar{t}$ is therefore obtained by solving

$$-\frac{1 + p(v_m)r}{v_m^2} + \frac{p'(v_m)r}{v_m} = 0.$$

Property of an Open, Unbounded Set

4605 [1954, 572]. *Proposed by D. J. Newman and W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

Given an open, unbounded set of positive reals. Prove that there exists a real number such that infinitely many integral multiples of it lie in the set.

I. *Solution by M. Golomb and S. H. Gould, Purdue University.* Let A_n, with $n=1, 2, 3, \cdots$, denote the set of positive reals p_n such that the given set S contains no multiple of p_n greater than np_n. Then the union A of the sets A_n is of the first category, so that its complement, being of the second category, is non-empty, as desired.

For, let $[a, b]$ be any interval in the positive reals and, for each n, choose t with $tb<(t-1)a$. Then, with $m=1, 2, \cdots$, the union $\{[ma, mb]\}$ of all multiples of $[a, b]$ contains every real $r>tb$. Thus, if A_n were dense in $[a, b]$, the multiples of A_n would be dense in (tb, ∞) and the set S, being open and unbounded, would contain infinitely many of them, in contradiction to the definition of A_n. Thus A_n is nowhere dense and A is of the first category.

Note: The set A_n is easily seen to be closed, so that A is an F_σ set of first category. Conversely, given such an A, it is easy to construct an open, unbounded set S such that S contains only finitely many multiples of any number in A.

II. *Solution by R. D. Anderson and N. J. Fine, University of Pennsylvania.* We prove somewhat more, that the set of numbers satisfying the condition is dense in the positive reals. Let S be the given open set, and let (a_1, b_1) be an arbitrary open interval, $0<a_1<b_1$. If $n>a_1/(b_1-a_1)$, the intervals (na_1, nb_1) and $((n+1)a_1, (n+1)b_1)$ overlap, and so the integral multiples of (a_1, b_1) cover the reals to the right of some point. Hence, given n_1, there exists $n_2>n_1$ such that (n_2a_1, n_2b_1) intersects S. Since S is open, there is an interval (a_2, b_2) such that $[a_2, b_2]\subset(a_1, b_1)$ and $[n_2a_2, n_2b_2]\subset S$. This process may be continued to get a nested sequence of closed intervals $[a_k, b_k]$ and an increasing sequence n_k such that $n_kx\in S$ for every $x\in[a_k, b_k]$, $k>1$. Any point in the non-vacuous intersection will satisfy the required condition.

Also solved by Y. H. Clifton, H. J. Cohen, C. D. Gorman, Horace Komm, Joseph Lehner, E. C. Milner, D. J. Newman, P. J. Owens, L. A. Ringenberg, Azriel Rosenfeld, and W. R. Scott.

$$\bar{t}(v) = \sum_{n=0}^{\infty} (T + nr)e^{-pT}\cdot\frac{(pT)^n}{n!} = T(1 + pr) = \frac{x}{v}(1 + p(v)r).$$

The velocity v_m for minimum $\bar{t}$ is therefore obtained by solving

$$-\frac{1 + p(v_m)r}{v_m^2} + \frac{p'(v_m)r}{v_m} = 0.$$

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Note: The set A_n is easily seen to be closed, so that A is an F_σ set of first category. Conversely, given such an A, it is easy to construct an open, unbounded set S such that S contains only finitely many multiples of any number in A.

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Also solved by Y. H. Clifton, H. J. Cohen, C. D. Gorman, Horace Komm, Joseph Lehner, E. C. Milner, D. J. Newman, P. J. Owens, L. A. Ringenberg, Azriel Rosenfeld, and W. R. Scott.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio

Mathematics for the Secondary School. By W. D. Reeve. New York, Henry Holt and Company. xii+547 pages. $5.95.

This new book seems to the present reviewer to be several books in one, combined rather successfully in an "integrated" attack on an important problem.

This book combines a text for the teacher of Mathematics methods, a manual for the curriculum planner, and a guide for the teacher in service, including the experienced teacher. It provides a comprehensive treatment of the changes made and in progress in the teaching of Mathematics, with some account of the pressures enforcing these changes. The book is heavily documented and quotes freely from authorities. There is excellent bibliographical information, which is, however, presented in footnotes rather than in an annotated collection.

There are several notable elements of strength. The necessity of adjustment to the demands of mass education receives due attention throughout the book. Especially strong individual sections deal with the building of a vocabulary and establishing precision in English language use, with methods of reasoning with examples of false or deceptive reasoning, and with the early introduction of the function concept.

Reeve supports a program of attention to the gifted student but is less fertile in suggestions than in the case of the slow learner or the student who does not anticipate the need for Mathematical knowledge. The present reviewer wishes the writer had not accepted a rather low minimum for the mathematical knowledge of Mathematics teachers. Cessation of the study of Mathematics with the completion of Calculus leaves the student without a balanced knowledge of Mathematics. The emphasis would be on analysis rather to the disadvantage of geometry and algebra. Some courses in Higher Algebra (say, Theory of Equations) and in Projective and Non-Euclidean Geometry would leave the student with a vastly enhanced competence in some of the areas Reeve emphasizes, *e.g.*, the nature of proof.

This book may be recommended for use as a text in "methods" classes but should be studied under the guidance of an instructor who has himself substantial mastery of the essential Mathematics. It may perhaps be recommended more strongly to the teacher in service as a valuable addition to his professional library.

F. C. OGG
Bowling Green State University

get necessary and sufficient conditions that a local analytic mapping of a neighborhood in a given surface M onto a neighborhood in another surface R can be extended over all of M to give a conformal homeomorphism of M onto a subdomain of R. When R is the sphere and M is a subdomain of R, these results lead to the Grunsky conditions for a mapping of M to be schlicht.

For $M \subset R$, an integral operator T which transforms differentials f' on M into differentials Tf' on R is defined as a scalar product over M of f' and a bilinear differential for R. A study is made of the spectral theory of this and related integral operators in chapter six. This leads to more relations between the domain functionals of M and of R. Chapter seven develops the variational formulas for the Green's function (and hence the other domain functionals) as the following deformations are made in the surface: (a) cutting a hole, (b) attaching a handle, (c) attaching a cross-cap, and (d) attaching a cell. The first three alter the topological type of the surface while (d) alters only the conformal type, so that (d) may be used to preserve the conformal class of a surface after application of the others. Chapter eight gives various applications of the variational methods; for example, the coefficient problem for schlicht functions.

The last chapter is devoted to putting this work in a still broader framework. A sketchy introduction gives the reader the definitions and basic facts about Kähler manifolds. The Green's and Neumann's operators are defined on finite Kähler manifolds and a close analogy to the 2-dimensional case is displayed. However the reproducing kernels, defined in terms of the Green's and Neumann's operators, lose many of their important features, showing that one cannot expect to generalize all that was valid on finite Riemann surfaces to Kähler manifolds.

George Springer
Northwestern University

NEW BOOKS RECEIVED

Calculus. By W. L. Hart. Boston, D. C. Heath and Company. 13+626 pages, 1955. $5.50.

Harmonic Analysis and the Theory of Probability. By Salomon Bochner. Berkeley and Los Angeles, University of California Press. viii+176 pages, 1955. $4.50.

Lectures on Functions of a Complex Variable. Edited by Wilfred Kaplan. Ann Arbor, University of Michigan Press, 1955. ix+435 pages. $10.00.

Trojkaty Pitagorejskie. By Waclaw Sierpinski. Warsaw, Panstwowe Wydawnictwo Naukowe, 1954, 94 pages.

Douze, Notre Dix Futur. By Jean Essig, Paris, Dunoid. 1955, 170 pages.

Fuenfstellige Tafeln der Kreis- und Hyperbelfunktionen. By Hayashi Keiichi. Berlin, Germany, Walter de Gruyter and Co.

Arytmetyka Teoretyczna. By Waclaw Sierpinski. Warsaw, Panstwowe Wydawnictwo Naukowe, 1955. 258 pages.

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Dzieje Mysli Kopernikowskiej. By T. Przypkowski. Warsaw, Wydawnictwo Ministerstwa Obrony Narodowej, 1954. 116 pages.

Machine Translation of Languages. Edited by W. N. Locke and A. D. Booth. New York, John Wiley and Sons, Inc. xii+243 pages, 1955. $6.00.

From Zero to Infinity. By Constance Reid. New York, Thomas Y. Crowell Co., 145 pages, 1955. $3.00.

On Uniquely Solvable Boolean Equations. By W. L. Parker and B. A. Bernstein. Berkeley and Los Angeles, University of California Press. 29 pages, 1955. $0.50.

Second Colloque sur les Equations Aux Derivees Partielles. Tenu a Bruxelles du 24 au 26 mai 1954. Paris, Georges Thone, Editeur, Mass et Cie, Publisher. 128 pages. 1.500 francs francais.

Advanced Calculus, An Introduction to Classical Analysis. By Louis Brand. New York, John Wiley and Sons, Inc. xii+574 pages, 1955. $8.50.

Basic Mathematics for General Education, Second Edition. By H. C. Trimble, L. C. Peck and F. C. Bolser. New York, Prentice-Hall, Inc., 1955. xiv+363 pages.

Fundamentals of Business Mathematics, Second Edition. By W. R. Van Voorhis and C. W. Topp. New York, Prentice-Hall, Inc., 1955. xi+452 pages.

Fundamental Concepts of Mathematics, Second Edition. By R. H. Moorman. Minneapolis, Minn., Burgess Publishing Company, 1955. iii+92 pages. $2.75.

Analytic Geometry, Third Edition. By F. H. Steen and D. H. Ballou. Boston, Ginn and Co., 1955. vii+225+19 pages.

Differential Equations, Second Edition. By Lester R. Ford. New York, McGraw-Hill Book Company, 1955. xii+291 pages. $5.00.

Arithmetic in General Education. By D. C. Duncan. Dubuque, Iowa. Wm. C. Brown Co., 1955. $2.25.

Representation Conforme et Transformations a Integrale de Dirichlet Bornee (*Cahiers Scientifiques, Fascicule XXII*). By Mme. Jacqueline Lelong-Ferrand. Paris, Gauthier Villars, 1955. Complete volume in 8(16–25) de vii-257 pages. $11.63.

Statistical Methods, Third Edition. By F. C. Mills. New York, Henry Holt and Company, 1955. xviii+842 pages. $6.90.

Principles of Mathematics. By C. B. Allendoerfer and C. O. Oakley. New York, McGraw-Hill Book Company, 1955. xv+448 pages. $5.00.

Advanced Calculus. By A. E. Taylor. New York, Ginn and Company, 1955. xiii+786 pages. $5.00.

Lessons in Elementary Analysis. By G. S. Mahajani. Poona, India, 1954. xiv+348 pages. Price Rs. 10-0-0.

Mathematics for Engineers, Part I, Ninth Edition Revised. By W. N. Rose. London, England, 1955. xiv+527 pages. 21s.

OBITUARY

RAYMOND CLARE ARCHIBALD

IN MEMORIAM

Raymond Clare Archibald was born in Colchester County, Nova Scotia, on October 7, 1875, the son of Abram Newcomb and Mary Mellish Archibald. When he was a small boy, his father died and Raymond was brought up by his mother. During his youth she held a position as teacher in the Mount Allison Ladies College at Sackville, New Brunswick, and Archibald was himself graduated from Mount Allison University when still but eighteen years of age. At this time he received an A.B. with first class honors in mathematics and a teacher's diploma in violin. Presently he continued his studies at Harvard where he was awarded a second bachelor's degree in 1896 and a master's degree in 1897. After one more year of graduate study there, he went to Germany for two years. The first was spent at the University of Berlin and the second at the University of Strasbourg, from which he received his doctor's degree in 1900. Later on (1909–1910) he studied for a year at the Sorbonne and still later (1922) for a short time at the University of Rome.

During the years 1894–95 and 1900–07 he taught in the Mount Allison Ladies College. Some of his duties were in mathematics, but in later years he used to refer more often to his teaching of the violin during this period. He was also given responsibilities in the library which he "developed from nothing to 12,000 volumes and catalogued by writing 30,000 cards by hand."

After a year as Professor of Mathematics at Acadia University in Wolfville, Nova Scotia, he was brought to Brown as instructor in 1908. He now felt that he must banish music almost completely from his life and devote all his energies to mathematics in order to make a success of his work.

When as a freshman in Brown University in 1915–16 I first came to know Archibald, he was my teacher. Then he was in the prime of life and full of energy. He was striking in appearance, his hair wavy and beginning to gray, worn a little longer than was generally the custom; his eyes large and expressive. Always carefully groomed, he wore a high starched collar and stiff detachable cuffs. Often a white edging on his waistcoat gave an added touch of elegance to his dress. To me he was an impressive figure among the Brown faculty.

Archibald firmly believed that the relationship between teacher and student should be personal and friendly and close. It was his custom to make appointments with each student in his classes for one or two half-hour conferences each semester. Thus I came to know him outside the classroom, and in these conferences he gave me encouragement in my work and fostered the growth of my interest in mathematics.

In 1908 the mathematical library at Brown, though well selected, was of small proportions. It was not difficult, after Archibald's introduction to library work at Mount Allison, to enlist his interest in developing and enlarging the mathematical library here. He threw himself into this task with a will, and in the

ever increasing field of usefulness and significance.

With the passage of the years many well deserved distinctions came to Archibald from institutions on both sides of the Atlantic. Honorary degrees, memberships in academies of science and foreign mathematical societies, all paid tribute to the high regard in which he was held by friends everywhere and in many cases acknowledged services generously performed and freely given. He was President of the Mathematical Association in 1922, Vice-President and Chairman of Section A, Mathematics, of the American Association for the Advancement of Science in 1928, and in 1937 was Vice-President and Chairman of Section L, History and Philosophy of Science.

Archibald was both a scholar of the old school and a gentleman of the old school as most of us now regard it. He was brought up in the classical tradition with much emphasis on Latin and Greek. He had a very remarkable memory, and he carried with him at all times an enormous store of factual information in the fields of his interest. His death on July 26, 1955, in Sackville brought to a close the life of a man who will long be remembered for his kindness, his unwillingness to compromise his standards, and his deep devotion to Brown University.

A minute adopted by the Faculty of Brown University, September 20, 1955; prepared by C. R. Adams with the assistance of Otto Neugebauer.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1955 Preliminary Actuarial Examination are as follows:

First Prize $200

Adler, David . . . . Rensselaer Polytechnic Institute

Additional Prizes of $100

Harris, Glen A., Jr. . . . . Yale University
Horn, William A. . . . . University of Cincinnati
McCullough, Roger S. . . . . University of Toronto
Mitchell, Barry M. . . . . University of Toronto

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Mitchell, Barry M. . . . . University of Toronto

Moskowitz, Lester	Brooklyn College
See, Gary N.	University of Michigan
Sprung, Donald W. L.	University of Toronto
Walton, Charles K.	University of Toronto

The Society of Actuaries has authorized a similar set of nine prizes for the 1956 examinations on Part 2.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. Language Aptitude Examination
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning).

Part 2. General Mathematics Examination.
(Algebra, trigonometry, coordinate geometry, differential and integral calculus).

Part 3. Special Mathematics Examination.
(Finite differences, probability and statistics).

The 1956 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 9, 1956 (tentative date). The closing date for application is March 15, 1956.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

Professor R. L. Swain of Teachers College at New Paltz, New York, is on leave for 1955–1956 on a Faculty Fellowship awarded by the Ford Fund for the Advancement of Education.

Butler University announces: Miss Anne J. Flanagan, formerly an assistant at Florida State University, has been appointed to an instructorship; Assistant Professor R. H. Oehmke received a research grant from the Air Force for an investigation in non-associative algebras.

Case Institute of Technology reports the following: Dr. J. T. Chu has been appointed Visiting Assistant Professor; Mr. J. S. Klein of Williams College and Mr. R. P. Knupke have been appointed to instructorships; Assistant Professor P. E. Guenther has been promoted to an associate professorship.

Indiana University announces the following: Assistant Professor Murray Rosenblatt of the Statistical Research Center, University of Chicago, has been appointed to an associate professorship; Dr. J. R. Blum has been promoted to an assistant professorship.

At Montana State College: Associate Professor Bernard Ostle has been promoted to a professorship; Mr. L. R. Amunrud, Mr. G. R. Ingram, and Miss

Beverly D. Swindlehurst have been appointed to instructorships; Dr. W. J. Swartz of Iowa State College has been appointed to an assistant professorship.

North Carolina State College reports: Assistant Professor G. C. Watson has been promoted to an associate professorship; Mr. H. A. Petrea has been promoted to an assistant professorship; Mrs. Martha J. Garren, previously a teacher in Wake County Public Schools, North Carolina, Mr. A. R. Marshall, formerly a teacher in the public schools of Maryland, Miss Carlotta P. Patton, and Mr. W. C. Turner of Sweet Briar College have been appointed to instructorships.

Stanford University reports the following: Dr. H. H. Levine, research fellow at Harvard University, has been appointed Visiting Associate Professor; Dr. Robert Osserman, previously at Harvard University, has been appointed Acting Assistant Professor; Associate Professor P. R. Garabedian has been promoted to a professorship; Acting Assistant Professor P. W. Berg has been promoted to an assistant professorship.

State University of Iowa announces that Associate Professors N. B. Conkwright, E. N. Oberg, and H. T. Muhly have been promoted to professorships.

Stevens Institute of Technology reports: Professor L. L. Merrill, head of the Department of Mathematics, has been named to the additional post of Dean of Faculty; Associate Professor L. Z. Pollara has been promoted to the position of Professor and Head of the Department of Chemistry and Chemical Engineering; Assistant Professor N. J. Rose has been promoted to an associate professorship.

At Syracuse University: Associate Professor K. L. Chung has been promoted to a professorship; Dr. Jerome Blackman and Mr. Cyrus Derman have been promoted to assistant professorships; Mrs. Marjorie Halpern of Lehigh University and Mr. D. M. Friedlen, previously a graduate assistant at the University of Illinois, have been appointed to instructorships.

Utah State Agricultural College reports the following appointments to the staff: S. W. Bingham, Max Connor, Kent Harris, Richard Mitchell, Phillip Monson, Kenneth Moosman, H. A. Mortensen, C. C. Nielsen, Wendell Pope, W. R. Rich, and H. B. Tingey. Associate Professor Joseph Elich is on sabbatical leave for the year 1955–1956.

University of Alberta announces the following: Professor E. S. Keeping has been promoted to the position of Head of the Department of Mathematics; Assistant Professors Leo Moser, Thorleif Fostvedt, and Allan Gibb have been promoted to associate professorships; Mr. J. R. Pounder of Dublin, Ireland, has been appointed to an assistant professorship; Dr. A. H. Lightstone, formerly a teaching fellow at the University of Toronto, has been appointed Lecturer; Dr. Irwin Guttman, previously a teaching fellow at the University of Toronto, has been appointed to an assistant professorship.

University of Arizona announces: Assistant Professor B. C. Meyer has been promoted to an associate professorship; Mr. M. W. Karlin, formerly a graduate student at the University of California at Los Angeles, and Mr. L. A. Kenna, previously an instructor at Tucson Senior High School, Arizona, have been

appointed to instructorships; Mrs. Georgia T. Hart, Mr. J. E. Lee, and Mr. J. E. Strang, formerly students at the University of Arizona, have been appointed to assistantships.

University of British Columbia reports the following: Associate Professor Eugene Leimanis has been promoted to a professorship; Assistant Professor T. E. Hull has been promoted to an associate professorship; Assistant Professor H. F. Davis, II, of Miami University has been appointed to an instructorship.

At the University of Buffalo: Dr. J. H. Hodges of Duke University has been appointed to an assistant professorship; Mr. R. T. J. Mahoney, formerly a student at the University, Mr. E. P. Rozycki, previously in military service, Mr. R. T. Sandberg, recently a student at Alfred University, and Mr. F. R. White, formerly a student at the University, have been appointed teaching fellows; Mr. R. G. Fryer and Mr. R. V. Nolan, previously teaching fellows at the University, have been promoted to instructorships; Professor V. E. Pound has retired with the title Professor Emeritus.

University of Delaware announces the following: Assistant Professor J. H. Barrett has been awarded a National Science Foundation grant for research in mathematics and is on leave for the academic year of 1955–1956 at Yale University; Dr. Madeline Alexander has been appointed Lecturer.

University of Georgia reports that Mr. C. D. Kirkland and Mr. Lewis McNair, previously assistants at the University, have been appointed to instructorships.

University of Nebraska announces that Dr. G. C. Cree of the University of Maryland and Mr. Richard Scheer have been appointed to instructorships.

At the University of New Mexico: Dr. Donald Dubois of Ohio State University and Dr. P. W. M. John of the University of Oklahoma have been appointed to assistant professorships.

University of Oklahoma reports the following: Associate Professor Arthur Bernhart has been promoted to a professorship; Assistant Professor R. V. Andree has been promoted to an associate professorship; Professor C. E. Springer has resigned from his position as Chairman of the Department of Mathematics and Astronomy in order to return to full time teaching; Associate Professor W. N. Huff has been appointed Chairman.

University of Oregon announces the following: Assistant Professor Herman Rubin of Stanford University has been appointed to an associate professorship; Dr. H. G. Tucker, formerly an associate at the University of California, has been appointed to an assistant professorship; Mr. H. S. Bear, previously a teaching assistant at the University of California, and Mr. H. G. H. Bartram of the University of Colorado have been appointed to instructorships; Assistant Professor Bertram Yood has been promoted to an associate professorship; Dr. R. L. San Soucie has been promoted to an assistant professorship.

University of Saskatchewan announces: Associate Professor Peter Scherk has been promoted to a professorship; Mr. Nathan Shklov, previously a special instructor, has been promoted to an assistant professorship; Mr. W. O. J. Moser

of the University of Toronto has been appointed to an instructorship.

At the University of Tennessee: Associate Professor W. S. Beckwith of the University of Georgia has been appointed Acting Assistant Professor; Dr. Shirley A. Burr, previously a teaching assistant at Cornell University, Dr. Carol G. Doss, and Mr. Karl Eide, recently teaching assistants at the University, and Assistant Professor F. J. Witt of Tennessee Polytechnic Institute have been appointed to instructorships; Assistant Professor H. W. Stephens of Ball State Teachers College has been appointed Consultant in Secondary Mathematics; Miss Ruth B. Hofstra has received a scholarship from Southern Regional Fellowships; Mr. J. A. Alexander and Mr. R. J. Smith have received National Science Foundation grants for research in Continuous Geometry; Mr. Ralph McWilliams has received a National Science Foundation scholarship.

University of Virginia reports: Professor C. L. Clark of Oregon State College has been appointed Visiting Professor; Mr. Marvin Rosenblum of the University of California has been appointed Acting Assistant Professor; Dr. W. E. Malbon of the University of Virginia has been promoted to an assistant professorship.

West Virginia University announces: Associate Professor M. L. Vest has been promoted to a professorship; Mr. R. A. Barron, Mrs. Lorna W. Carlin, Miss Elaine Cook, Mr. Edwin Duda, Miss Constance M. Foley, and Mrs. Anna L. Weihrer have been appointed to instructorships; Associate Professor Margaret B. Cole has retired.

Dr. J. E. Adney, Jr., of Ohio State University has been appointed to an assistant professorship at Purdue University.

Mr. W. O. Alexander, Jr., formerly a graduate student at the University of Houston, has been appointed to an assistant professorship at the University of Corpus Christi.

Mr. B. W. Antliff of the University of Saskatchewan has been appointed to an instructorship at Canadian Services College, Royal Roads, British Columbia.

Dr. H. A. Antosiewicz of American University has a position as a mathematician in the United States Department of Commerce, National Bureau of Standards, Washington, D. C.

Dr. W. A. Beck, previously an assistant at Purdue University, has been appointed to an assistant professorship at Bucknell University.

Mr. A. H. Blessing of the University of Buffalo is employed now by the Bell Aircraft Corporation, Niagara Falls, New York.

Mr. L. L. Brassaw, Jr., formerly a teaching fellow at the University of Buffalo, is now with the Bell Aircraft Corporation, Niagara Falls, New York.

Professor H. K. Brown of Northeastern University has a position as chief mathematician for Avco Manufacturing Corporation, Everett, Massachusetts.

Captain L. G. Campbell, previously an instructor at the United States Naval Academy, has been appointed to an instructorship at the United States Air Force Academy.

Associate Professor C. L. Carroll, Jr., of North Carolina State College has accepted a position as a mathematician at the Office of Scientific Research, Air

Research and Development Command, United States Air Force, Baltimore, Maryland.

Mr. L. E. Carville, formerly a mathematician for Army Map Service, Washington, D. C., is an engineer for Engineering and Research, Riverdale, Maryland.

Mrs. Ella F. Casey, recently a teacher in Mobile Public Schools, Alabama, is teaching at Wheatland-Chili Central School, Scottsville, New York.

Mr. F. A. Ceney, Jr., previously a student at Southern Illinois University, is now a mathematics teacher at East Peoria Community High School, Illinois.

Assistant Professor Peter Chiarulli of Brown University has accepted a position as a mathematician with the National Bureau of Standards, Washington, D. C.

Dr. T. S. Chihara, formerly a graduate teaching assistant at Purdue University, has been appointed to an assistant professorship at Seattle University.

Mr. R. R. Christensen of Northrop Aircraft, Hawthorne, California, has accepted a position as a member of the technical staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Assistant Professor J. G. Christiano of the University of Pittsburgh has been promoted to an associate professorship.

Associate Professor D. E. Christie of Bowdoin College has been promoted to a professorship.

Dr. E. J. Cogan of Pennsylvania State University has been appointed to an instructorship at Dartmouth College.

Assistant Professor Haskell Cohen of the University of Tennessee has been appointed to an assistant professorship at Louisiana State University.

Dr. W. J. Coles of the University of Wisconsin has a position as an analyst in the Department of Defense, Washington, D. C.

Dr. G. E. Collins, formerly a teaching assistant at Cornell University, has a position as a mathematician with I.B.M. Corporation, New York City.

Associate Professor H. D. Colson of State Teachers College, Bemidji, Minnesota, has been appointed to an associate professorship at Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor E. M. Cook of Northeastern University has been promoted to an associate professorship.

Dr. R. J. Dickson, Jr., has accepted a position as a research engineer with Lockheed Aircraft Corporation, Burbank, California.

Professor W. J. Dixon of the University of Oregon has been appointed to a professorship in the Department of Preventive Medicine and Public Health, University of California at Los Angeles.

Associate Professor Mary P. Dolciani of Vassar College has been appointed to an assistant professorship at Hunter College.

Mr. E. L. Dolney of the University of Alaska has been appointed to an assistant professorship at the School of Mines and Metallurgy, University of Missouri.

Assistant Professor Eldon Dyer of the University of Georgia has been appointed to an assistant professorship at Johns Hopkins University.

Dr. D. E. Edmonson, previously research instructor at Tulane University, has been appointed to an assistant professorship at Southern Methodist University.

Visiting Assistant Professor Joanne Elliott of Brown University has been appointed to an assistant professorship at Barnard College, Columbia University.

Dr. M. P. Epstein of the University of California has been appointed to an assistant professorship at Johns Hopkins University.

Assistant Professor Trevor Evans of Emory University has been promoted to an associate professorship.

Assistant Professor J. K. Everton of Utah State Agricultural College has a position with the Sandia Corporation, Albuquerque, New Mexico.

Visiting Professor Howard Eves of the University of Maine has been appointed to a professorship at the University.

Mr. D. L. Faas, previously a student at the University of Houston, has a position as a junior physicist with the Shell Development Company, Houston, Texas.

Assistant Professor J. V. Finch of Beloit College has been promoted to an associate professorship.

Mr. C. H. Finnie, Jr., has accepted a position as a mathematician at the Arnold Engineering Development Center, Tullahoma, Tennessee.

Mr. A. J. Flynn, previously an analyst at Eureka Williams Corporation, Bloomington, Illinois, has a position as an engineering assistant with General Electric Company, Bloomington.

Assistant Professor Kenneth Fowler of the University of Arizona has been appointed to a professorship at Teachers College at New Paltz, New York.

Professor C. H. Frick of Mary Washington College is on leave for the year 1955–1956 and is Head of Ballistics and Statistical Theory Branch, United States Naval Proving Ground, Dahlgren, Virginia.

Mr. W. B. Fritz of the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as a senior engineer-numerical analyst with Westinghouse Electric Corporation, Westinghouse Air-Arm Division, Baltimore, Maryland.

Assistant Professor R. E. Fullerton of the University of Wisconsin has been appointed to an associate professorship at the University of Maryland.

Miss D. Joan Gallagher, formerly a student at the University of Colorado, has a position as a mathematician with Douglas Aircraft Company, Santa Monica, California.

Assistant Professor Marion Goddard of Elmira College has been appointed to an instructorship at Baldwin-Wallace College.

Mr. A. J. Goldman, previously a graduate student at Princeton University, has been appointed to an instructorship at the University.

Assistant Professor Oscar Goldman of Brandeis University has been promoted to an associate professorship.

Mr. R. G. Green, recently a student at Abilene Christian College, is teaching in Merkel High School, Texas.

Mr. W. H. Gregory, formerly a teacher at Hayward Union High School, California, has been appointed to an instructorship at Monterey Peninsula College, California.

Assistant Professor N. G. Gunderson of the University of Rochester has been promoted to an associate professorship.

Mr. J. P. Gwin, recently a graduate student at Louisiana State University, is teaching at Wentworth Military Academy, Lexington, Missouri.

Assistant Professor B. F. Hadnot of Florida State University has a position as a mathematician with I.B.M. Corporation, Atlanta, Georgia.

Dr. C. J. A. Halberg, Jr., formerly a graduate student at the University of California at Los Angeles, has been appointed to an instructorship at the University of California at Riverside.

Mr. M. N. Haller, previously an engineer for Capehart-Farnsworth Company, Fort Wayne, Indiana, has a position now as an engineer with North Electric Company, Galion, Ohio.

Miss Kathleen Hamlin, formerly a student at Wayne University, has a position as associate scientist at Lockheed Aircraft Corporation, Van Nuys, California.

Dr. R. E. Heaton, recently a graduate student at Indiana University, has been appointed to an assistant professorship at the University of Richmond.

Assistant Professor C. H. Heinke of Capital University has been promoted to an associate professorship.

Mr. J. E. Hoffman, previously a teaching assistant at the University of Oklahoma, has been promoted to an instructorship.

Mr. Bernard Jacobson, formerly a graduate assistant at Michigan State University, has been promoted to an instructorship at the University.

Assistant Professor L. A. Jehn of the University of Dayton has been promoted to an associate professorship.

Professor E. R. Johnston of Wisconsin State College, Whitewater, has been appointed to an associate professorship at Purdue University.

Mr. C. E. Kelley of Wentworth Military Academy has been appointed to an assistant professorship at Central Missouri State College.

Assistant Professor R. J. Koch of Louisiana State University has been appointed Visiting Assistant Professor at Tulane University.

Mr. C. W. Koiner has a position as a physicist at the United States Naval Ordnance Test Station, China Lake, California.

Dr. E. Kreyszig of Stanford University has been appointed to the staff of the University of Ottawa.

Assistant Professor R. E. Lee of School of Mines and Metallurgy, University of Missouri, has been promoted to an associate professorship.

Dr. W. W. Leutert, formerly chief of the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, is now in charge of the Computing Center, Lockheed Missile Systems Division, Van Nuys, California.

Associate Professor P. E. Lewis of North Carolina State College has a position as a senior research engineer with Consolidated-Vultee Aircraft Corporation, San Diego, California.

Mrs. Sally I. Lipsey, previously a graduate student at Columbia University, has been appointed Lecturer at the University.

Professor Lee Lorch of Fisk University has been appointed to a professorship at Philander Smith College.

Dr. R. D. Luce, recently a fellow at the Center for Advanced Study in Behavioral Sciences, Menlo Park, California, has been appointed to an assistant professorship in mathematical statistics and sociology at Columbia University.

Dr. E. A. Maier, formerly on the research staff of Giustina Brothers Lumber Company, Eugene, Oregon, has been appointed to an assistant professorship at Pacific Lutheran College.

Mr. R. G. McDermot, previously an assistant at the University of Pittsburgh, has been appointed to an instructorship at the University.

Mr. K. A. McGown, formerly a teacher in Boonton High School, New Jersey, is teaching at Leonia High School, New Jersey.

Mr. R. A. McHaffey, previously a graduate assistant at Iowa State College, has been appointed Instructor in Chemistry at Newark College of Engineering.

Mr. R. J. McQuillin, recently a student at the College of Puget Sound, has a position as an assistant engineer with Pacific Telephone and Telegraph Company, Tacoma, Washington.

Assistant Professor L. J. Montzingo, Jr., of Roberts Wesleyan College has been promoted to an associate professorship.

Miss Vivian Morgan, formerly a student at the State College of Washington, is now a mathematician for North American Aviation Company, Los Angeles, California.

Professor R. K. Morley of Worcester Polytechnic Institute has retired with the title of Professor Emeritus.

Mr. G. E. Neu, formerly an instructor at the University of Buffalo, has a position with the I.B.M. Corporation, Buffalo, New York.

Miss Masako Oba, previously a student at the University of California at Los Angeles, has a position as a mathematician at Douglas Aircraft Company, Santa Monica, California.

Associate Professor J. M. H. Olmsted of the University of Minnesota has been promoted to a professorship.

Mr. J. L. Olpin of Eastern Arizona Junior College, Thatcher, Arizona, has been appointed to an assistant professorship at Brigham Young University.

Dr. F. P. Palermo of Princeton University has been appointed to an instructorship at the University of Michigan.

Assistant Professor J. M. Perry of Clarkson College of Technology has been promoted to an associate professorship.

Dr. G. M. Petersen of the University of Oklahoma has accepted an appointment as research fellow at University College, Swansea, Wales.

Assistant Professor R. P. Peterson, Jr., of the University of California, Riverside, has accepted a position as Head of the Operations Research Section, Controllers Department, Bank of America, San Francisco, California.

Mr. Nick Popiel, Jr., formerly a mathematician at Vinco Corporation, Detroit, Michigan, has a position as design analyst for the Ford Motor Company, Dearborn, Michigan.

Assistant Professor Valdemars Punga of Rensselaer Polytechnic Institute has been appointed to an associate professorship at Hartford Graduate Center of the Institute, East Windsor Hill, Connecticut.

Dr. Rimhak Ree of the University of British Columbia has been appointed Lecturer at Montana State University.

Mr. A. E. Richmond of Multnomah College has a position as an engineer at Tektronix, Inc., Portland, Oregon.

Assistant Professor J. D. Riley of the University of Kentucky has been appointed to an assistant professorship at Iowa State College.

Associate Professor D. L. Robb of Baldwin-Wallace College has been promoted to a professorship.

Mr. J. T. Robinson of Johns Hopkins University has been appointed to an instructorship at Canisius College.

Assistant Professor F. Virginia Rohde of the University of Florida has been promoted to an associate professorship.

Assistant Professor P. G. Rooney of the University of Alberta has been appointed to an assistant professorship at the University of Toronto.

Dr. M. E. Rose has accepted a position as a mathematician with the Office of Naval Research, Washington, D. C.

Miss Anne Rychlicki, previously a student at Alliance College, has a position as a research technician with the Ford Motor Company, Dearborn, Michigan.

Mr. G. X. Saltarelli, formerly a mathematician at Bell Aircraft Corporation, Niagara Falls, New York, is now a teacher at Amherst Central Senior High School, Snyder, New York.

Mr. J. A. Schumaker of Grinnell College has been appointed to an assistant professorship at New Jersey State Teachers College, Montclair.

Mr. R. C. Scott, previously an assistant at Lehigh University, has been promoted to an instructorship.

Dr. C. E. Sealander, formerly principal mathematician at Battelle Memorial Institute, Columbus, Ohio, has a position as a research specialist with the Boeing Airplane Company, Seattle, Washington.

Mr. C. R. Seliger, previously a teaching assistant at Rutgers University, has a position as an associate engineer at the Glenn L. Martin Company, Baltimore, Maryland.

Mr. E. G. Walsh, previously an engineering mathematician for Vinco Corporation, Detroit, Michigan, is now a design analyst for Ford Motor Company, Dearborn, Michigan.

Dr. W. H. Warner, formerly a research associate at Brown University, has been appointed to an assistant professorship at the University of Minnesota.

Dr. D. V. V. Wend of Iowa State College has been appointed to an assistant professorship at the University of Utah.

Assistant Professor R. F. Williams of Florida State University has been appointed to an instructorship at the University of Wisconsin.

Dr. F. L. Wolf of Carleton College has been promoted to an assistant professorship.

Assistant Professor F. H. Young of Montana State College has been appointed to an assistant professorship at Portland State College.

Assistant Professor J. L. Zemmer, Jr., of the University of Missouri has been promoted to an associate professorship.

Mr. J. M. Shaheen of Angola, Indiana, died on July 14, 1955, while serving in the United States Army.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE SIXTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The sixteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 3, 1956. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio, by a postcard request. Application blanks will be mailed out early in January. All applications must be filed with the Director not later than February 10, 1956. If

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Classroom Notes: D. H. Ballou, Herman Betz, W. F. Cheney, Kenneth Cooke, John Dyer-Bennet, William Feller, Tomlinson Fort, William C. Fox, Philip Franklin, Gordon Fuller, Leonard Gillman, Michael Golomb, E. A. Guillemin, F. B. Hildebrand, M. A. Hill, F. B. Householder, Witold Hurewicz, R. C. James, R. E. Johnson, Wilfred Kaplan, J. B. Kelly, M. S. Klamkin, Morris Kline, R. E. Langer, E. B. Leach, Walter Leighton, W. J. LeVeque, Robert Levit, L. H. Loomis, W. S. Loud, C. C. MacDuffee, F. H. Miller, Max Morris, M. E. Munroe, F. J. Murray, C. O. Oakley, Gordon Pall, Moses Richardson, R. F. Rinehart, J. T. Rule, A. H. Sprague, D. J. Struik, A. E. Taylor, H. P. Thielman, G. L. Walker, R. J. Walker, Morgan Ward, C. R. Wylie, R. C. Yates, Oscar Zariski.

CALENDAR OF FUTURE MEETINGS

Thirty-ninth Annual Meeting, Rice Institute, Houston, Texas, December 30, 1955.

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Geneva College, Beaver Falls, Pennsylvania, Spring, 1956.

ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.

INDIANA, Wabash College, Crawfordsville, May 5, 1956.

IOWA, Grinnell College, Grinnell, April 20–21, 1956.

KANSAS, University of Wichita, April 21, 1956.

KENTUCKY

LOUISIANA-MISSISSIPPI, McNeese State College, Lake Charles, Louisiana, February 17–18, 1956.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Catholic University, Washington, D. C., December 3, 1955.

METROPOLITAN NEW YORK, Stevens Institute of Technology, Hoboken, New Jersey, April 28, 1956.

MICHIGAN, University of Michigan, Ann Arbor, March, 1956.

MINNESOTA

MISSOURI, Fontbonne College, St. Louis, Spring, 1956.

NEBRASKA, University of Nebraska, Lincoln, April 21, 1956.

NEW ENGLAND

NORTHERN CALIFORNIA, Stanford University, Stanford, January 14, 1956.

OHIO, April 14, 1956.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.

PHILADELPHIA

ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.

SOUTHEASTERN, University of Georgia, Athens, March 16–17, 1956.

SOUTHERN CALIFORNIA, Pomona College, Claremont, March 17, 1956.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanical Arts, Las Cruces, April, 1956.

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